MA0301 ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2019

Exam 1

Exercise 1 (Sets):	15 points
Exercise 2 (Relations):	15 points
Exercise 3 (Induction):	20 points
Exercise 4 (Functions):	15 points
Exercise 5 (Graphs):	15 points
Exercise 6 (Boolean algebra):	10 points
Exercise 7 (Finite state automata)	10 points

Total: <u>100 points</u>

Note: In each of the exercises 1.1, 2.1, 4.1, 5.1, exactly one answer is correct.

Exercise 1. Sets (15 points)

(1) **(3 points)** *Suppose* $U = \mathbb{N} = \{1, 2, 3, 4, 5, 6, \ldots\}$. *Let* $A = \{1, 2, 3, 4\}$ *and* $B := \{2, 3, 8, 9\}$.

Which of the three statements is correct?

I) The symmetric difference $A \triangle B = \{1, 4, 8, 9\}$.

II) The symmetric difference $A \triangle B = U - (A \cup B)$.

III) The symmetric difference $A \triangle B = \{2, 3\}$.

(2) **(5 points)** *A* fundamental product *of the sets* $A_1, A_2, \ldots, A_n$ *is defined to be a set of the form* $A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_n^{\epsilon_n}$, *where* $A_i^{\epsilon_i}$ *is either the set* A_i *or its complement* $\overline{A_i}$.

a) *List all fundamental products of three sets* A_1, A_2, A_3

b) *Find the number of fundamental products of* m *sets* $A_1, A_2, \ldots, A_m$

(3) **(7 points)** *Write down the definition of the cartesian product of two sets. Show that for three sets* A, B, C *the following equality of sets holds:*

$$A \times (B - C) = (A \times B) - (A \times C).$$

Solution 1. 1) I is correct.

2a) $A_1 \cap A_2 \cap A_3$, $\quad \overline{A_1} \cap A_2 \cap A_3$, $\quad A_1 \cap \overline{A_2} \cap A_3$, $\quad A_1 \cap A_2 \cap \overline{A_3}$,
$\overline{A_1} \cap \overline{A_2} \cap A_3$, $\quad A_1 \cap \overline{A_2} \cap \overline{A_3}$, $\quad \overline{A_1} \cap A_2 \cap \overline{A_3}$, $\quad \overline{A_1} \cap \overline{A_2} \cap \overline{A_3}$

2b) Observe that the set $A_1^{\epsilon_1}$ in a fundamental product $A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_m^{\epsilon_m}$ can be either A_1 or its complement $\overline{A_1}$. The same holds for the sets $A_2^{\epsilon_2}, \ldots, A_m^{\epsilon_m}$. Hence, there are 2^m such fundamental products.

3) $A \times B := \{(a, b) \mid a \in A, b \in B\}$

Let $(x, y) \in (A \times B) - (A \times C)$. This is equivalent to $(x, y) \in (A \times B)$ and $(x, y) \notin (A \times C)$. This is equivalent to $x \in A \wedge y \in B$ and $\neg(x \in A \wedge y \in C) \Leftrightarrow (x \in A \wedge y \in B) \wedge (x \notin A \vee y \notin C) \Leftrightarrow (x \in A \wedge y \in B \wedge x \notin A) \vee (x \in A \wedge y \in B \wedge y \notin C) \Leftrightarrow (x \in A \wedge y \in B \wedge y \notin C) \Leftrightarrow x \in A \wedge y \in (B - C) \Leftrightarrow (x, y) \in A \times (B - C)$.

Date: May 24, 2019.

Exercise 2. Relations (15 points)

(1) **(3 points)** *Which of the three statements is correct?*

I) A partial order P on a set X is reflexive, symmetric, and transitive.

II) A partial order P on a set X is reflexive, anti-symmetric, and transitive.

III) A partial order P on a set X is anti-reflexive, anti-symmetric, and transitive.

(2) **(4 points)** *Show that:*

$$R = \{(a,a),(b,b),(c,c),(d,d),(e,e),(a,b),(a,c),(a,d),(a,e),(b,c),(e,d),(e,c)\}$$

defines a partial order on $A := \{a,b,c,d,e\}$ and draw the corresponding Hasse diagram.

(3) **(4 points)** *Let $B := \{2,3,4,16\}$ be ordered by divisibility. Find the maximal and minimal elements of B.*

(4) **(4 points)** *Let $C := \{a,b,c,d,e,f,g,h\}$. For each of the following families of subsets of C, determine whether or not it is a partition of C. If it is a partition of C draw the graphical representation of the corresponding equivalence relation.*

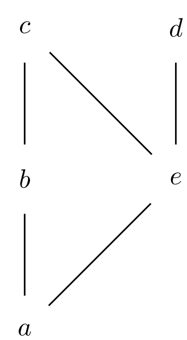
a) $\{\{a,c,e,g\},\{b,d\},\{h,f,c\}\}$

b) $\{\{a,c,e,g\},\{d,f,c\}\}$

c) $\{\{a,c,e,g\},\{b,h\},\{d,f\}\}$

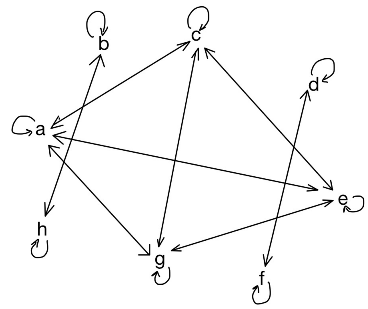
Solution 2. 1) II) is correct.

2) Hasse diagram for partial order R on $A := \{a,b,c,d,e\}$



3) The maximal elements are 3 and 16. The minimal elements are 2 and 3.

4) a) is not partition since the subsets are not disjoint; b) is not a partition since no subset contains the element b; c) is a partition



Exercise 3. Induction (20 points)

(1) **(5 points)** *Use induction to show that*

$$\sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} = \frac{n}{2n+1}.$$

(2) **(7 points)** *Determine for which natural numbers we have* $n^2 \geq 2n+1$.

(3) **(8 points)** *Let* c_n *be the sequence defined by* $c_0 = 1 = c_1$, $c_2 = 3$ *and for* $n > 0$

$$c_{n+2} = 3c_{n+1} - 3c_n + c_{n-1}.$$

Show that for $n \geq 0$

$$c_n = n^2 - n + 1.$$

Solution 3. 1) base step: $n = 1$: $\frac{1}{3} = \frac{1}{2+1}$. Assume that the statement holds for $n > 0$.

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{(2k-1)(2k+1)} &= \sum_{k=1}^{n} \frac{1}{(2k-1)(2k+1)} + \frac{1}{(2(n+1)-1)(2(n+1)+1)} \\
&= \frac{n}{2n+1} + \frac{1}{(2(n+1)-1)(2(n+1)+1)} \\
&= \frac{n}{2n+1} + \frac{1}{(2n+1)(2n+3)} \\
&= \frac{2n^2+3n+1}{(2n+1)(2n+3)} \\
&= \frac{(n+1)(2n+1)}{(2n+1)(2n+3)} \\
&= \frac{n+1}{2n+3} = \frac{n+1}{2(n+1)+1}.
\end{aligned}$$

Hence, the statement is true for all $n > 0$.

2) Observe that $1^2 = 1 < 3$ and $2^2 = 4 < 5$, but $3^2 = 9 > 7$. Hence, the inequality is true for $n = 3$, which serves also as our base step. Now, assume that $n^2 \geq 2n+1$ and consider

$$(n+1)^2 = n^2 + 2n + 1 \geq 2n + 1 + 2n + 1 = 2n + 2 + 2n \geq 2n + 2 + 1 = 2(n+1) + 1.$$

3) Given $c_0 = 1 = c_1$, $c_2 = 3$ and for $n > 2$ shifting the recursion we have

$$c_n = 3c_{n-1} - 3c_{n-2} + c_{n-3}.$$

We want to show that $c_n = n^2 - n + 1$.
Base step: $n = 0, 1, 2$ we have

$$c_0 = 1 = 0^2 - 0 + 1, \quad c_1 = 1 = 1^2 - 1 + 1, \quad c_2 = 3 = 2^2 - 2 + 1.$$

Ind. hypothesis: we assume that for a fixed n, if $0 \leq i \leq n$, then $c_i = i^2 - i + 1$.
Ind. step: to obtain c_{n+1}, we can use c_n, c_{n-1}, c_{n-2}. We have $c_n = n^2 - n + 1$, $c_{n-1} = (n-1)^2 - (n-1) + 1$, $c_{n-2} = (n-2)^2 - (n-2) + 1$. With this we obtain

$$c_{n+1} = 3c_n - 3c_{n-1} + c_{n-2} = 3(n^2 - n + 1) - 3((n-1)^2 - (n-1) + 1) + (n-2)^2 - (n-2) + 1.$$

This then yields

$$3n^2 - 3n + 3 - 3n^2 + 9n - 9 + n^2 - 5n + 7 = n^2 + n + 1 = n^2 + 2n + 1 - (n+1) + 1 = (n+1)^2 - (n+1) + 1.$$

Exercise 4. Functions (15 points)

(1) **(3 points)** *Let A, B be sets. Suppose that f is a subset of $A \times B$. Which of the three statements is correct?*

I) The set f defines a function if and only if each element $a \in A$ appears as the first coordinate in at least one ordered pair of f.

II) The set f defines a function if and only if each element $b \in B$ appears as the second coordinate in exactly one ordered pair of f.

III) The set f defines a function if and only if each element $a \in A$ appears as the first coordinate in exactly one ordered pair of f.

(2) **(5 points)** *Define the function $f(x) := 2x - 3$ from $\mathbb{R}$ to $\mathbb{R}$. Show that F is surjective and injective. Find its inverse function f^{-1}.*

(3) **(7 points)** *Let $f : A \to B$ and $g : B \to C$ be given. Show that if $g \circ f : A \to C$ is injective, then f is injective.*

Solution 4. 1) III) is correct.

2) Both injectivity and surjectivity are clear. The inverse is $f^{-1}(x) = \frac{x+3}{2}$.

3) Suppose that f is not injective. This means that there exist elements $a, b \in A$, $a \neq b$, for which $f(a) = f(b)$. With this we obtain that $(g \circ f)(a) = g(f(a)) = g(f(b)) = (g \circ f)(b)$, which implies that $(g \circ f)$ is not injective, contradicting the hypothesis. Hence, with $(g \circ f)$ injective f must be injective, too.

Exercise 5. Graphs (15 points)

(1) **(3 points)** *Which of the three statements is correct?*

I) Let $G = G(V, E)$ be a (multi-)graph. Then G has a closed Euler trail if and only if it is connected and every vertex of G has either degree 3 or degree 5.

II) Let $G = G(V, E)$ be a (multi-)graph. Then G has a closed Euler trail if and only if it is connected and every vertex of G has even degree.

III) Let $G = G(V, E)$ be a (multi-)graph. Then G has a closed Euler trail if and only if it is connected and G has exactly two vertices of odd degree.

(2) **(5 points)** *Let $V = \{a, b, c, d\}$. Determine which of the following graphs $G = G(V, E)$ has an Euler trail or Euler circuit.*

a) $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}\}$.
b) $E = \{\{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{d, a\}\}$.
c) $E = \{\{a, b\}, \{c, d\}, \{b, a\}, \{c, c\}, \{d, c\}\}$.

(3) **(7 points)** *Find the number of edges of the complete graph K_n, $n \geq 1$.*

Solution 5. 1) II

2) Find the degree of each vertex.

a) All vertices have even degree: Euler circuit

b) All vertices have degree three: no Euler circuit; no Euler trail

c) The graph is not connected.

3) Each pair of vertices determines an edge. There are $\frac{n!}{2!(n-2)!}$ ways of selecting two vertices out of n. Hence, there are $= n(n-1)/2$ edges in K_n, $n > 0$.

Exercise 6. Boolean algebra (10 points)

(1) **(5 points)** *Let B be a Boolean algebra and let a be any element in B. Show that if $a + x = 1$ and $a \cdot x = 0$, then $x = \bar{a}$.*

(2) **(5 points)** *Let B be a Boolean algebra. Show that for any elements $a, b \in B$ we have that i) $a + b = b$ is equivalent to ii) $a \cdot b = a$.*

Solution 6. 1) We know that $\bar{a} = \bar{a} + 0 = \bar{a} + a \cdot x = (\bar{a} + a) \cdot (\bar{a} + x) = 1 \cdot (\bar{a} + x) = (\bar{a} + x)$. We also know that $x = x + 0 = x + (a \cdot \bar{a}) = (x + a) \cdot (x + \bar{a}) = 1 \cdot (x + \bar{a}) = x + \bar{a}$. Therefore $x = x + \bar{a} = \bar{a}$.

2) Suppose that $a \cdot b = a$. The absorption law tell us that $b = b + (a \cdot b) = b + a = a + b$. Now suppose that $a + b = b$. The absorption law yields $a = a \cdot (a + b) = a \cdot b$. Therefore $a + b = b$ iff $a \cdot b = a$.

Exercise 7. Finite state automata (10 points)

(1) **(5 points)** *Let* $\Sigma := \{a, b\}$ *and define the language* $L := \{a^m b^n \mid m, n > 0\}$. *Construct an automaton* A' *which will accept the language* $L(A')$.

(2) **(5 points)**

a) *Draw the table for the automaton* A *in Fig. 1.*

b) *Find the language* $L(A)$ *accepted by the automaton* A *in Fig. 1.*

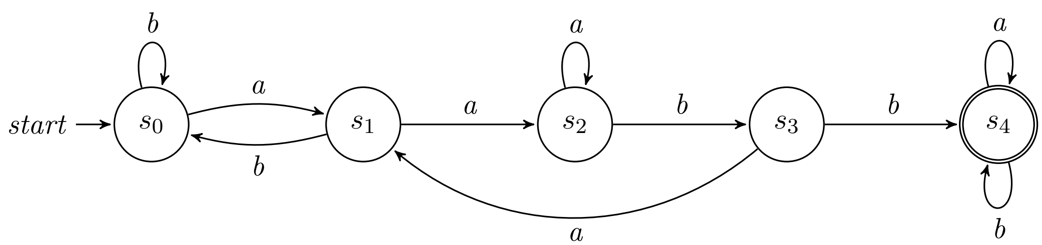


FIGURE 1. The automaton A.

Solution 7. 1)

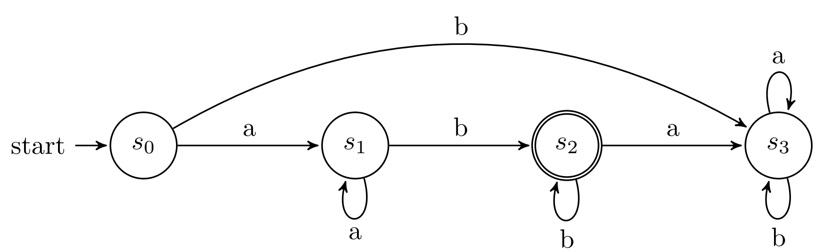


FIGURE 2. The automaton A' accepting $L := \{a^m b^n \mid m, n > 0\}$.

2)

A	ν	
	a	b
s_0	s_1	s_0
s_1	s_2	s_0
s_2	s_2	s_3
s_3	s_1	s_4
s_4	s_4	s_4

Words that reach the accepting state s_4 in the automaton A in Fig. 1., and stay there are exactly the words that contain the sequence $aabb$ as a subword. The language $L(A)$ consists of those words.