MA0301 ELEMENTARY DISCRETE MATHEMATICS
NTNU, SPRING 2020

ENGLISH VERSION

EXAM 1

Exercise 1 (Sets):	10 points
Exercise 2 (Logic):	10 points
Exercise 3 (Relations):	10 points
Exercise 4 (Induction):	20 points
Exercise 5 (Functions):	15 points
Exercise 6 (Graphs):	15 points
Exercise 7 (Combinatorics):	8 points
Exercise 8 (Finite state automata & machines)	12 points

Total: 100 points

Note: In each of the multiple choice exercises exactly one answer is correct.

Exercise 1. Sets (10 points)

(1) **(1 point)** *D_n is the set of positive integers which divide exactly the positive integer n. Which of the three statements is correct?*

A) $D_{60} \cap D_{84} = D_{12}$.

B) $D_{60} \cap D_{84} = D_6$.

C) $D_{60} \cap D_{84} = \emptyset$.

(2) **(1 point)** *Which of the three statements is correct?*

I) Let $X := \{x_1, x_2, x_3, x_4, x_5\}$ and $Y := \{y_1, y_2, y_3, y_4\}$. The cardinality of the powerset of the cartesian product of X and Y is larger than $2^{|X||Y|+1}$.

II) Let $X := \{x_1, x_2, x_3, x_4, x_5\}$ and $Y := \{y_1, y_2, y_3, y_4\}$. The cardinality of the powerset of the cartesian product of X and Y is 1048576.

III) Let $X := \{x_1, x_2, x_3, x_4, x_5\}$ and $Y := \{y_1, y_2, y_3, y_4\}$. The cardinality of the powerset of the cartesian product of X and Y is smaller than 1038576.

(3) **(3 points)** *Let X, Y, and Z be sets. Demonstrate that $\overline{X \cap Y \cap Z} = \overline{X} \cup \overline{Y} \cup \overline{Z}$ by showing that each side is a subset of the other side.*

(4) **(5 points)** *Consider the set $A = \{1,2,3,4,5,6,7,8,9\}$. Let $P_1 = \{\{1,3,5,7,9\},\{2,4,6,8\}\}$ and $P_2 = \{\{1,2,3,4\},\{5,7\},\{6,8,9\}\}$ be two partitions of A. Compute the following set*

$$P_3 := \{P_{1i} \cap P_{2j} \mid i = 1,2,\ j = 1,2,3\} \backslash \emptyset,$$

where P_{11}, P_{12} and P_{21}, P_{22}, P_{23} are the blocks of P_1 and P_2, respectively. In words, the set P_3 consists of all intersections between the blocks of P_1 and P_2, excluding the empty set. Show that P_3 is a partition of A.

Date: May 29, 2020.

Solution 1. *(1) The answer is A). By definition, $D_{60} \cap D_{84}$ is the set of common divisors of 60 and 84. Since the greatest common divisor of 60 and 84 is 12, and any other common divisor divides 12, we have that $D_{60} \cap D_{84} = D_{12}$.*

(2) The answer is II). Note that $|X| = 5$ and $|Y| = 4$. The cartesian product of X and Y is defined as the set of ordered pairs of the form (x, y), such that $x \in X$ and $y \in Y$. Note that there are $5 \cdot 4 = 20$ of these ordered pairs, and then $|X \times Y| = 20$. Now, we know that the cardinality of the power set of a set of cardinality n is given by 2^n. Then, the desired cardinality is equal to $2^{20} = 1048576$.

(3) We use the corresponding definitions of complement, intersection and union of sets:

$$\begin{aligned}
x \in \overline{X \cap Y \cap Z} \quad &\Leftrightarrow \quad x \notin X \cap Y \cap Z \qquad (\textit{definiton of complement}) \\
&\Leftrightarrow \quad x \notin X \vee x \notin Y \vee x \notin Z \qquad (\textit{DeMorgan's Law}) \\
&\Leftrightarrow \quad x \in \overline{X} \vee x \in \overline{Y} \vee x \in \overline{Z} \qquad (\textit{definiton of complement}) \\
&\Leftrightarrow \quad x \in \overline{X} \cup \overline{Y} \cup \overline{Z} \qquad (\textit{definiton of union})
\end{aligned}$$

The chain of right implications allows us to conclude that $x \in \overline{X \cap Y \cap Z} \Rightarrow x \in \overline{X} \cup \overline{Y} \cup \overline{Z}$, which means that $\overline{X \cap Y \cap Z} \subseteq \overline{X} \cup \overline{Y} \cup \overline{Z}$. On the other hand, the chain of left implications establishes that $\overline{X} \cup \overline{Y} \cup \overline{Z} \subseteq \overline{X \cap Y \cap Z}$. Hence $\overline{X \cap Y \cap Z} = \overline{X} \cup \overline{Y} \cup \overline{Z}$ as we wanted to show.

(4) First compute the sets of the collection P_3:

$$\begin{aligned}
P_{11} \cap P_{21} &= \{1,3,5,7,9\} \cap \{1,2,3,4\} = \{1,3\}, \\
P_{11} \cap P_{22} &= \{1,3,5,7,9\} \cap \{5,7\} = \{5,7\}, \\
P_{11} \cap P_{23} &= \{1,3,5,7,9\} \cap \{6,8,9\} = \{9\}, \\
P_{12} \cap P_{21} &= \{2,4,6,8\} \cap \{1,2,3,4\} = \{2,4\}, \\
P_{12} \cap P_{22} &= \{2,4,6,8\} \cap \{5,7\} = \emptyset, \\
P_{12} \cap P_{23} &= \{2,4,6,8\} \cap \{6,8,9\} = \{6,8\}.
\end{aligned}$$

Hence

$$P_3 = \{\{1,3\}, \{5,7\}, \{9\}, \{2,4\}, \{6,8\}\}.$$

Now we show that P_3 is a partition of A. This is by definition since all the elements of P_3 are pairwise disjoint sets and their union is exactly A:

$$\{1,3\} \cup \{5,7\} \cup \{9\} \cup \{2,4\} \cup \{6,8\} = A.$$

Exercise 2. Logic (10 points)

(1) **(1 point)** *Which of the three statements is correct?*

A) The negation of $\exists x \forall y\ (p(x,y) \wedge \neg q(x,y))$ *is* $\exists x \forall y\ (p(x,y) \vee q(x,y))$.

B) The negation of $\exists x \forall y\ (p(x,y) \wedge \neg q(x,y))$ *is* $\forall x \exists y\ (p(x,y) \vee q(x,y))$.

C) The negation of $\exists x \forall y\ (p(x,y) \wedge \neg q(x,y))$ *is* $\forall x \exists y\ (p(x,y) \Rightarrow q(x,y))$.

(2) **(2 points)** *Which of the three statements is correct?*

I) For primitive statements p and q, the statement $\neg p \vee (q \Rightarrow (p \wedge q))$ *is a tautology.*

II) For primitive statements p and q, the statement $\neg p \vee (q \Rightarrow (p \wedge q))$ *is a contradiction.*

III) For primitive statements p and q, the statement $\neg p \vee (q \Rightarrow (p \wedge q))$ *is unsatisfiable.*

(3) **(3 points)** *Use a truth table to show that* $p \Rightarrow (q \vee r)$ *is logically equivalent to* $(p \Rightarrow q) \vee (p \Rightarrow r)$.

(4) **(4 points)** *For primitive statements* p, q, r, s, *use the laws of logic to simplify the statement*

$$(\neg q \vee (((\neg r \wedge p) \wedge q) \vee (p \wedge (r \wedge q)))) \Rightarrow s.$$

Solution 2. *(1) The answer is C). By DeMorgan's Law and negation of the quantifiers* $\exists$ *and* $\forall$, *we have that*

$$\begin{aligned} \neg\left(\exists x \forall y\ (p(x,y) \wedge \neg q(x,y))\right) &\equiv \forall x \neg\left(\forall y\ (p(x,y) \wedge \neg q(x,y))\right) \\ &\equiv \forall x \exists y\ \neg(p(x,y) \wedge \neg q(x,y)) \\ &\equiv \forall x \exists y\ (\neg p(x,y) \vee \neg\neg q(x,y)) \\ &\equiv \forall x \exists y\ (\neg p(x,y) \vee q(x,y)) \\ &\equiv \forall x \exists y\ (p(x,y) \Rightarrow q(x,y)), \end{aligned}$$

where in the last step we used that $\neg p \vee q \equiv p \Rightarrow q$.

(2) The answer is I). Note that $q \Rightarrow (p \wedge q) \equiv \neg q \vee (p \wedge q) \equiv (\neg q \vee p) \wedge (\neg q \vee q) \equiv \neg q \vee p$ *and* $\neg p \vee (q \Rightarrow (p \wedge q)) \equiv \neg p \vee \neg q \vee p \equiv (p \vee \neg p) \vee \neg q \equiv T \vee \neg q \equiv T$, *where* T *is a tautology.*

(3) The corresponding truth table is the following:

p	q	r	$q \vee r$	$p \Rightarrow (q \vee r)$	$p \Rightarrow q$	$p \Rightarrow r$	$(p \Rightarrow q) \vee (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

Since the fifth and eighth columns coincide, we conclude that $p \Rightarrow (q \vee r) \equiv (p \Rightarrow q) \vee (p \Rightarrow r)$.

(4) Let us to simplify the left hand-side of the conditional. Notice that

$$\begin{aligned}((\neg r \wedge p) \wedge q) \vee (p \wedge (r \wedge q)) &\equiv ((p \wedge q) \wedge \neg r) \vee ((p \wedge q) \wedge r) \quad \textit{(commutative and associative law)} \\ &\equiv (p \wedge q) \wedge (r \vee \neg r) \quad \textit{(distributive law)} \\ &\equiv p \wedge q \quad \textit{(since } r \vee \neg r \textit{ is a tautology)}\end{aligned}$$

Then

$$\begin{aligned}\neg q \vee (((\neg r \wedge p) \wedge q) \vee (p \wedge (r \wedge q))) &\equiv \neg q \vee (p \wedge q) \\ &\equiv (\neg q \vee p) \wedge (\neg q \vee q) \quad \textit{(distributive law)} \\ &\equiv \neg q \vee p \quad \textit{(since } q \vee \neg q \textit{ is a tautology)} \\ &\equiv q \Rightarrow p.\end{aligned}$$

Hence

$$\big(\neg q \vee (((\neg r \wedge p) \wedge q) \vee (p \wedge (r \wedge q)))\big) \Rightarrow s \equiv (q \Rightarrow p) \Rightarrow s.$$

Exercise 3. Relations (10 points)

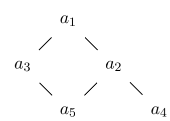
(1) **(2 points)** *Which of the three statements is correct?*

A) Consider the positive integers together with the relation $R := \{(n,m) \mid n+m \text{ even}\}$. *Then R defines a poset on the positive integers.*

B) Consider the positive integers together with the relation $R := \{(n,m) \mid n+m \text{ even}\}$. *Then R defines an equivalence relation on the positive integers.*

C) Consider the positive integers together with the relation $R := \{(n,m) \mid n+m \text{ even}\}$. *Then R defines a linear order on the positive integers.*

(2) **(3 points)** *Consider the following Hasse diagram*



corresponding to the poset (A,R). Write down A and $R \subseteq A \times A$.

(3) **(5 points)** *Determine the minimal elements of the set*

$$S = \{\{1\},\{2,3\},\{1,4\},\{3,4\},\{3,5\},\{1,2,4\},\{2,4,5\},\{1,2,3,4,5\}\}.$$

with the proper subset order $\subsetneq$.

Solution 3. *(1) The answer is B). Let us show that R is reflexive, symmetric and transitive.*

- *Reflexivity: Let n be a positive integer. Notice that $2n = n+n$ is obviously an even number, then $(n,n) \in R$. Hence R is reflexive.*
- *Symmetry: Let n,m be positive integers such that $(n,m) \in R$. By definition, we have that $n+m$ is an even number. Since $n+m = m+n$, we have that $m+n$ is an even number and hence $(m,n) \in R$. Hence R is symmetric. This in particular implies that R cannot be a partial order or a linear order.*
- *Transitivity: Let n,m,p be positive integers such that $(n,m),(m,p) \in R$. By definition, we have that $n+m$ and $m+p$ are even numbers. Then their sum $n+2m+p$ is an even number, let us say that it has the form $2r$ for some r positive integer. Then we have that*

$$n+p = 2r - 2m = 2(r-m).$$

Then we have that $n+p$ is an even number and hence $(n,p) \in R$.

From above, we conclude that R is an equivalence relation on the positive integers.

(2) According to the diagram, we have that a_5Ra_3, a_5Ra_2, a_4Ra_2, a_3Ra_1 and a_2Ra_1. By considering transitivity, we also have the pairs a_5Ra_1 and a_4Ra_1. Finally, by writing down the reflexive pairs, we have that the poset (A,R) is given by

$$A = \{a_1,a_2,a_3,a_4,a_5\},$$

$$R = \{(a_1,a_1),(a_2,a_2),(a_3,a_3),(a_4,a_4),(a_5,a_5),(a_5,a_3),$$
$$(a_5,a_2),(a_5,a_1),(a_4,a_2),(a_4,a_1),(a_3,a_1),(a_2,a_1)\}.$$

(3) Notice that $\{1,4\}$, $\{1,2,4\}$ *and* $\{1,2,3,4,5\}$ *cannot be minimal elements of* S *because* $\{1\} \in S$ *and* $\{1\} \subsetneq \{1,4\}, \{1,2,4\}, \{1,2,3,4,5\}$. *The remaining five elements of the set*

$$m(S) = \{\{1\}, \{2,3\}, \{3,4\}, \{3,5\}, \{2,4,5\}\}$$

are the minimal elements of S *since, given* $A \in m(S)$ *there is no other* $B \in S$ *such that* $B \subsetneq A$. *Hence the list of minimal elements of* S *is given by* $m(S)$.

Exercise 4. Induction (20 points)

(1) **(3 points)** *Use induction to show that for $m > 0$*

$$\sum_{k=1}^{m} 2^{k-1}k^2 = 2^m(m(m-2)+3) - 3.$$

(2) **(4 points)** *Show that if m is a positive integer, then the number $x_m := 8^m - 14m + 27$ is divisible by 7.*

(3) **(5 points)** *Show that for $m > 0$*

$$\sum_{k=1}^{m} k^3 = \Big(1+2+3+\cdots+m\Big)^2.$$

(4) **(8 points)** *Let $(F_0 = 0, F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, \ldots)$, $F_n = F_{n-1} + F_{n-2}$, $n > 1$, be the Fibonacci numbers. Use the identity $F_{n+1}^2 - F_{n-1}^2 = F_{2n}$, $n > 0$, to show that for $m > 0$*

$$\sum_{i=1}^{m} F_{4i-2} = F_{2m}F_{2m}.$$

Solution 4. *(1) Base case: For $m = 1$, we have that $\sum_{k=1}^{1} 2^{k-1}k^2 = 2^0 1^2 = 1 = 4 - 3 = 2^1(1(1-2)+3) - 3$. Hence, the formula holds for $m = 1$. For the induction hypothesis, we will assume that the formula holds for a positive integer $m \geq 1$. We will show that the formula also holds for $m + 1$. Indeed, by splitting the sum and using the induction hypothesis we have*

$$\begin{aligned}
\sum_{k=1}^{m+1} 2^{k-1}k^2 &= \sum_{k=1}^{m} 2^{k-1}k^2 + 2^m(m+1)^2 \\
&= 2^m(m(m-2)+3) - 3 + 2^m(m+1)^2 \quad \textit{(by induction hypothesis)} \\
&= 2^m((m+1)^2 + m^2 - 2m + 3) - 3 \\
&= 2^m((m+1)^2 + m^2 - 2m - 3 + 6) - 3 \quad \textit{(by adding up } 0 = -3+3\textit{)} \\
&= 2^m((m+1)^2 + (m+1)(m-3) + 6) - 3 \\
&= 2^m((m+1)(m+1+m-3) + 6) - 3 \\
&= 2^m((m+1)(2(m-1)) + 6) - 3 \\
&= 2^{m+1}((m+1)((m+1)-2) + 3) - 3.
\end{aligned}$$

Hence the formula holds for $m + 1$. By mathematical induction, we conclude that the formula holds for any $m > 0$.

(2) Base case: For $m = 1$, we have that $x_1 = 8^1 - 14 \cdot 1 + 27 = 21$, which is divisible by 7. For the induction hypothesis, assume that there is a positive integer m such that x_m is divisible by 7. For the induction step, we will show that x_{m+1} is divisible by 7. Indeed, by hypothesis we have that $x_m = 7k$ for some positive integer k. Then

$$x_{m+1} = 8^{m+1} - 14(m+1) + 27 = (8^m - 14m + 27) + 8^{m+1} - 8^m - 14 = 7k + 8^m(8-1) - 14 = 7(k + 8^m - 2),$$

and hence we conclude that x_{m+1} is divisible by 7. By mathematical induction, we conclude that x_m is divisible by 7 for any positive integer $m > 0$.

(3) For the base case, we have that $\sum_{k=1}^{1} k^3 = 1^3 = 1 = 1^2$*. Then the formula holds for* $m = 1$*. For the induction hypothesis, we assume that the formula holds for a positive integer* $m > 0$*. We will prove that the formula holds for* $m+1$*. Recall that*

$$1 + 2 + \cdots + m = \frac{m(m+1)}{2}.$$

By splitting the sum and using the induction hypothesis we have

$$\begin{aligned}
\sum_{k=1}^{m+1} k^3 &= \sum_{k=1}^{m} k^3 + (m+1)^3 \\
&= \frac{m^2(m+1)^2}{4} + (m+1)^3 \\
&= (m+1)^2 \left(\frac{m^2 + 4(m+1)}{4} \right) \\
&= (m+1)^2 \left(\frac{m^2 + 4m + 4}{4} \right) \\
&= \frac{(m+1)^2(m+2)^2}{4} \\
&= \frac{(m+1)^2((m+1)+1)^2}{4} \\
&= \left(1 + 2 + \cdots + m + (m+1)\right)^2 .
\end{aligned}$$

Hence the formula holds for $m+1$*. By mathematical induction, we conclude that the formula holds for any positive integer* $m > 0$*.*

(4) For the base case, note that $\sum_{k=1}^{1} F_{4k-2} = F_2 = 1 = F_2^2$*. Hence the formula holds for* $m = 1$*. For the induction hypothesis, assume that the formula holds for a positive integer* $m > 0$*. We will show that the formula holds for* $m+1$*. By induction hypothesis and the previous identity, we have:*

$$\begin{aligned}
\sum_{i=1}^{m+1} F_{4i-2} &= F_{2m}^2 + F_{4(m+1)-2} \\
&= F_{2m}^2 + F_{2(2m+1)} \\
&= F_{2m}^2 + F_{2m+1+1}^2 - F_{2m+1-1}^2 \\
&= F_{2m}^2 + F_{2(m+1)}^2 - F_{2m}^2 \\
&= F_{2(m+1)}^2.
\end{aligned}$$

Hence the formula holds for $m+1$*. By mathematical induction, we conclude that the formula holds for any positive integer* $m > 0$*.*

Exercise 5. Functions (15 points)

(1) **(1 point)** *Which of the three statements is correct?*

A) If the functions $f : A \to B$ and $g : B \to C$ are surjective, then the composite $g \circ f : A \to C$ is surjective.

B) If the functions $f : A \to B$ and $g : A \to B$ are surjective, then the composite $g \circ f : A \to B$ is surjective.

C) Show that if the functions $f : A \to B$ and $g : B \to C$ are surjective, then the composite $f \circ g : B \to C$ is surjective.

(2) **(2 points)** *Which of the three statements is correct?*

I) The function $h : \mathbb{R} \to [0, 1)$, $t \mapsto h(t) := t - \lfloor t \rfloor$, is injective. Here $[0, 1) := \{x \in \mathbb{R} \mid 0 \le x < 1\}$.

II) The function $h : \mathbb{R} \to [0, 1)$, $t \mapsto h(t) := t - \lfloor t \rfloor$, is not surjective. Here $[0, 1) := \{x \in \mathbb{R} \mid 0 \le x < 1\}$.

III) The function $h : \mathbb{R} \to [0, 1)$, $t \mapsto h(t) := t - \lfloor t \rfloor$, is not injective. Here $[0, 1) := \{x \in \mathbb{R} \mid 0 \le x < 1\}$.

(3) **(3 points)** *Which of the three statements is correct?*

i) The inverse function of the function $g(x) = \frac{6-4x}{14-10x}$ is $g^{-1}(x) = \frac{7x+3}{5x-2}$.

ii) The inverse function of the function $g(x) = \frac{2x-3}{5x-7}$ is $g^{-1}(x) = \frac{7x-3}{5x+2}$.

iii) The inverse function of the function $g(x) = \frac{3-2x}{7-5x}$ is $g^{-1}(x) = \frac{7x-3}{5x-2}$.

(4) **(4 points)** *Define the function G on the positive integers recursively:*

$$G(1) = 0 \quad \text{and for } n > 1, \qquad G(n) = G(\lfloor n/2 \rfloor) + 1.$$

Compute $G(27)$, $G(26)$ and $G(25)$. Is G an injective function?

(5) **(5 points)** *Consider a function $f : A \to B$ such that $f(A) = B$. Define for $b \in B$ the set $f^{-1}(b) := \{a \in A \mid f(a) = b\} \subseteq A$. Show that $P := \{f^{-1}(b) \mid b \in B\}$ defines a partition of A.*

Solution 5. *(1) The answer is A). The composition functions in B) and C) make no sense in general. In order to prove A), take $f : A \to B$ and $g : B \to C$ two surjective functions, and consider their composite $g \circ f : A \to C$. Let $c \in C$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. In the same way, since f is surjective, there exists $a \in A$ such that $f(a) = b$. Then*

$$g \circ f(a) = g(f(a)) = g(b) = c.$$

We have shown that for any $c \in C$, there is $a \in A$ such that $g \circ f(a) = c$. Hence $g \circ f$ is surjective.

(2) The answer is III). Consider the function $h : \mathbb{R} \to [0, 1)$ given by $h(t) = t - \lfloor t \rfloor$. Note that h is surjective since for $t \in [0, 1) \subset \mathbb{R}$, then $\lfloor t \rfloor = 0$ and $h(t) = t$. On the other hand, the function is not injective. If n is a positive integer, then $n = \lfloor n \rfloor$. In particular, $h(0) = 0 - 0 = 0 = 1 - 1 = h(1)$.

(3) The answer is iii). If $g(x) = \frac{3-2x}{7-5x}$. Then check that $g(\frac{7x-3}{5x-2}) = x = g^{-1}(\frac{3-2x}{7-5x})$. The computation of the inverse goes as follows

$$\begin{aligned} y = \frac{3-2x}{7-5x} \quad &\Leftrightarrow \quad (7-5x)y = 3-2x \\ &\Leftrightarrow \quad 7y - 5xy = 3 - 2x \\ &\Leftrightarrow \quad 2x - 5xy = 3 - 7y \\ &\Leftrightarrow \quad x(2-5y) = 3 - 7y \\ &\Leftrightarrow \quad x = \frac{7y-3}{5y-2}. \end{aligned}$$

Hence $g^{-1}(x) = \frac{7x-3}{5x-2}$.

(4) By definition of G we have

$$G(27) = G(13) + 1 = G(6) + 1 + 1 = G(3) + 1 + 1 + 1 = G(1) + 1 + 1 + 1 + 1 = 0 + 4 = 4.$$

$$G(26) = G(13) + 1 = 4.$$

$$G(25) = G(12) + 1 = G(6) + 1 + 1 = 4.$$

We observe that G is not injective since $G(27) = G(26)$ and $26 \neq 27$.

(5) We have to show that the elements of P are pairwise disjoint and their union is A. Indeed,

$$\bigcup_{b \in B} f^{-1}(b) \quad = \quad f^{-1}\left(\bigcup_{b \in B} \{b\}\right) = f^{-1}(B) = A,$$

by properties of the inverse image. On the other hand, consider $b, c \in B$. Then

$$f^{-1}(b) \cap f^{-1}(c) = f^{-1}(\{b\} \cap \{c\}) = \begin{cases} f^{-1}(b) & \text{if } b = c \\ \emptyset & \text{if } b \neq c \end{cases}$$

Hence the elements of P are pairwise disjoint. We conclude that P defines a partition of A.

Exercise 6. Graphs (15 points)

(1) **(1 point)** *Which of the three statements is correct?*

<u>A</u>) *Consider a graph $G(V, E)$ with $|E| = 6$. The minimum number of vertices necessary for G to be planar is four.*

<u>B</u>) *Consider a graph $G(V, E)$ with $|E| = 6$. The minimum number of vertices necessary for G to be planar is five.*

<u>C</u>) *Consider a graph $G(V, E)$ with $|E| = 6$. The minimum number of vertices necessary for G to be planar is six.*

(2) **(1 point)** *Which of the three statements is correct?*

<u>I</u>) *Euler's formula says that in a plane drawing of a connected planar graph $G(V, E)$, the numbers of vertices, $|V(G)|$, edges, $|E(G)|$, and regions, $|R(G)|$, of G satisfy: $|V(G)| - |E(G)| + |R(G)| = 2$.*

<u>II</u>) *Euler's formula says that in a plane drawing of a connected planar graph $G(V, E)$, the numbers of vertices, $|V(G)|$, edges, $|E(G)|$, and regions, $|R(G)|$, of G satisfy: $|V(G)| + |E(G)| + |R(G)| = 2$.*

<u>III</u>) *Euler's formula says that in a plane drawing of a connected planar graph $G(V, E)$, the numbers of vertices, $|V(G)|$, edges, $|E(G)|$, and regions, $|R(G)|$, of G satisfy: $|V(G)| - |E(G)| - |R(G)| = 2$.*

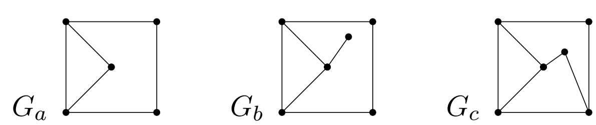
(3) **(2 points)** *Which of the three statements is correct?*

<u>D</u>) *Let $G(V, E)$ be a finite connected graph. Its spanning trees have $|V|$ edges.*

<u>E</u>) *Let $G(V, E)$ be a finite connected graph. Its spanning trees have $|E|$ edges.*

<u>F</u>) *Let $G(V, E)$ be a finite connected graph. Its spanning trees have $|V| - 1$ edges.*

(4) **(2 points)** *Check Euler's formula for the graphs G_a, G_b, G_c below:*



(5) **(4 points)** *Consider the following matrix*

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

and draw the corresponding directed graph $G(V, E)$.

(6) **(5 points)** *Let $G(V, E)$ be a finite connected planar graph with at least three vertices. Show that the graph $G(V, E)$ has at least one vertex of degree smaller than six.*

Solution 6. *(1) The answer is A). Recall that if $G(V, E)$ is a planar graph with $|V| \geq 3$ then we have that $|E| \leq 3|V| - 6$. If $|E| = 6$, then $6 \leq 3|V| - 6$ which implies that $12 \leq 3|V|$, or equivalently, $4 \leq |V|$.*

(2) The answer is I). Just remember the celebrated Euler's formula.

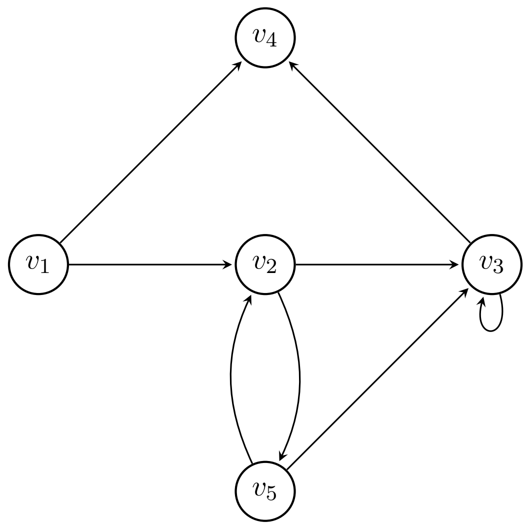
(3) The answer is F). It is a very known results that a connected graph with $|V|$ vertices is a tree if and only if $|E| = |V| - 1$. In particular, any spanning tree of a finite connected graph with $|V|$ vertices has $|V| - 1$ edges.

(4)

- *G_a. It is easy to see that $|V| = 5$, $|E| = 6$ and $|R| = 3$. Then $5 - 6 + 3 = 2$.*
- *G_b. In this case we have that $|V| = 6$, $|E| = 7$ and $|R| = 3$. Then $6 - 7 + 3 = 2$.*
- *G_c. In this case we have that $|V| = 6$, $|E| = 8$ and $|R| = 4$. Then $6 - 8 + 4 = 2$.*

Euler's formula is verified in all the cases.

(5) Recall that the matrix $A = (a_{ij})$ associated to a directed graph $G(V, E)$ is defined by $a_{ij} = 1$ if and only if there is an edge from v_i to v_j in G, and $a_{ij} = 0$ if there is no such edge. According to the matrix, the corresponding directed graph is the following:



(6) We proceed by contradiction. Assume that all the vertices of G have at least degree 6. Recalling the formula of the sum of the degrees of a graph, we have that

$$2|E| = \sum_{v \in V} \deg(v) \geq \sum_{v \in V} 6 = 6|V|.$$

On the other hand, since $|V| \geq 3$ and the graph is connected and planar, we have that $|E| \leq 3|V| - 6$. Combining this inequality in the above one, we have

$$3|V| \leq |E| \leq 3|V| - 6 \quad \Rightarrow \quad 6 \leq 3|V| - 3|V| = 0,$$

which is clearly a contradiction. Hence, our initial assumption is not true. We conclude that at least one vertex has degree smaller than six.

Exercise 7. Combinatorics (8 points)

(1) **(1 point)** *Which of the three statements is correct?*

A) The number of permutations of the letters in XAYEXAZZ is 5040.

B) The number of permutations of the letters in XAYEXAZZ is 10080.

C) The number of permutations of the letters in XAYEXAZZ is 40320.

(2) **(2 points)** *Which of the three statements is correct?*

I) The binomial formula says that in the expansion of $(2x+3y^2)^5$ *one finds the term* $280x^4y^2$.

II) The binomial formula says that in the expansion of $(2x+3y^2)^5$ *one finds the term* $220x^4y^2$.

III) The binomial formula says that in the expansion of $(2x+3y^2)^5$ *one finds the term* $240x^4y^2$.

(3) **(5 points)** *Find the number of permutations formed from the letters of the word HEGAHEPP, where the two H's are next to each other.*

Solution 7. *(1) The answer is A). Note that there are 8 letters, then we have* $8!$ *permutations. However, we must divide by the number of times that we are extra counting due to the repetition of letters. In this case, we have two* X*s, two* A*s and two* Z*s. Then we are counting twice three times. Therefore the number of different permutations is* $8!/2^3 = 5040$.

(2) The answer is III). Recall the binomial formula

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

In our case, we have $n = 5$, $a = 2x$ *and* $b = 3y^2$. *Then*

$$(2x+3y^2)^5 = \sum_{k=0}^{5} \binom{5}{k} 2^k 3^{5-k} x^k y^{10-2k}.$$

Then, the therm in the monomial x^4y^2 *will appear with the index* $k = 4$. *The respective coefficient is given by*

$$\binom{5}{4} 2^4 3^1 = 240.$$

(3) Consider the pair HH as a single letter. Then there are seven letters, two of which are P and two of which are E. Hence there are $7!/(2!2!)$ *many permutations.*

Exercise 8. Finite state automata and machines (12 points)

(1) **(1 point)** *Which of the three statements is correct?*

A) Let $\Sigma = \{0, 1, 2\}$. *The word* $w = 012$ *belongs to* $L(r)$ *for the regular expression* $r = 0^* \vee (1 \vee 2)^*$.

B) Let $\Sigma = \{0, 1, 2\}$. *The word* $w = 012$ *belongs to* $L(r)$ *for the regular expression* $r = 0^*(1 \vee 2)^*$.

C) Let $\Sigma = \{0, 1, 2\}$. *The word* $w = 012$ *belongs to* $L(r)$ *for the regular expression* $r = 0^*(1 \vee 2)$.

(2) **(2 points)** *Draw the state diagram of the finite state machine* M *corresponding to the transition table*

M	ν		ω	
	0	1	0	1
s_0	s_1	s_2	0	1
s_1	s_3	s_1	1	2
s_2	s_1	s_0	2	0
s_3	s_0	s_2	2	0

What is the output corresponding to the input sequence 00101001101*?*

(3) **(4 points)** *What is the language* L *accepted by the automaton* A_1 *in Figure 1.*

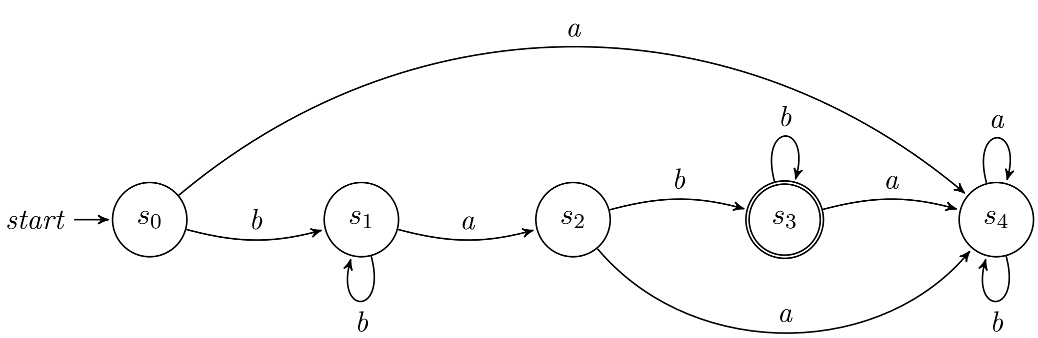
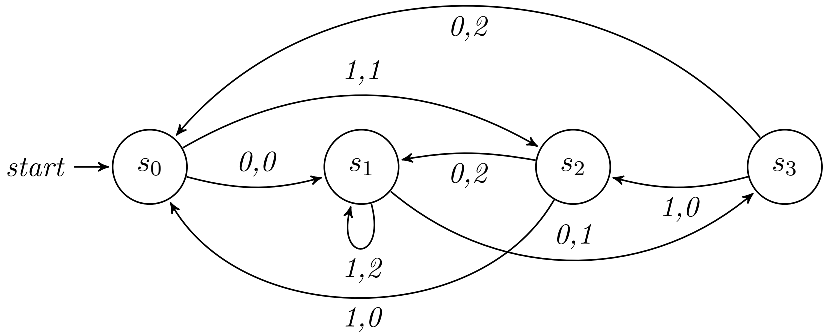


FIGURE 1. The automaton A_1.

(4) **(5 points)** *Let* $B = \{a, b\}$. *Draw an automaton* A_2 *with three states, that accepts those words made of letters from* B*, where the number of* b*'s can be exactly divided by three.*

Solution 8. *(1) The answer is B). According to the definition, the language of the regular expression* $r = 0^*(1 \vee 2)^*$ *is defined by strings with any occurrences of 0 and then any occurrences of 1 or 2. It is easy to see then that the word* $w = 012$ *belongs to* $L(r)$.

(2) The corresponding diagram is the following



The output corresponding to the input sequence 00101001101 *is* 01022121002 *and the machine stops in state* s_1*.*

(3) Observe that s_4 *is a sink. If the string starts with* a*, then we reach the sink. So, any accepted string must start with* b *and going to* s_1*. After that, we can get any occurrences of* b *since it stays in* s_1*. Then we got an* a *and move to* s_2*. If we got one more* a*, then we reach* s_4*. So, we have to get a* b *to move to* s_3*. There we can get any occurrences of* b *since it stays in* s_3*. However, if we read an* a *in* s_3*, we go to* s_4*. Hence the language* L *accepted by* A_1 *is given by* $\{b^m a b^n \mid m, n \geq 1\}$*.*

(4) A proposed automaton A_2 *which fits in the conditions of the exercise is the following:*

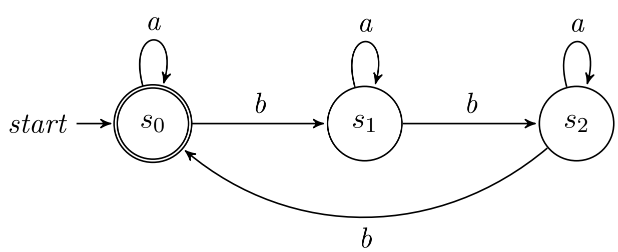


FIGURE 2. The automaton A_2.