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EXAM IN TIØ4120 OPERASJONSANALYSE GRUNNKURS

Wednesday 3 December 2008
Time: 09:00 – 13:00
English
Allowed material:
C – Approved calculator permitted (HP 30S, Citizen SR-270X)

Deadline for examination results: 5 January 2009

Some of these exercises may have alternative ways to solve them. Just because they are not mentioned here, does not mean they do not exist.

Exercise 1 (25%)
A company has a weekly production of soap and shampoo using 2 different raw materials. The amount of available raw materials per week is limited. One ton of soap requires 3 tons of raw material A and 3 tons of raw material B. It takes 4 tons of raw material A and 1 ton of raw material B to produce 1 ton of shampoo.

In addition to the two products above, the production process results in a byproduct. Each week, at least 20 tons of the byproduct have to be produced. Each week, the company has access to 70 tons of raw material A and 19 tons of raw material B.

The company has formulated the following LP for maximizing profits given the conditions above:

$$\max 26x_1 + 15x_2$$

subject to

$$\begin{aligned} 4x_1 + x_2 &\geq 20 \\ 3x_1 + 4x_2 &\leq 70 \\ 3x_1 + x_2 &\leq 19 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a) Interpret x_1 and x_2 in the optimization problem.
x_1 – number of tons of soap produced
x_2 – number of tons of shampoo produced

b) How many tons of the byproduct do you get from producing one ton of soap and shampoo, respectively?
The first constraint defines the production of the byproduct. We see that producing1 ton of soap results in 4 tons of the byproduct, whereas producing 1 ton shampoo gives 1 ton of the byproduct.

c) Solve the problem with the Simplex-algorithm.
The problem is transformed into standard form in order to solve it with the Simplex-algorithm. In addition to slack/surplus-variables, we introduce an artificial variable a_1 in the first constraint. This allows us to use the primal simplex.

$$\max 26x_1 + 15x_2 - Ma_1$$

subject to

$$\begin{aligned} 4x_1 + x_2 - s_1 + a_1 &= 20 \\ 3x_1 + 4x_2 + s_2 &= 70 \\ 3x_1 + x_2 + s_3 &= 19 \\ x_1, x_2, s_1, s_2, s_3, a_1 &\geq 0 \end{aligned}$$

The initial starting solution is $x_1 = x_2 = s_1 = 0, a_1 = 20, s_2 = 70, s_3 = 19$.

The pivot element is marked with [ ].

c_B	c_j / x_B	x_1	x_2	s_1	s_2	s_3	a_1	b_i
	c_j	26	15	0	0	0	$-M$	
c_B	x_B	x_1	x_2	s_1	s_2	s_3	a_1	b_i
$-M$	a_1	[4]	1	-1			1	20
0	s_2	3	4		1			70
0	s_3	3	1			1		19
	z_j	$-4M$	$-M$	M	0	0	$-M$	$-20M$
	$c_j - z_j$	$26 + 4M$	$15 + M$	$-M$	0	0	0	
26	x_1	1	0.25	-0.25			X	5
0	s_2		3.25	0.75	1		X	55
0	s_3		[0.25]	0.75		1	X	4
	z_j	26	6.5	-6.5	0	0	X	130
	$c_j - z_j$	0	8.5	6.5	0	0	X	
26	x_1	1		-1		-1	X	1
0	s_2			-9	1	-13	X	3
15	x_2		1	3		4	X	16
	z_j	26	15	19	0	34	X	266
	$c_j - z_j$	0	0	-19	0	-34	X	

We reached the optimal solution as all entries in the $c_j - z_j$-row are negative. The optimal solution is given as $x_1 = 1, x_2 = 16, s_2 = 3$. Profit is 266.

d) Formulate the dual problem.

The dual problem can be formulated in two ways. The first one is by transforming the first constraint of the primal problem into a $\leq$-constraint:

$$\min -20y_1 + 70y_2 + 19y_3$$

subject to

$$\begin{aligned} -4y_1 + 3y_2 + 3y_3 &\geq 26 \\ -y_1 + 4y_2 + y_3 &\geq 15 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$

Alternatively, we can keep the first constraint and define the first dual variable as strictly negative.

$$\min 20y_1 + 70y_2 + 19y_3$$

subject to

$$\begin{aligned} 4y_1 + 3y_2 + 3y_3 &\geq 26 \\ y_1 + 4y_2 + y_3 &\geq 15 \\ y_1 &\leq 0 \\ y_2, y_3 &\geq 0 \end{aligned}$$

e) How much can you change the profit from producing shampoo before the optimal solution changes?

If we change the profit from producing shampoo by Δ, the $c_j - z_j$-row in the optimal Simplex-tableau changes as follows:

$c_j - z_j$	0	0	-19-3Δ	0	-34-4Δ

The solution remains optimal as long as $-19 - 3\Delta \leq 0$ and $-34 - 4\Delta \leq 0$. The first inequality results in $\Delta \geq -6.33$, the second one in $\Delta \geq -8.5$. The optimal solution will not change as long as $c_2 \geq 8.67$.

f) Instead of producing the byproduct, the company has the chance to buy some the byproduct in an external market. How much is the company willing to pay for 1 ton of the byproduct? How many tons should the company buy at this price?
The shadow price of constraint 1 is given in the optimal Simplex-tableau. The company would pay 19 for each ton of byproduct. To determine how much they would be willing to buy at this price, we have to check the interval for which this price is valid (we are only interested in the lower end of the interval). We perform a sensitivity analysis for changing the right hand side of constraint 1.

Constraint 1 is a $\geq$-contraint. That means that decreasing the RHS weakens this constraint, whereas increasing the RHS means tightening it (in contrast to a $\leq$-contraint). We therefore have to switch the formulas given on the slideset for slack-/surplus-variables not in the basis.

We are interested in how much we can decrease the RHS of constraint 1 (as this is the amount bought in the market). Calculating

$$\min\left\{-\frac{b_i}{A_{ij}}|A_{ij} < 0, i = 1\dots m\right\} = -\frac{3}{-9} = 0.33$$

tells us how much we can decrease the RHS. We can decrease the RHS to 19.67. The company is willing to buy as much 0.33 tons at a price of 19.

Exercise 2 (25%)
Emil runs a small business, carving wooden figures in his spare time. He produces big and small figures. The big ones sell for 50 NOK a piece, whereas customers are willing to pay 40 NOK for a small figure. It takes him 3 hours and 2dm^3 of wood to produce a small figure, whereas he can finish a big one in just 2 hours using 5dm^3 of wood. He has 20 hours and 35dm^3 of wood available every week.

He needs some help determining how many big and small wooden figures he should carve each week in order to maximize the sales revenues (assume that all figures will be sold).

a) Formulate the integer optimization problem.
We define the following decision variables:
x_1 – number of big figures carved
x_2 – number of small figures carved

The optimization is the given by

$$\max 50x_1 + 40x_2$$

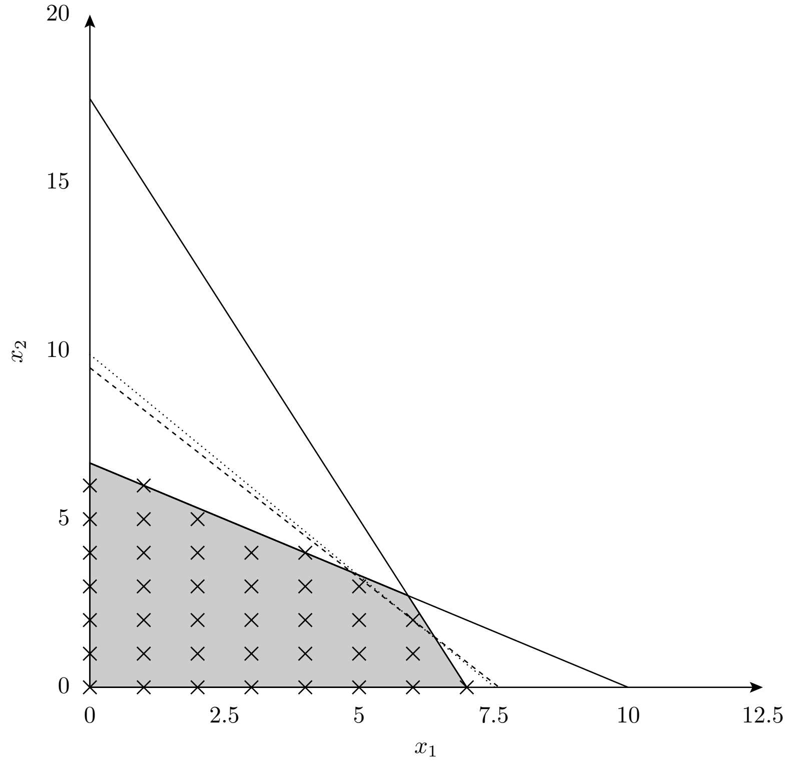
subject to

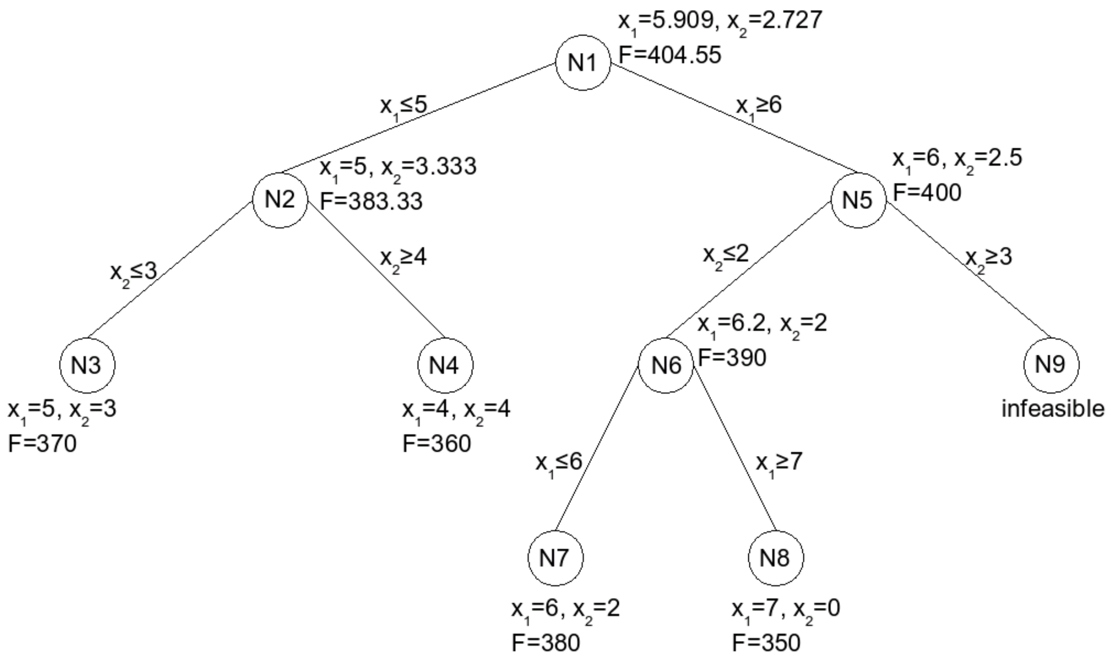
$$\begin{aligned} 2x_1 + 3x_2 &\leq 20 \\ 5x_1 + 2x_2 &\leq 35 \\ x_1, x_2 &\geq 0 \text{ and integer} \end{aligned}$$

b) Draw the feasible region of the problem and solve it graphically.
The dashed line is the objective function of the integer problem. The optimal solution is given as $x_1 = 6, x_2 = 2$.

c) Solve the problem using Branch-and-Bound. Use the figure from b) to solve the LP-relaxations.
The optimal solution is given in Node 7 with $x_1 = 6$ and $x_2 = 2$. Total income is 380 NOK.

Assume now, that Emil values his spare time at 5 NOK/hour and has to spend 3 NOK per dm^3 wood.


20
15
10
5
0
0
2.5
5
7.5
10
12.5
x_1
x_2

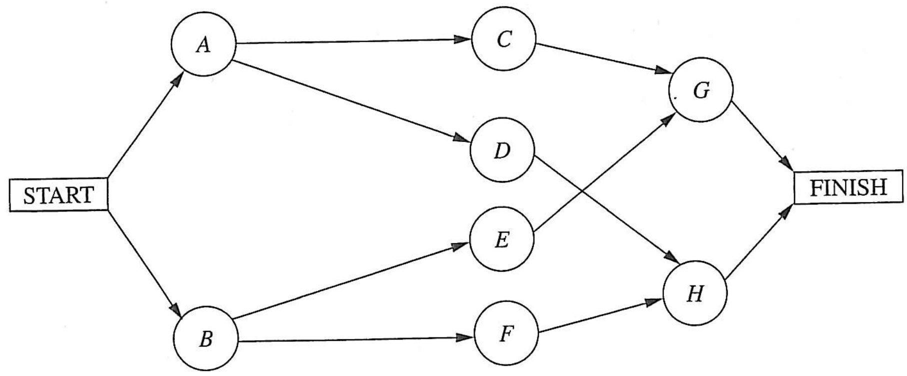

N1
x1=5.909, x2=2.727
F=404.55
x1≤5
x1≥6
N2
x1=5, x2=3.333
F=383.33
N5
x1=6, x2=2.5
F=400
x2≤3
x2≥4
x2≤2
x2≥3
N3
x1=5, x2=3
F=370
N4
x1=4, x2=4
F=360
N6
x1=6.2, x2=2
F=390
N9
infeasible
x1≤6
x1≥7
N7
x1=6, x2=2
F=380
N8
x1=7, x2=0
F=350

d) How would the optimal solution change if Emil wants to maximize profits instead of revenues? Use the figure from b).
Profit is defined as revenues minus costs. It costs Emil 25 NOK to produce a big figure $(2 \cdot 5 + 5 \cdot 3)$ and 21 NOK to produce a small figure $(3 \cdot 5 + 2 \cdot 3)$. The new objective function is

$$\max 25x_1 + 19x_2.$$

As the dotted lined shows, the optimal solution is unchanged. Optimal profit is 188.

Exercise 3 (20%)
Consider the following project network:



The activity durations are given as

Activity	a	m	b
A	28	32	36
B	22	28	34
C	26	36	46
D	14	16	18
E	32	32	32
F	12	16	26
G	12	16	26
H	16	20	24

a is the optimistic estimate for the activity duration, m is the most likely activity duration, and b is the pessimistic estimate for the activity duration.

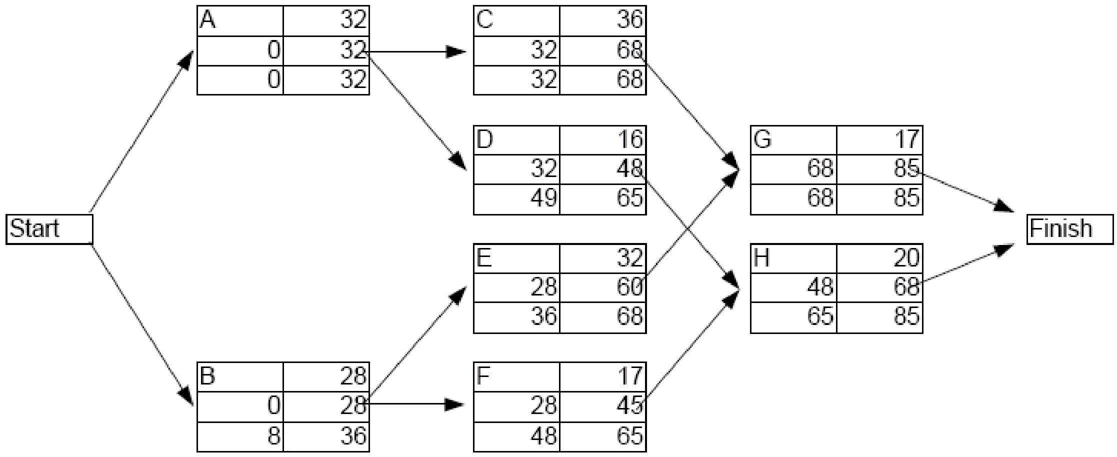
The following formulas might be useful: $t = \dfrac{a + 4m + b}{6}$, $v = \left(\dfrac{b-a}{6}\right)^2$

a) Determine the earliest possible finish time of the project and the activities on the critical path.
First, we have to calculate the expected duration and the variance of each activity. Using the formulas given above results in the following table (see next page):

We can now draw the project network, determine the finishing time, and the activities on the critical path.

The critical path is given by activities A-C-G.

Activity	a	m	b	t	v
A	28	32	36	32	1.78
B	22	28	34	28	4
C	26	36	46	36	11.11
D	14	16	18	16	0.44
E	32	32	32	32	0
F	12	16	26	17	5.44
G	12	16	26	17	5.44
H	16	20	24	20	1.78



b) What is the probability that the project will be finished after 80 weeks? Use the table for the normal distribution on page 8.
The activities on the critical path have a duration of 85 weeks and a variance of 18.33. The standard deviaton is 4.281. With these values, we can calculate the z-value needed for determining the correct probability p.

$$z = \frac{x - \mu}{\sigma} = \frac{80 - 85}{4.281} = -1.168.$$

The standard normal distribution is symmetric, so we can use the z-value 1.168 with table. We can read the probability 0.379 from the normal distribution table. As the original z-value is -1.168, the correct probability p is

$$p = 0.5 - 0.379 = 0.121.$$

The project is finished after 80 weeks with a probability of 0.121.

c) If you were to crash the project, which activities would you start with? What do you have to consider when crashing these activities?
In order to shorten the total project duration, the duration of activities on the critical path have to be shortened first. When doing this, one has to make sure that no other activities become part of the critical path. In this case, these new activities have to be shortened to further decrease total project duration.

Exercise 4 (20%)
A gas station with only one gas pump employs the following policy: if a customer has to wait, the price is 10 NOK per liter; if she does not have to wait, the price is 12 NOK per liter.

Customers arrive according to a Poisson process with a mean rate of 10 per hour. Service times at the pump have an exponential distribution with a mean of 5 minutes. Arriving customers always wait until they can eventually buy gasoline.

Each customer purchases on average 60 liters of gasoline.

a) Calculate the probability that there is no customer in the system, the expected amount of time a customer waits in the queue, and the expected amount of customers at the gas station (at the pump and in the queue).
The arrival rate is given as $\lambda = 10$, the service rate is given as $\mu = 12$ (both in customers/hour). With this information we can calculate the probability of the no customers being in the system:

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \frac{10}{12} = 0.167.$$

The expected waiting time in the queue W_q is given as

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)} = \frac{10}{12(12 - 10)} = \frac{5}{12}.$$

The expected waiting time in the queue is 25 minutes (arrival rate and service rate are given in customers per hour). The expected amount of customers at the gas station is given by

$$L = \frac{\lambda}{\mu - \lambda} = \frac{10}{12 - 10} = 5.$$

b) Determine the expected price of gasoline per liter and the expected income of operating the gas station for 10 hours
If no customer is in the system (i.e. the system is idle), we can sell gasoline for 12 NOK/liter to the next customer. The expected price p of gasoline is given by

$$p = P_0 \cdot 12 + (1 - P_0) \cdot 10 = 0.167 \cdot 12 + 0.833 \cdot 10 = 10.33 \text{ NOK}.$$

We can expected 10 customers per hour, during a 10-hour day this makes 100 customers buying 60 liters each. Expected income i is then given as

$$i = 6000 \cdot 10.33 = 61980 \text{ NOK}.$$

The gas pump is getting old and no longer as reliable as it used to be. Assume that the service times are distributed according to the following probability distribution:

Service time [min]	Probability
4.0	0.40
5.0	0.30
6.0	0.20
7.0	0.10

c) Simulate 1 hour of operations for the gas station and determine the expected price of one liter gasoline. Use the last column of random numbers of the table on page 9.
We start by defining the ranges for the random numbers defining the service times:

Service time [min]	Probability	Cum. Prob.	Random Numbers
4.0	0.40	0.4	0-39
5.0	0.30	0.7	40-69
6.0	0.20	0.9	70-89
7.0	0.10	1.0	90-99

With the random numbers from the last column, we get the following service times:

Random number	00	46	49	80	41	61	94	89	87
Service time	4	5	5	6	5	5	7	6	6

Arrival times are given as 10 customers per hour, so we expect a customer every 6 minutes. We assume that the system is empty in the beginning with the first customer arriving at time 6. We can now set up a table of events for the hour of simulated operations:

Time	Event	No. at pump	No. waiting	Price
6	arrival	1	0	12
10	finished	0	0	
12	arrival	1	0	12
17	finish	0	0	
18	arrival	1	0	12
23	finish	0	0	
24	arrival	1	0	12
30	finish	0	0	
30	arrival	1	0	12
35	finish	0	0	
36	arrival	1	0	12
41	finish	0	1	
42	arrival	1	0	12
48	arrival	1	1	10
49	finish	1	0	
54	arrival	1	1	10
55	finish	1	0	
60	arrival	1	1	10

Of the 10 arriving customers, 3 have to wait and therefore pay only 10 NOK/liter. The average price of gasoline during this hour is 11.40 NOK/liter.

d) How reliable are the results from the simulation?
Simulation results have to be confirmed using several runs, as a single simulation may not be representative. We see here that the expected gasoline price from the simulation is higher than what is calculated in part b). This can be possible as the empirical service time distribution is skewed towards shorter service time, thus potentially reducing the length of the queue. Whether the expected price of gasoline is correct has to be confirmed by additional simulations.

Exercise 5 (10%)

a) Which algorithms can you use to solve the linear assignment problem?
The linear assignement problem can be solved by Branch-and-Bound (variables are binary), the Simplex algorithm (due to the properties of the problem), transportation algorithms (Stepping-Stone, MODI, as the LAP is a special case of the transportation problem), and the Hungarian method.

b) Which algorithm would you use for solving the linear assignment problem? Explain why you prefer it over the others.
The Hungarian method is the preferred method for solving linear assigment problems. The solution to the LAP is highly degenerate, only n of $2n-1$ basic variables are non-zero. This can cause problems with exchanging one basic variable with value 0 for another. The Hungarian method focuses on finding the non-zero variables rather than feasible basic solutions.