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EXAM IN TIØ4120 OPERASJONSANALYSE GRUNNKURS

Saturday 19 December 2009
Time: 09:00 – 13:00
English
Allowed material:
C – Approved calculator permitted (HP 30S, Citizen SR-270X)

Deadline for examination results: 19 January 2010

Some of these exercises may have alternative ways to solve them. Just because they are not mentioned here, does not mean they do not exist.

Exercise 1 (30%)
A company produces one single product type. Four different production facilities are available with the following variable production costs 4, 5, 5, and 1 (in 1000 NOK/ton). The company has committed to producing exactly 100 tons in each period. Additionally, the company has committed to using at least 40 tons of waste product A in each period. The four production facilities consume 0.3, 0.5, 0.6, and 0.2 tons of A to produce one ton of the finished product.

a) Formulate the company's cost minimization problem as a linear programming problem.

Defining decisions variables:
x_i – amount produced at facility i (in tons).
With that we get the following cost minimization problem:

$$\min 4x_1 + 5x_2 + 5x_3 + x_4$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 100 && (1)\\ 0.3x_1 + 0.5x_2 + 0.6x_3 + 0.2x_4 &\geq 40 && (2)\\ x_i &\geq 0 && (3) \end{aligned}$$

b) What can you say about the number of used production facilities without solving the problem?

The optimal solution will be one of the feasible basic solutions. As we have 2 constraints, the basic solution will have two at most two non-zero variables. Thus, two facilities will be producing in the optimal solution.

c) Solve the problem with the Simplex-algorithm.

In order to solve the problem with the Simplex algorithm, we have to bring it into standard (augmented) form. For this we subtract a surplus variable s from constraint (2). In addition, we add an artificial variable a to constraint (1) and penalize using this variable with M in the objective function. It is not necessary to use an artificial variable in constraint (1). See below for how to solve the problem without it.

The original problem in standard form:

$$\min 4x_1 + 5x_2 + 5x_3 + x_4 + Ma$$

subject to

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + a &= 100 && (4)\\ 0.3x_1 + 0.5x_2 + 0.6x_3 + 0.2x_4 - s &= 40 && (5)\\ x_i, a, s &\geq 0 && (6) \end{aligned}$$

As initial solution we set $x_1 = x_2 = x_3 = x_4 = 0$. This leads to the following initial Simplex tableau. Note that $a = 100, s = -40$ is an infeasible starting solution, so we use the Dual Simplex to solve our problem. The pivot element is marked with [ ].

	c_j	4	5	5	1	M	0	
c_B	x_B	x_1	x_2	x_3	x_4	a	s	b_i
M	a	1	1	1	1	1		100
0	s	$-\frac{3}{10}$	$-\frac{1}{2}$	$[-\frac{3}{5}]$	$-\frac{1}{5}$		1	-40
	z_j	M	M	M	M	M	0	$100M$
	$z_j - c_j$	$M-4$	$M-5$	$M-5$	$M-1$	0	0	
5	x_3	1	1	1	1	1		100
0	s	$\frac{3}{10}$	$\frac{1}{10}$		$\frac{2}{5}$		1	20
	z_j	5	5	5	5	5	0	500
	$z_j - c_j$	1	0	0	4	$5-M$	0	
5	x_3	$\frac{1}{4}$	$\frac{3}{4}$	1		$-\frac{1}{2}$	$-\frac{5}{2}$	50
1	x_4	$\frac{3}{4}$	$\frac{1}{4}$		1	$\frac{3}{2}$	$\frac{5}{2}$	50
	z_j	2	4	5	1	-1	-10	300
	$z_j - c_j$	-2	-1	0	0	$-1-M$	-10	

Note that we could have removed the artificial variable from the Simplex tableau once it left the basis. However, it will come in handy for the sensitivity analysis in question e) of this exercise. (Question e) can be solved without this artificial variable!)

As mentioned before, it is not necessary to introduce the artificial variable. In this case, we solve the following problem:

$$\min 4x_1 + 5x_2 + 5x_3 + x_4$$

subject to

$$x_1 + x_2 + x_3 + x_4 = 100 \tag{7}$$

$$0.3x_1 + 0.5x_2 + 0.6x_3 + 0.2x_4 - s = 40 \tag{8}$$

$$x_i, s \geq 0 \tag{9}$$

When choosing a starting solution, we now have to pick one of the x_i as part of the basis. We choose x_4 as part of the starting solution (the next steps for setting up the initial tableau are the same for all variables). Unfortunately, we cannot just use the coefficients from constraint (8) as we will need a unity vector under x_4 in the initial simplex tableau. We therefore have to remove x_4 from constraint (8). From constraint (7), we see that $x_4 = 100 - x_1 - x_2 - x_3$. Putting this one into constraint (8), we get

$$0.1x_1 + 0.3x_2 + 0.4x_3 - s = 20.$$

We can now pick the following starting solution: $x_1 = x_2 = x_3 = 0$ and $x_4 = 100, s = -20$. Again, this is an infeasible starting solution, requiring the Dual Simplex to solve the problem. The following tableau is set up according to Hillier/Lieberman.

	x_1	x_2	x_3	x_4	s	b_i
Z	-3	-4	-4	0	0	100
x_4	1	1	1	1		100
s	$-\frac{1}{10}$	$-\frac{3}{10}$	$[-\frac{2}{5}]$		1	-20
Z	-2	-1	0	0	-10	300
x_4	$\frac{3}{4}$	$\frac{1}{4}$		1	$\frac{5}{2}$	50
x_3	$\frac{1}{4}$	$\frac{3}{4}$	1		$-\frac{5}{2}$	50

d) How much does the supplier of the waste product have to pay to the company so that they are willing to use one more ton of A?

The shadow price (reduced cost) of s in the optimal Simplex tableau tells us how the objective function changes if the RHS of constraint (5) changes. Thus, the supplier of the waste product has to pay (at least) 10000 NOK.

e) How much can you increase production without starting new facilities or stopping production in plants found in c)?

In order to determine when we start/stop production facilities, we can analyze the a-column in the optimal Simplex tableau. We see that we can change the RHS of constraint (1) in the interval $50 \cdot \frac{2}{3} \leq \Delta \leq 50 \cdot \frac{2}{1} \Rightarrow 33.3 \leq \Delta \leq 100$. The production amount can be increased by 100 units.

If we solved the problem without the artificial variable, we see that facility 4 is cheaper than facility 3. Thus, an increase in production we lead to an increase in x_4 (as we want to minimize production costs). Once all waste product A is consumed at facility 4, we will stop production at facility 3. From constraint (2) we see that facility 4 consumes 40 tons of A once production reaches 200 units. Thus, we can increase production by 100 units before stopping facility 3.

Assume that the production costs at facilities 1 and 4 both increase with d (in 1000 NOK/ton).

f) How much can you change d without changing the solution from c)?

Let's update the optimal Simplex tableau to reflect the change in production costs:

	c_j	$4+d$	5	5	$1+d$	M	0	
c_B	x_B	x_1	x_2	x_3	x_4	a	s	b_i
5	x_3	$\frac{1}{4}$	$\frac{3}{4}$	1		$-\frac{1}{2}$	$-\frac{5}{2}$	50
$1+d$	x_4	$\frac{3}{4}$	$\frac{1}{4}$		1	$\frac{3}{2}$	$\frac{5}{2}$	50
	z_j	$2+\frac{3}{4}d$	$4+\frac{1}{4}d$	5	$1+d$	$-1+\frac{3}{2}d$	$-10+\frac{5}{2}d$	300
	$z_j - c_j$	$-2-\frac{1}{4}d$	$-1+\frac{1}{4}d$	0	0	$-1+\frac{3}{2}d-M$	$-10+\frac{5}{2}d$	

This solution is optimal as long as all $z_j - c_j \leq 0$. This results in the following conditions:

$-2 - \frac{1}{4}d \leq 0 \Rightarrow d \geq -8$

$-1 + \frac{1}{4}d \leq 0 \Rightarrow d \leq 4$

$-1 + \frac{3}{2}d - M \leq 0$. This condition will always be satisfied as M is sufficiently large.

$-10 + \frac{5}{2}d \leq 0 \Rightarrow d \leq 4$

The solution remains optimal as long as $d \leq 4$.

Exercise 2 (20%)
Suppose a company has two machines for manufacturing a product. Machine 1 makes two units per hour, while machine 2 makes three per hour. The company has an order for 100 units. Energy restrictions dictate that only one machine can operate at one time. The company has 40 hours of regular machining time, but overtime is available. It costs 40 NOK to run machine 1 for one hour, whereas machine 2 costs 50 NOK. The company has the following goals:

1. Meet the demand of 100 unit exactly.
2. Use the 40 hours of machining time.
3. Minimize costs

a) Formulate the goal programming problem.

Defining decisions variables:
x_i – hours machine i is in use.
Let's first formulate the different goals:

1. Meet the demand of 100 unit exactly: $2x_1 + 3x_2 + d_1^- - d_1^+ = 100$
2. Use the 40 hours of machining time: $x_1 + x_2 + d_2^- - d_2^+ = 40$
3. Minimize costs: $40x_1 + 50x_2 + d_3^- - d_3^+ = 0$

With respect to goal 3: In order to minimize the costs, we introduce a sufficiently low costs level that we want to achieve. The goal programming problem can then be formulated as:

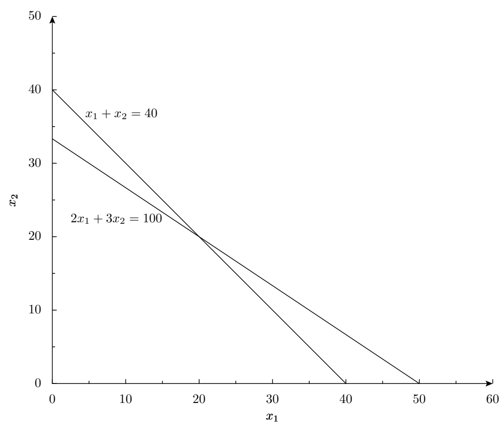
$$\min P_1(d_1^- + d_1^+), P_2(d_2^- + d_2^+), P_3(d_3^+)$$

subject to:

$$\begin{aligned} 2x_1 + 3x_2 + d_1^- - d_1^+ &= 100 \\ x_1 + x_2 + d_2^- - d_2^+ &= 40 \\ 40x_1 + 50x_2 + d_3^- - d_3^+ &= 0 \\ x_i, d_i^-, D_i^+ &\geq 0 \end{aligned}$$

b) Solve the problem graphically.

As deviations in both directions are penalized for goal 1 and 2, the solution of will be at the intersection of the lines describing the first 2 constraints. The third goal is not evaluated as the solution is already determined.



The solution is given as $x_1 = x_2 = 20$ with total production costs of 1800.

Assume now, that the company introduces a new goal: limiting overtime usage to 10 hours. This new goal is assigned priority 2.

c) Formulate the new goal programming problem.

The new goal can be formulated as $d_2^+ + d_4^- - d_4^+ = 10$. However, this is not a standard formulation for a goal programming problem. As either d_2^- or d_2^+ is equal to zero, we can express $d_2^+ = x_1 + x_2 - 40$. With this expression we can reformulate our new goal as $x_1 + x_2 + d_4^- - d_4^+ = 50$. This results in the following problem:

$$\min P_1(d_1^- + d_1^+), P_2(d_4^+), P_3(d_2^- + d_2^+), P_4(d_3^+)$$

subject to:

$$\begin{aligned} 2x_1 + 3x_2 + d_1^- - d_1^+ &= 100 \\ x_1 + x_2 + d_2^- - d_2^+ &= 40 \\ 40x_1 + 50x_2 + d_3^- - d_3^+ &= 0 \\ x_1 + x_2 + d_4^- - d_4^+ &= 50 \\ x_i, d_i^-, D_i^+ &\geq 0 \end{aligned}$$

d) Among the methods used in multicriteria decision making, we also find the Analytical Hierarchy Process (AHP). Explain the difference between Goal Programming and AHP.

Goal programming uses goals and constraints that can be formulated as mathematical expressions. Thus, it is well suited for problems that can be quantified, giving an answer to the question "how much?". When solving a goal programming problem, one tries to satisfy the goals in a hierarchical manner, satisfying the goal with the highest priority first. Goals that are already satisfied will never be violated when a goal with a lower priority is satisfied.

The Analytical Hierarchy Process (AHP) is based on pairwise comparisons of alternatives with respect to different criteria. These comparisons are used to describe how much an alternative is preferred over a different one. The criteria are also compared to one another to determine the importance of each criteria. The best alternative can then be calculated. As the comparisons are based on subjective evaluations, AHP answers to the question "which one?".

Exercise 3 (25%)
A company has 4 production facilities with limited capacity to produce one single product. The company has three different suppliers that can only supply a limited amount of raw materials. It takes one unit of raw material to produce one unit of finished product.
The raw materials are not identical, neither are the production facilities. Hence, the variable production cost depends both on the raw material r and production facility p (See Table 1). The

Table 1: Variable production cost in 1000 NOK

	$p=1$	$p=2$	$p=3$	$p=4$
$r=1$	5	4	9	5
$r=2$	6	3	6	4
$r=3$	4	2	6	2

capacities at the production facilities are given as 30, 60, 30, and 30 units, respectively. For the next planning period, the company can acquire raw materials in the amount of 65, 65, and 50 units. The company has committed to producing 150 units during the next planning period. Assume that the company wishes to minimize the variable production costs in the next planning period.

a) Formulate the planning problem as an ordinary transportation problem.

In the ordinary transportation problem, $\sum_i a_i = \sum_j b_j$. In our case, the supply of raw material exceeds the production capacity, so we need to introduce a dummy facility with a capacity equal to the difference between supply and capacity, i.e. 30. Production costs at the dummy facility at 0. We can now define the model elements.

Sets
- $\mathcal{R}$ Raw material types, $\mathcal{R} = \{1, 2, 3\}$
- $\mathcal{P}$ Production facilities, $\mathcal{P} = \{1, 2, 3, 4, 5\}$

Indices
- r Raw material type, $r \in \mathcal{R}$
- p Production facility, $p \in \mathcal{P}$

Data
- C_{rp} Cost of using raw material r at facility p, see Table 1
- a_r Availability of raw material r, $a_r = \{65, 65, 30\}$
- b_p Capacity at facility p, $b_p = \{30, 60, 30, 30, 30\}$

Decision variables
- x_{rp} Amount of raw material r used at production facility p

With this notation, we can now formulate the ordinary transportation problem:

$$\min \sum_{r\in\mathcal{R}} \sum_{p\in\mathcal{P}} C_{rp} x_{rp}$$

subject to:

$$\begin{aligned}
\sum_{p\in\mathcal{P}} x_{rp} &= a_r \quad r \in \mathcal{R} \\
\sum_{r\in\mathcal{R}} x_{rp} &= b_p \quad p \in \mathcal{P} \\
x_{rp} &\geq 0 \quad r \in \mathcal{R}, p \in \mathcal{P}
\end{aligned}$$

b) Solve the problem.

We first find a starting solution with the Northwest-Corner-Method:

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	
$r=1$	5 30	4 35	9	5	0	65
$r=2$	6	3 25	6 30	4 10	0	65
$r=3$	4	2	6	2 20	0 30	50
	30	60	30	30	30	

We now find the optimal solution using the MODI-method. Reduced costs for the are given in the upper right corner of each cell, the stepping-stone path is given in the lower left corner.

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	u_i
$r=1$	5 30	4 $-\Delta$ 35	9 2	5 0	0 -3 $+\Delta$	1
$r=2$	6 2	3 $+\Delta$ 25	6 30	4 $-\Delta$ 10	0 -2	0
$r=3$	4 2	2 1	6 2	2 $+\Delta$ 20	0 $-\Delta$ 30	-2
v_j	4	3	6	4	2	

We reallocate 10 units according to our stepping-stone path and update the reduced costs.

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	u_i
$r=1$	5 30	4 $-\Delta$ 25	9 2	5 3	0 $+\Delta$ 10	0
$r=2$	6 2	3 35	6 30	4 3	0 1	-1
$r=3$	4 -1	2 -2 $+\Delta$	6 -1	2 30	0 $-\Delta$ 20	0
v_j	5	4	7	2	0	

We now reallocate 20 units and update the reduced costs again.

	$p=1$	$p=2$	$p=3$	$p=4$	$p=5$	u_i
$r=1$	5 30	4 5	9 2	5 1	0 30	0
$r=2$	6 2	3 35	6 30	4 1	0 1	-1
$r=3$	4 1	2 20	6 1	2 30	0 2	-2
v_j	5	4	7	4	0	

All reduced costs are positive: we have found the optimal solution. Total transportation cost is 555.

c) In case the price for the finished product drops, the company would like to reduce its production commitment. In which facility should the company reduce production first?

The company should reduce production first in facility 3 as production costs are highest here (alternatively: facility 3 has the highest reduced cost, v_3). In an ordinary transportation problem, we usually cannot reduce production amounts at one place without increasing it other places. However, we can simply update production capacities and ship the excess raw materials to the dummy facility.

d) A problem with the production equipment causes the costs to increase with 1000 NOK at facility 1. What are the consequences for the optimal solution found in c)?

NB! The question above should have referred to the solution obtained in b)! Therefore, we have two possible solutions depending on how the question above is interpreted.

With respect to question b):
We update the production costs and recalculate the reduced costs.

	$p = 1$	$p = 2$	$p = 3$	$p = 4$	$p = 5$	u_i
$r = 1$	6 30	4 5	9 2	5 1	0 30	0
$r = 2$	7 2	3 35	6 30	4 1	0 1	-1
$r = 3$	5 1	2 20	6 1	2 30	0 2	-2
v_j	6	4	7	4	0	

The change in production costs causes an increase in v_1, but the reduced costs are still positive. Thus, the optimal solution is not affected.

With respect to question c):
Production at facility 1 is now as expensive as facility 3. It is therefore possible to reduce production at facility 1 instead. The savings are identical to the ones achieved by a reduction at facility 3.

Exercise 4 (15%)
A company uses 4 units of energy, x_e, to produce 1 ton of finished product y. The price for one unit of energy, w_e, depends on the total amount of energy consumed in each period and is given by the following function:

$$w_e = 50 + 0.01x_e.$$

The company can only use 500 units of energy in each period. In addition to the energy, the company needs raw materials for 2200 NOK to produce 1 ton of the finished product. These raw materials have unlimited availability.
The price p for the finished product y in the market depends on the production amount and is given by the following function:

$$p = 4200 - 0.4y.$$

Assume that company wishes to find the profit-maximizing production plan.

a) Formulate the planning problem as nonlinear optimization problem.

Profit is defined as sales revenues-production costs. Sales revenues are given by $p \cdot y$, production costs are given as $2200y + w_e \cdot x_e$. Substituting p and w_e in these expression, we can formulate the following non-linear optimization problem:

$$\max (4200 - 0.4y)y - 2200y - (50 + 0.01x_e)x_e$$

subject to:

$$\begin{aligned} \frac{1}{4}x_e - y &= 0 \\ x_e &\leq 500 \\ x_e, y &\geq 0 \end{aligned}$$

b) Solve the planning problem with the Lagrangian multiplier method.

We assign a Lagrangean multiplier to each of the constraints, remove them from the problem and a penalty term for violating the constraint to the objective function. T

$$\mathcal{L} = 2000y - 0.4y^2 - 50x_e - 0.01x_e^2 - \lambda_1 \left(\frac{1}{4}x_e - y\right) - \lambda_2 (x_e - 500)$$

Calculating 1st derivatives and setting them equal to zero results in:
$\frac{\partial \mathcal{L}}{\partial y} = 2000 - 0.8y + \lambda_1 = 0 \Rightarrow \lambda_1 = -1900$
$\frac{\partial \mathcal{L}}{\partial x_e} = -50 - 0.02x_e - \frac{1}{4}\lambda_1 - \lambda_2 = 0 \Rightarrow \lambda_2 = 535$
$\frac{\partial \mathcal{L}}{\partial \lambda_1} = \frac{1}{4}x_e - y = 0 \Rightarrow y = 125$
$\frac{\partial \mathcal{L}}{\partial \lambda_2} = x_e - 500 = 0 \Rightarrow x_e = 500$
Total profit is 216250.

c) Which other methods for solving nonlinear optimization problems can be used? Explain their advantages and disadvantages.

In addition to the Lagrangian multiplier method, we discussed solving unconstrained, single- and multi-variable problems that can be solved by setting the first derivative(s) to zero and determine which of the stationary points is the optimal solution. Problems with few variables and only equality constraints can be solved by substitution, reducing the problem to an unconstrained single-variable problem. This is a very easy way of solving the NLP, but works only for problems with 2 variables and 1 constraint. A more general way of solving NLPs is to use the Karush-Kuhn-Tuck conditions. This approach works for all non-linear programming problems, but can be very time-consuming as the resulting set of equations may be quite complex.

Exercise 5 (10%)
Consider the following cost matrix:

$$C = \begin{bmatrix} 4 & 5 & 7 & 3 & 6 \\ 1 & 3 & 5 & 8 & 4 \\ 0 & 6 & 5 & 7 & 2 \\ 3 & 5 & 6 & 3 & 6 \\ 9 & 3 & 4 & 3 & 4 \end{bmatrix}$$

Solve the Linear Assignment Problem with a suitable method such that the assignment costs are minimized.
We use the Hungarian Method to solve the Linear Assignment problem. We start by subtracting the smallest element in each row from all elements in that row:

$$C^* = \begin{bmatrix} 1 & 2 & 4 & 0 & 3 \\ 0 & 2 & 4 & 7 & 3 \\ 0 & 6 & 5 & 7 & 2 \\ 0 & 2 & 3 & 0 & 3 \\ 6 & 0 & 1 & 0 & 1 \end{bmatrix}$$

As we do not have a sufficient amount of zeros in the cost matrix to find assignments using only zero-elements, we subtract the smallest element in each column from that column:

$$\bar{C} = \begin{bmatrix} 1 & 2 & 3 & 0 & 2 \\ 0 & 2 & 3 & 7 & 2 \\ 0 & 6 & 4 & 7 & 1 \\ 0 & 2 & 2 & 0 & 2 \\ 6 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We now cover all zeros by the smallest number of horizontal and vertical lines (see above). As we only need 3 lines to cover all zeros, we cannot yet make the assignments. We now subtract the smallest of the uncovered elements from the uncovered elements and add it to the elements covered by 2 lines:

$$\bar{C} = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 0 & 1 & 2 & 7 & 1 \\ 0 & 5 & 3 & 7 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 7 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We need 4 lines to cover all zero elements, so we have to repeat the procedure.

$$\bar{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 7 & 0 \\ 1 & 5 & 3 & 8 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 2 & 0 \end{bmatrix}$$

We finally need 5 lines to cover all zeros and we can make our assignments. There exist several possible assignments with a total costs of 15, one of them is $x_{12} = x_{21} = x_{35} = x_{44} = x_{53} = 1$.