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Department of Mathematical Sciences

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Examination in TMA4110/TMA4115 Calculus 3

English
Thursday August 8, 2013
Time: 09:00 – 13:00
Grades ready by August 29, 2013

Permitted aids (Code C): Specified, simple calculator (HP 30S or Citizen SR-270X)
Rottmann: *Matematisk formelsamling*

All answers must be justified, except in Problem 5, and your calculations should be detailed enough to clearly indicate your line of argument. Each of the problems 1a, 1b, 2a, 2b, 3a, 3b, 4a, and 4b counts 10%, and each of the problems 5a–h counts 2.5%. No partial credit will be given in problems 5a–h.

Problem 1 Given the matrix $A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 2 & 8 & 4 \\ 2 & 1 & 9 & 2 \end{bmatrix}$.

a) Write the solution set of the matrix equation $A\mathbf{x} = \begin{bmatrix} 2 \\ -8 \\ 1 \end{bmatrix}$ in parametric vector form.

b) Find an orthonormal basis for the row space $\mathrm{Row}(A)$.

Problem 2

a) Diagonalize the matrix $A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$ (that is, find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$).

b) Make a change of variable, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, that transforms the quadratic form $x_1^2 + 6x_1x_2 + x_2^2$ into a quadratic form with no cross-product term (that is, find a matrix B such that if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, then $x_1^2 + 6x_1x_2 + x_2^2 = ay_1^2 + by_2^2$ for a suitable choice of constants a and b).

Problem 3 Let $y_1(t) = t$ and $y_2(t) = t\ln(t)$ be two solutions of the differential equation

$$t^2y'' - ty' + y = 0$$

on the interval $(0, \infty)$.

a) Show that the set $\{y_1(t), y_2(t)\}$ is a fundamental set of solutions for the above equation on the interval $(0, \infty)$.

b) Find the general solution to the differential equation $t^2y'' - ty' + y = t$, where $0 < t < \infty$.

Problem 4

a) The matrix $A = \begin{bmatrix} -1 & 0 & 1 \\ 1 & -2 & 0 \\ 0 & 2 & -1 \end{bmatrix}$ has 3 eigenvalues. One eigenvalue is $-2 + i$, and $\begin{bmatrix} 1+i \\ 1-i \\ -2 \end{bmatrix}$ is an eigenvector corresponding to $-2 + i$. Find the other two eigenvalues and a corresponding eigenvector for each of these eigenvalues.

b) Three tanks contain salt water. Tank 1 holds 10 litres, tank 2 holds 5 litres, and tank 3 holds 10 litres. Salt water flows from tank 1 to tank 2 at a rate of 10 litres per second, and from tank 2 to tank 3 at a rate of 10 litres per second, and from tank 3 to tank 1 at a rate of 10 litres per second. Suppose that initially tank 1 contains 10 grammes of salt, tank 2 contains 6 grammes of salt, and tank 3 contains 4 grammes of salt. Suppose also that the tanks are stirred so that the salt in each tank is evenly distributed. How much salt does each of the three tanks contains after $\pi/2$ seconds?

Problem 5 *You do not have to give reasons for your answers for this problem.*

a) For each of the following 4 statements, determine whether it is true or not.

1. The equation $x_2 = x_1(2 + \sqrt{2}i)$ is linear.
2. The differential equation $y'' + 16y = e^{-4t} + 3\sin(4t)$ is linear.
3. The differential equation $t^2 y''(t) + 3ty'(t) - 3y(t) = 0$ is linear.
4. The transformation T from $\mathbb{R}$ to $\mathbb{R}$ given by $T(x) = x^2 + x$ is linear.

b) For each of the following 4 statements, determine whether it is true or not.

1. The three lines $2x_1 + x_3 = 1$, $-2x_1 + x_2 = 3$ and $x_1 + x_3 = 1$ have exactly one point in common.
2. If $\mathbf{a}_1 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$, then $\mathbf{b}$ belongs to $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$.
3. If $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, then $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \mathbb{R}^4$.
4. The following vectors are linearly independent: $\begin{bmatrix} 2 \\ -4 \\ 8 \end{bmatrix}$, $\begin{bmatrix} 4 \\ -6 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

c) For each of the following 4 statements, determine whether it is true or not.

1. If A is an $m \times n$ matrix, B is an $n \times m$ matrix, and $AB = 0$, then we must either have that $A = 0$ or that $B = 0$.
2. If C and D are $n \times n$-matrices, then it must be the case that $(C + D)(C - D) = C^2 - D^2$.
3. If E is an invertible matrix, then $(2E)^{-1} = 2E^{-1}$.
4. If F is an 2×2 matrix and the equation $F\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has a unique solution, then F must be invertible.

d) Let A be an $n \times n$ matrix and k a scalar. For each of the following 4 statements, determine whether it is true or not.

1. $\det(kA) = k^n \det(A)$.
2. If $\det(A) = 2$, then $\det(A^2) = 4$.
3. $\det(A^T) = \det(A)$.
4. If A is invertible, then $\det(A)\det(A^{-1}) = 0$.

e) Let V be a vector space different from the zero vector space, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ be vectors in V. For each of the following 4 statements, determine whether it is true or not.

1. If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is linearly independent, then it must be the case that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a basis for V.
2. If $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\} = V$, then some subset of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is a basis for V.
3. If $\dim(V) = p$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ cannot be linearly independent.
4. If $\dim(V) = p$, then $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ must be a basis for V.

f) Let A be an $n \times n$ matrix. For each of the following 4 statements, determine whether it is true or not.

1. If A is invertible and 1 is an eigenvalue of A, then 1 must also be an eigenvalue of A^{-1}.
2. If $\mathbf{v}$ is an eigenvector of A, then $\mathbf{v}$ must also be an eigenvector of A^2.
3. If A has fewer than n distinct eigenvalues, then A cannot be diagonalizable.
4. If every vector in $\mathbb{R}^n$ can be written as a linear combination of eigenvectors of A, then A must be diagonalizable.

g) Let $\mathbf{v}$ and $\mathbf{u}$ be vectors in $\mathbb{R}^n$ and let W be a subspace of $\mathbb{R}^n$. For each of the following 4 statements, determine whether it is true or not.

1. The distance between $\mathbf{v}$ and $\mathbf{u}$ is $\|\mathbf{u} - \mathbf{v}\|$.
2. If $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent, then $\mathbf{u}$ and $\mathbf{v}$ must be orthogonal to each other.
3. If $\mathbf{v}$ coincides with its orthogonal projection onto W, then $\mathbf{v}$ must belong to W.
4. No vector in $\mathbb{R}^n$ can belong to both W and $W^\perp$ (the orthogonal complement of W).

h) Let A be an $n \times n$ matrix. For each of the following 4 statements, determine whether it is true or not.

1. If $A^T = A$ and if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$ which satisfy that $A\mathbf{u} = 5\mathbf{u}$ and $A\mathbf{v} = 2\mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ must be orthogonal to each other.
2. If $A = PDP^T$ where $P^T = P^{-1}$ and D is a diagonal matrix, then A must be symmetric.
3. If A is symmetric and all the eigenvalues of A are positive, then the quadratic form $\mathbf{x} \to \mathbf{x}^T A\mathbf{x}$ must be positive definite.
4. If A is symmetric and λ is an eigenvalue of A, then the dimension of the eigenspace of A corresponding to λ must be equal to the multiplicity of λ as a root of the characteristic polynomial of A.