NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for
MA0301 Elementær diskret matematikk

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Date Signature

Problem 1

a) How many non-negative integer solutions does the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 21$$

have?

b) How many of these solutions satisfy the additional requirements $x_2 \geq 2$ and $x_3 \geq 3$?

c) How many of the solutions to (a) satisfy $x_4 \leq 4$ as an additional requirement?

Solution 1 *a) There are* $\binom{21+5-1}{21} = \binom{25}{21}$ *non-negative integer solutions to this equation.*

b) There are $\binom{16+5-1}{16} = \binom{20}{16}$ *solutions to (a) that also satisfy* $x_2 \geq 2$ *and* $x_3 \geq 3$.

c) There are $\binom{16+5-1}{16} = \binom{20}{16}$ *solution to (a) that satisfy* $x_4 \geq 5$. *Therefore, there are* $\binom{25}{21} - \binom{20}{16}$ *solutions to (a) that satisfy* $x_4 \leq 4$.

Problem 2

a) Determine whether or not the following statement is a tautology, by either proving it or giving a counterexample:

$$\big((p \to q) \wedge (r \to \neg q)\big) \to (p \wedge r)$$

b) Negate the statement given in (a). (In your final answer, the $\neg$ symbol may only appear directly in front of p, q and/or r.)

c) Establish the validity of the following argument using the rules of inference

$$\begin{array}{rl} & p \to \neg q \\ & \neg r \vee q \\ & r \\ \hline \therefore & \neg p \end{array}$$

Solution 2 *a) Using the truth values $p = 0$, $q = 0$, $r = 0$, we find that $(p \to q)$ is true and $(r \to \neg q)$ is true. Therefore $\big((p \to q) \wedge (r \to \neg q)\big)$ is true.*

On the other hand, the statement $p \wedge r$ is false, and hence $\big((p \to q) \wedge (r \to \neg q)\big) \to (p \wedge r)$ is false. Therefore, the statement is not a tautology.

b) We negate the statement and work the negation through the brackets:

$$\neg\Big(\big((p \to q) \wedge (r \to \neg q)\big) \to (p \wedge r)\Big) \iff \big((p \to q) \wedge (r \to \neg q)\big) \wedge \neg(p \wedge r)$$
$$\iff \big((p \to q) \wedge (r \to \neg q)\big) \wedge (\neg p \vee \neg r)$$

c) Using the rules of inference, we find that:

(1)	$p \to \neg q$	*premise*
(2)	$\neg r \vee q$	*premise*
(3)	r	*premise*
(4)	q	(2) + (3) + *rule of disjunctive syllogism*
(5)	$\therefore \neg p$	(1) + (4) + *modus tollens*

Problem 3 Let $r \in \mathbb{R}$ with $r \neq 1$. Use induction to prove that

$$\sum_{i=0}^{n} r^i = \frac{1 - r^{n+1}}{1 - r}$$

for all $n \in \mathbb{Z}^+$.

Solution 3 *We prove the formula by induction.*

Basic step: *For $n = 1$ we have*

$$\sum_{i=0}^{1} r^i = r^0 + r^1 = 1 + r$$

and

$$\frac{1 - r^{1+1}}{1 - r} = \frac{1 - r^2}{1 - r} = \frac{(1 + r)(1 - r)}{1 - r} = 1 + r.$$

Therefore, the formula holds for $n = 1$.

Now let $n \in \mathbb{Z}^+$ be arbitrary, and assume as an ***Induction Hypothesis*** *that*

$$\sum_{i=0}^{n} r^i = \frac{1-r^{n+1}}{1-r}.$$

Induction Step*: Using the induction hypothesis, we now prove the formula for $n+1$. We have*

$$\begin{aligned}
\sum_{i=0}^{n+1} r^i &= \left(\sum_{i=0}^{n} r^i\right) + r^{n+1} \\
&= \frac{1-r^{n+1}}{1-r} + r^{n+1} && \text{by the induction hypothesis} \\
&= \frac{1-r^{n+1}}{1-r} + \frac{(1-r)r^{n+1}}{1-r} \\
&= \frac{1-r^{n+1}}{1-r} + \frac{r^{n+1}-r^{n+2}}{1-r} \\
&= \frac{1-r^{n+1}+r^{n+1}-r^{n+2}}{1-r} \\
&= \frac{1-r^{n+2}}{1-r} \\
&= \frac{1-r^{(n+1)+1}}{1-r}
\end{aligned}$$

This shows that the formula also holds for $n+1$, which completes our induction proof. The formula must therefore be valid for all $n \in \mathbb{Z}^+$.

Problem 4

a) Give 6 different strings in the language $\{00\}\{101\}^* \cup \{011\}\{0\}^*\{01\}$

b) Construct a finite state machine that recognises this language (with $\{0,1\}$ as input and output alphabet).

Solution 4 *a) There are many possible answers to this question. For example, take the strings 00, 00101, 00101101, 01101, 011001, 0110001.*

b) The Finite State Machine drawn in figure 1 recognises this language. There are also other valid solutions possible.

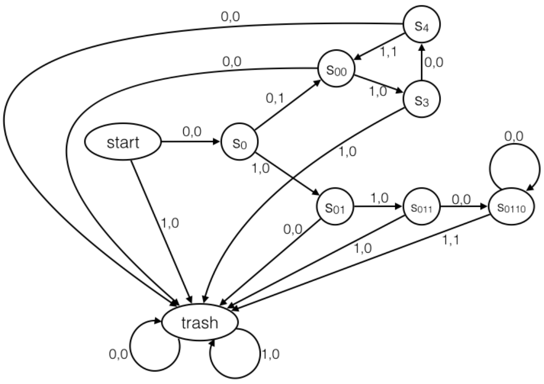

Figure 1: This figure belongs to Solution 4

Problem 5

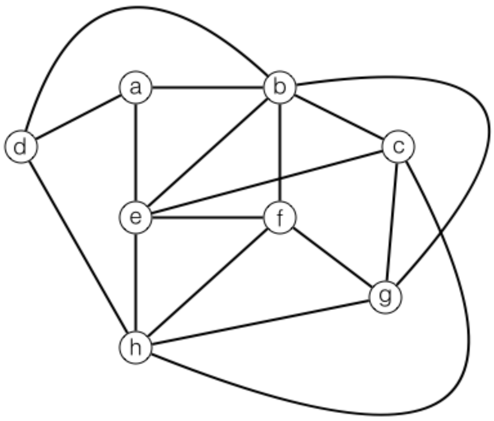

Figure 2: This figure belongs to Oppgave 5

a) Find a rooted spanning tree with root a for this graph using a Breadth First Search Algorithm and vertex order a,b,c,d,e,f,g,h.

b) Study the graph in Figure 1 and determine whether or not is it planar. Prove your answer.

c) Does the graph in Figure 1 have an Euler Trail? Prove your answer.

Solution 5 *a) We execute the Breadth First Search Algorithm with root a and vertex order a,b,c,d,e,f,g,h.*

We start the tree with only vertex a, and queue $Q = (a)$. *Next, we add vertices b, d and e to our tree with edges* $\{a, b\}$, $\{a, d\}$ *and* $\{a, e\}$. *Our new queue will be* $Q = (b, d, e)$. *We will now add c, f and g to our tree with edges* $\{b, c\}$, $\{b, f\}$ *and* $\{b, g\}$. *Our new queue will be* $Q = (d, e, c, f, g)$. *Finally, we add vertex h to our tree with edge* $\{d, h\}$. *Our queue will become* $Q = (e, c, f, g, h)$. *Visiting the vertices left in the queue will not give us any new vertices to add to our tree, since all vertices have already been added. See figure 3 for the BFS spanning tree.*

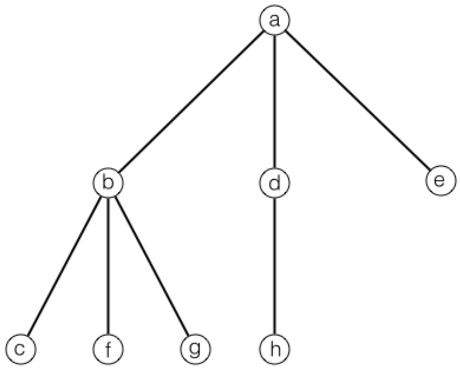

Figure 3: This figure belongs to Solution 5a

b) This graph is not planar. Consider the subgraph drawn in Figure 4, obtained by removing edges $\{h, f\}$, $\{b, f\}$, $\{a, b\}$, $\{a, d\}$, $\{a, e\}$ *and vertex a. This graph can be obtained from* K_5 *by elementary subdivisions: draw* K_5 *and label its vertices* b, c, e, g, h *and apply an elementary subdivision to edges* $\{e, g\}$ *and* $\{b, h\}$.

We see that our original graph contains a subgraph which is homeomorphic to K_5. *Therefore, it cannot be planar.*

c) Notice that vertices d and a have degree 3, and vertices e and h have degree 5. Since the graph has more than two vertices of odd degree, it cannot have an Euler Trail.

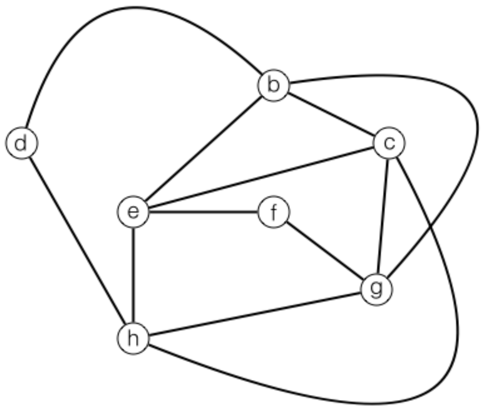

Figure 4: This figure belongs to Solution 5b

Problem 6 Let A be the set of all functions from $\mathbb{Z}^+$ to $\{1, 2, 3\}$.

a) Give the three properties of an equivalence relation. You may give the names of these properties, or their definitions.

b) Define a relation $\mathcal{R}_1$ on A by setting $f\mathcal{R}_1 g$ if and only if $f(5) = g(5)$. Prove that $\mathcal{R}_1$ is an equivalence relation.

c) Define a relation $\mathcal{R}_2$ on A by setting $f\mathcal{R}_2 g$ if and only if there exists an $n \in \mathbb{Z}^+$ such that $f(n) = g(n)$. Prove that $\mathcal{R}_2$ is not an equivalence relation.

Solution 6 *a) An equivalence relation $\mathcal{R}$ on a set A has to be*

- *reflexive ($\forall a \in A : a\mathcal{R}a$)*
- *symmetric ($\forall a, b \in A :$ if $a\mathcal{R}b$, then $b\mathcal{R}a$), and*
- *transitive ($\forall a, b, c \in A :$ if $a\mathcal{R}b$ and $b\mathcal{R}c$, then $a\mathcal{R}c$).*

b) We need to prove that $\mathcal{R}_1$ is reflexive, symmetric and transitive.

- *Reflexivity: If $f \in A$, then naturally we have $f(5) = f(5)$. Hence $f\mathcal{R}_1 f$. This proves reflexivity.*
- *Symmetry: If $f, g \in A$ and $f\mathcal{R}_1 g$, then $f(5) = g(5)$. This implies that $g(5) = f(5)$, so $g\mathcal{R}_1 f$. Therefore, the relation is symmetric.*

- *Transitivity: If* $f, g, h \in A$ *with* $f\mathcal{R}_1 g$ *and* $g\mathcal{R}_1 h$*, then we know that* $f(5) = g(5)$ *and* $g(5) = h(5)$*. This implies that* $f(5) = h(5)$*, so* $f\mathcal{R}_1 h$*. This proves transitivity.*

Since all three properties are satisfied, $\mathcal{R}_1$ *is an equivalence relation.*

c) *Define the function* $f\colon \mathbb{Z}^+ \to \{1, 2, 3\}$ *by* $f(n) = 1$ *for all* $n \in \mathbb{Z}^+$*.*

Define the function $g\colon \mathbb{Z}^+ \to \{1, 2, 3\}$ *by* $g(1) = 1$ *and* $g(n) = 2$ *for all* $n \geq 2$ *with* $n \in \mathbb{Z}^+$*.*

Define the function $h\colon \mathbb{Z}^+ \to \{1, 2, 3\}$ *by* $h(n) = 2$ *for all* $n \in \mathbb{Z}^+$*.*

Then f*,* g*, and* h *are elements of* A*. Since* $f(1) = 1 = g(1)$*, we have* $f\mathcal{R}_2 g$*. And as* $g(2) = h(2)$*, we have* $g\mathcal{R}_2 h$*. Now note that for all* $n \in \mathbb{Z}^+$ *we have* $f(n) = 1 \neq 2 = h(n)$*, which implies that* $f \not\mathcal{R}_2 h$*. Therefore, the relation* $\mathcal{R}_2$ *is not transitive. This means that it cannot be an equivalence relation.*