Problem 1

Find all solutions of $-(z+i)^3 = 2+2i$ giving your answers in standard form and draw them on the complex plane

Solution: We have

$$(z+i)^3 = -2-2i$$

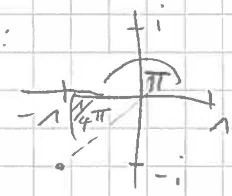
Hence z is a number of the form $\sqrt[3]{-2-2i} - i$

We first compute the 3rd roots of $(-2-2i)$

Note that the polar form of $-2-2i$

is determined by $r = 2\sqrt{1^2+1^2} = 2\sqrt{2} = \sqrt{8}$

and $\Theta = -\frac{3}{4}\pi$ Sketch:



i.e. $-2-2i = \sqrt{8}\, e^{-\frac{3}{4}\pi i}$

Thus the third roots of $-2-2i$ compute as:

$$z_1 = \sqrt[3]{\sqrt{8}}\, e^{-\frac{1}{4}\pi i} \qquad z_2 = \sqrt[3]{\sqrt{8}}\, e^{\left(-\frac{1}{4}\pi + \frac{2\pi}{3}\right)i}$$

$$z_3 = \sqrt[3]{\sqrt{8}}\, e^{\left(-\frac{1}{4}\pi + \frac{4\pi}{3}\right)i}$$

Note that $\sqrt[3]{\sqrt{8}} = \sqrt{2}$ so we find in standard form

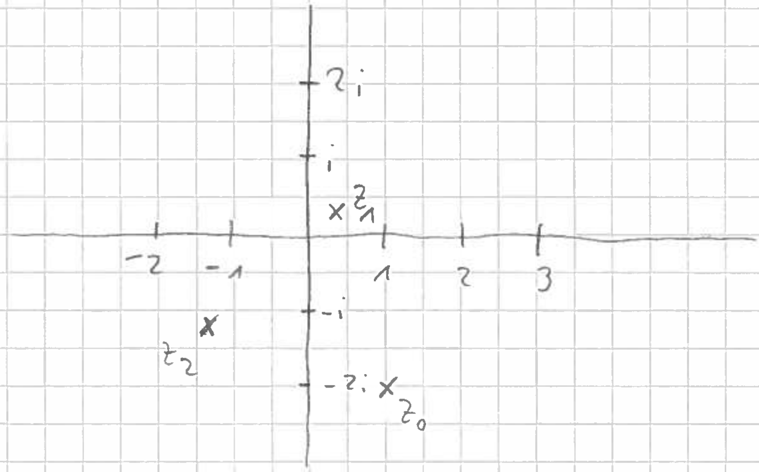
$$z_1 = \sqrt{2}\left(\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}\right) = 1 - i$$

$$z_2 = \sqrt{2}\left(\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}\right)\left(-\frac{1}{2} + i\frac{1}{2}\sqrt{3}\right) = \frac{(-1+\sqrt{3}) + i(\sqrt{3}+1)}{2}$$

$$z_3 = \sqrt{2}\left(\frac{1}{2}\sqrt{2} - i\frac{1}{2}\sqrt{2}\right)\left(-\frac{1}{2} - i\frac{1}{2}\sqrt{3}\right) = \frac{(-1-\sqrt{3}) + i(1-\sqrt{3})}{2}$$

Hence $z = 1-2i$ or $z = \frac{(-1+\sqrt{3}) + i(\sqrt{3}-1)}{2}$ or $z = \frac{(-1-\sqrt{3}) + i(-1-\sqrt{3})}{2}$

Drawing these points in the complex plane we obtain



2i
i
z_1
-2
-1
1
2
3
-i
z_2
-2i
z_0

Problem 2

Solve the initial value problem:

$$y'' - y' - 6y = te^{3t} \qquad y(0) = 0,\ y'(0) = 0$$

Solution:

i) Solve the homogeneous differential equation.

We obtain the polynomial equation

$$\lambda^2 - \lambda - 6 = 0$$

Clearly $\lambda_1 = 3$ is a root of the polynomial and we compute $(\lambda^2 - \lambda - 6) : (\lambda - 3) = \lambda + 2$

whence $\lambda_2 = -2$ is the other root of the polynomial

We obtain a fundamental system of solutions as

$$y_1(t) = e^{3t} \quad \text{and} \quad y_2(t) = e^{-2t}$$

ii) Construct a particular solution for the differential equation. For the right hand side we have a trial solution: The right hand side is of the form

Polynomial $\cdot\, e^{3t}$ (Note that 3 is a root of the polynomial associated to the equation!)

So we try: $y_p(t) = (At + B) \cdot te^{3t}$

$$y_p'(t) = (2At + B)e^{3t} + 3(At^2 + Bt)e^{3t}$$

$$y_p''(t) = 2Ae^{3t} + 3(2At + B)\,{}^{3t} + 3(2At + B)e^{3t} + 9(At^2 + Bt)e^{3}$$

We plug these derivatives into the equation (and divide by e^3 to obtain

$$2A + 3(2At + B) + 3(2At + B) + 9(At^2 + Bt)$$
$$-\left[2At + B + 3(At^2 + Bt)\right] - 6(At^2 + Bt) = t$$

Simplify and compare coefficients to arrive at

$$2A + 5B = 0$$
$$10A = 1$$
Thus $A = \frac{1}{10}$ and $B = -\frac{2}{50}$

We obtain the general solution of the differential equation:

$$y(t) = \left(\frac{1}{10}t^2 - \frac{2}{50}t + c_1\right)e^{3t} + c_2 e^{-2t}$$ with c_1, c_2 constant

Now from $0 = y(0) = c_1 + c_2$

and $0 = y'(0) = -\frac{2}{50} + 3c_1 - 2c_2$

we derive $c_2 = -c_1$

$c_1 = \frac{1}{125}$ and $c_2 = -\frac{1}{125}$

Thus the solution of the initial value problem is

$$\left(\frac{1}{10}t^2 - \frac{2}{50}t + \frac{1}{125}\right)e^{3t} - \frac{1}{125}e^{2t}$$

Problem 3

Find a particular solution of the ODE

$$y'' + 2y' + y = t^{-2} e^{-t}$$

Solution: Since we do not know a trial function for the right hand side given, we can not apply the method of undetermined coefficients.

To apply variation of parameter we compute a fundamental system of solutions for the homogeneous equation first

i) We obtain the Polynomial equation

$$(\lambda+1)^2 = \lambda^2 + 2\lambda + 1 = 0$$

So $\lambda = -1$ is a double root of the polynomial

ii) A fundamental solution for $y'' + 2y' + y = 0$ is given by

$$c_1 e^{-t} + c_2 t e^{-t} \quad \text{with } c_1, c_2 \text{ constant}$$

Now we set

$$y_p(t) = v_1(t) e^{-t} + v_2(t) t e^{-t}$$

with unknown functions v_1, v_2.

To compute v_1, v_2 compute the Wronskian

$$W(e^{-t}, te^{-t}) = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix} = e^{-2t}$$

Now we solve

$$v_1(t) = \int \frac{-te^{-t}(t^{-2}e^{-t})}{e^{-2t}} dt = -\int \frac{1}{t} dt = -\ln(|t|)$$

$$v_2(t) = \int \frac{e^{-t}(t^{-2}e^{-t})}{e^{-2t}} dt = \int t^{-2} dt = -t^{-1}$$

So we obtain as particular solution of the ODE:

$$y_p(t) = -\ln(|t|) e^{-t} \underbrace{- e^{-t}}_{\text{can be omitted since it is just a hom. solution}}$$

Problem 4

Find all solutions of the following linear system

$$
\begin{aligned}
5x_1 + 10x_2 - 3x_3 + 12x_4 + 8x_5 &= -9 \\
2x_1 + 4x_2 - x_3 + 5x_4 + x_5 &= 1 \\
x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= -1 \\
-x_1 - 2x_2 - 3x_4 + 2x_5 &= -6
\end{aligned}
$$

Solution: We use Gaussian elimination on the augmented matrix associated to the linear system:

$$
\begin{bmatrix} 5 & 10 & -3 & 12 & 8 & -9 \\ 2 & 4 & -1 & 5 & 1 & 1 \\ 1 & 2 & -1 & 2 & 1 & -1 \\ -1 & -2 & 0 & -3 & 2 & -6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & -1 \\ 0 & 0 & 2 & 2 & 3 & -4 \\ 0 & 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & -1 & -1 & 3 & -7 \end{bmatrix}
$$

$$
\rightsquigarrow \begin{bmatrix} 1 & 2 & -1 & 2 & 1 & -1 \\ 0 & 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 & -10 \\ 0 & 0 & 0 & 0 & 2 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

We observe that x_1, x_3 and x_5 are basic variables
x_2, x_4 are free variables

From the basic variables we get the particular solution

$$\vec{y}_p = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ -2 \end{bmatrix}$$

and from the free variables we obtain solutions for the homogeneous system:

$$\vec{y}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \vec{y}_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

Now all solutions to the linear system are given by $\vec{y}_p + c_1\vec{y}_1 + c_2\vec{y}_2$ $c_1, c_2 \in$

Problem 5

Let $V \subseteq \mathbb{R}^4$ be the subspace spanned by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Find the orthogonal projection of $\begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \end{bmatrix}$ onto V.

Solution: We first need an orthogonal basis of V to compute the orthogonal projection.

Run Gram-Schmidt algorithm on the generating set of V:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} - \frac{\vec{v}_1 \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} - \frac{12}{10} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 4 \\ 3 \\ -7 \\ 4 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix} - \frac{\vec{v}_1 \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}}{10} \cdot \vec{v}_1 - \frac{\vec{v}_2 \cdot \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}}{\vec{v}_2 \cdot \vec{v}_2} \cdot \vec{v}_2$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ (so $\begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}$ does not contribute anything and we drop it from the computation)

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{\vec{v}_1 \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}{10} \vec{v}_1 - \frac{\vec{v}_2 \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}}{\vec{v}_2 \cdot \vec{v}_2} \cdot \vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

again we get nothing new, so an orthogonal basis for V is given by $\{\vec{v}_1, \vec{v}_2\}$

From the lecture we know that the orthogonal projection of $\vec{y} = \begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \end{bmatrix}$ is given by the formula

$$\frac{\vec{y} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 + \frac{\vec{y} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$= \frac{2}{10} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + \frac{\frac{1}{5} \cdot 18}{18} \frac{1}{5} \begin{bmatrix} 4 \\ 3 \\ -7 \\ 4 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} + \frac{1}{5} \begin{bmatrix} 4 \\ 3 \\ -7 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

Problem 6

(a) The matrix $A = \begin{bmatrix} .3 & .6 \\ .7 & .4 \end{bmatrix}$ is a stochastic matrix and so has a steady state vector which is an eigen vector of A with eigen value 1. Find another eigen value of A and its corresponding eigenvector.

Solution: i) Compute the eigen values (we know already $\lambda_1 = 1$ is an eigenvalue)

$$\det(A - \lambda I_2) = \begin{vmatrix} .3-\lambda & .6 \\ .7 & .4-\lambda \end{vmatrix} = (.3-\lambda)(.4-\lambda) - .42$$

$$= \lambda^2 - .7\lambda - .3$$

Divide the Polynomial by $\lambda - 1$ (since we know the EV 1)

$$(\lambda^2 - .7\lambda - .3) : (\lambda - 1) = \lambda + .3$$

So $\lambda_2 = -.3$ is the other eigenvalue

ii) Compute an eigen vector for $\lambda_2 = -.3$

$$A - .3 I_2 = \begin{bmatrix} .6 & .6 \\ .7 & .7 \end{bmatrix} \underset{\text{Gauß}}{\rightsquigarrow} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

We obtain $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ as an eigen vector for the eigenvalue $\lambda_2 = -.3$

(b) Let $\vec{q}$ be ste -state vector for A. Starting with $\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and defining $\vec{v}_{k+1} = A\,\vec{v}_k$, how many iterations are needed to estimate $\vec{q}$ accurate to 2-decimal places?

Solution: To answer the question we need to compute the steady state vector $\vec{q}$:

First compute an eigen vector to the eigen value $\lambda_1 = 1$

$$A - \lambda_1 I_2 = \begin{bmatrix} -.7 & .6 \\ .7 & -.6 \end{bmatrix}$$

Gaussian elemination now yields

$$A - \lambda_1 I_2 \rightsquigarrow \begin{bmatrix} -1 & -\frac{6}{7} \\ 0 & 0 \end{bmatrix} \quad \text{so} \quad \vec{v}_1 = \begin{bmatrix} 6 \\ 7 \end{bmatrix} \text{ is an}$$

eigenvector for $\lambda_1 = 1$

The steady state vector $\vec{q}$ must be a probability vector and an eigenvector for the eigenvalue 1.

Thus we derive $\vec{q} = \frac{1}{13}\begin{bmatrix} 6 \\ 7 \end{bmatrix}$ (then the entries of $\vec{q}$ sum up to 1, i.e. $\vec{q}$ is a probability vector)

With the help of a calculator we obtain

$$\vec{q} \approxeq \begin{bmatrix} .4615 \\ .5385 \end{bmatrix}$$

We compute now iteratively

$$\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{v}_1 = A \cdot \vec{v}_0 = \begin{bmatrix} .3 \\ .7 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} .51 \\ .49 \end{bmatrix}$$

$$\vec{v}_3 = \begin{bmatrix} .447 \\ .553 \end{bmatrix} \quad \vec{v}_4 = \begin{bmatrix} .4659 \\ .5341 \end{bmatrix}$$

We can stop now because a comparison of $\vec{v}_4$ with $\vec{q}$ shows that the first two decimals are accurate.

Thus we need a minimum of 4 (or more) iterations to estimate $\vec{q}$ accurate to 2-decimal places.

Problem 7

Find the eigenvalues and eigenvectors of $\begin{bmatrix} -5 & 4 \\ -4 & 5 \end{bmatrix}$ and solve

$$x_1' = -5x_1 + 4x_2$$
$$x_2' = -4x_1 + 5x_2$$

with initial conditions $x_1(0) = x_2(0) = 3$

i) Find the eigenvalues of $A = \begin{bmatrix} -5 & 4 \\ -4 & 5 \end{bmatrix}$

$$\det(A - \lambda I_2) = \begin{vmatrix} -5-\lambda & 4 \\ -4 & 5-\lambda \end{vmatrix} = (-5-\lambda)(5-\lambda) + 16$$

$$= -9 + \lambda^2 = \lambda^2 - 9 = (\lambda+3)(\lambda-3)$$

Eigenvalues : $\lambda_1 = -3 \quad \lambda_2 = 3$

ii) Eigenvectors (Use Gaussian elimination)

$$(A + 3I_2) = \begin{bmatrix} -2 & 4 \\ -4 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is eigenvector for λ_1

$$(A - 3I_2) = \begin{bmatrix} -8 & 4 \\ -4 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is eigen-vector for λ_2

This settles the first part of the question.

To solve the system we observe that it can be written in matrix form as

$$\vec{x}' = A\vec{x} \qquad \left(\text{with } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \vec{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}\right)$$

From the computation of the eigenvalues we see that A is diagonalizable (as a 2×2 matrix with two distinct real eigenvalues!)

Thus we apply the solution formula from the lecture for diagonalizable systems:

The general solution to the system of differential equations is

$$\vec{x}(t) = c_1 \exp(\lambda_1 t)\vec{v_1} + c_2 \exp(\lambda_2 t)\vec{v_2}$$

$$= c_1 e^{-3t}\begin{bmatrix}2\\1\end{bmatrix} + c_2 e^{3t}\begin{bmatrix}1\\2\end{bmatrix} \text{ with } c_1, c_2 \text{ constant}$$

To obtain the solution of the initial value problem we insert the initial values:

$$\begin{bmatrix}3\\3\end{bmatrix} = \vec{x}(0) = c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\2\end{bmatrix}$$

$$= \begin{bmatrix}2c_1 + c_2\\ c_1 + 2c_2\end{bmatrix}$$

Solving for c_1, c_2 we obtain $c_1 = c_2 = 1$

Thus the solution to the initial value problem is

$$e^{-3t}\begin{bmatrix}2\\1\end{bmatrix} + e^{3t}\begin{bmatrix}1\\2\end{bmatrix}$$

Problem 8

Using a substitution of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ write the quadratic form $4x_1^2 + 24x_1x_2 + 11x_2^2$ in the form $ay_1^2\ by_2^2$ and sketch the set
$\{4x_1^2 + 24x_1x_2 + 11x_2^2 = 20\}$.

Solution:

Associated to the quadratic form $4x_1^2 + 24x_1x_2 + 11x_2^2$ is the symmetric matrix

$$A = \begin{bmatrix} 4 & 12 \\ 12 & 11 \end{bmatrix}$$

to compute P (and a and b) we have to diagonalize A:

i) Compute Eigenvalues:

$$\det(A - \lambda I_2) = \begin{vmatrix} 4-\lambda & 12 \\ 12 & 11-\lambda \end{vmatrix} = (4-\lambda)(11-\lambda) - 12^2$$

$$= \lambda^2 - 15\lambda - 144$$

$$= (\lambda - 20)(\lambda + 5)$$

so eigenvalues are $\lambda_1 = 20$ and $\lambda_2 = -5$

ii) Compute Eigenvectors

for $\lambda_1 = 20$: $A - 20I_2 = \begin{bmatrix} -16 & 12 \\ 12 & 9 \end{bmatrix}$ (use Gaussian elimination)

$\rightsquigarrow \begin{bmatrix} 1 & -3/4 \\ 0 & 0 \end{bmatrix}$ Thus $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ is an eigenvector for λ_1

for $\lambda_2 = -5$ $A - (-5)I_2 = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} \underset{\text{Gauß}}{\rightsquigarrow} \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \vec{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is an eigenvector for λ_2

We obtain now the ~~unit~~ eigenvectors $\vec{w}_1 = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ for λ_1

and $\vec{w_2} = \begin{bmatrix} -4/5 \\ 3/5 \end{bmatrix}$

Now $P = [\vec{w_1} \;\; \vec{w_2}] = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ follows

(since $A = P \begin{bmatrix} 20 & 0 \\ 0 & -5 \end{bmatrix} P^T$)

In particular, we obtain $a = 20$

$b = -5$

Whence for $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ we find the expression

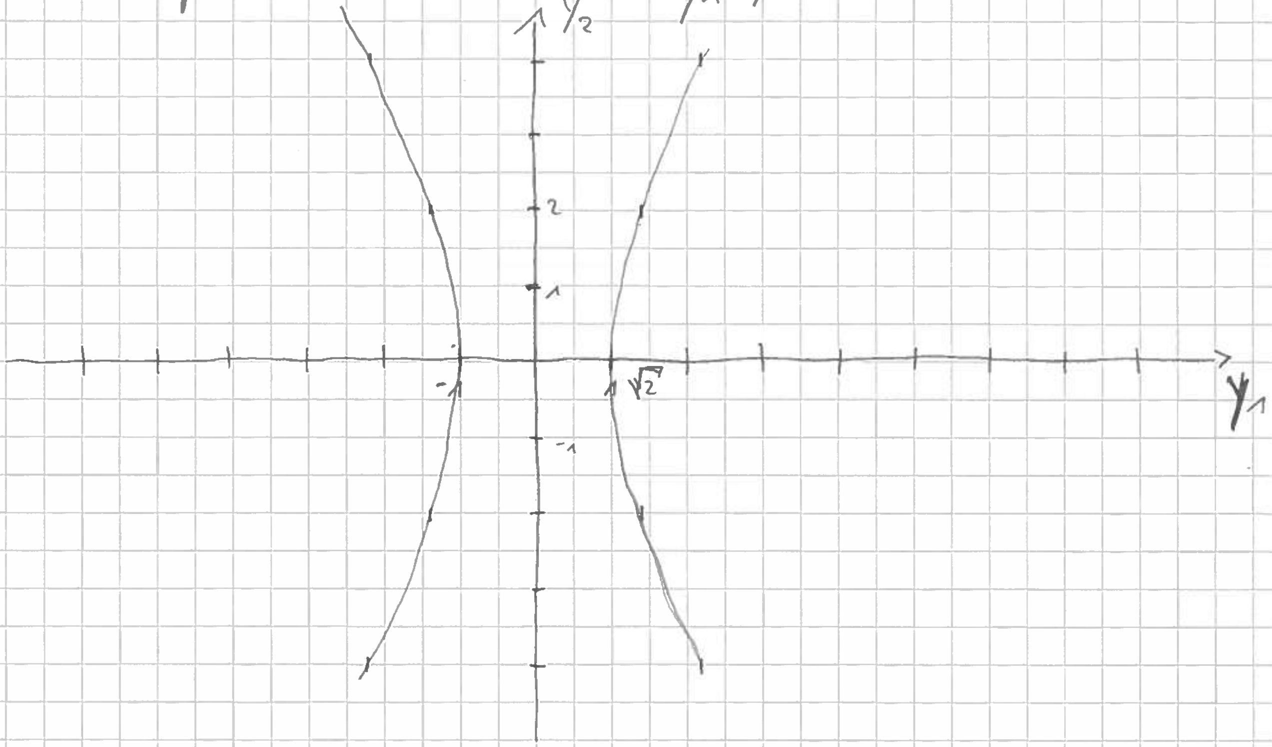
$20\, y_1^2 - 5\, y_2^2$ for the quadratic form $4x_1^2 + 24x_1x_2 + 11x_2^2$

Computing in the coordinates y_1, y_2 we find for

$$20\, y_1^2 - 5\, y_2^2 = 20$$

$$\Rightarrow \quad y_1 = \sqrt{1 + \tfrac{1}{4} y_2^2} \quad \underline{\text{or}} \quad y_1 = -\sqrt{1 + \tfrac{1}{4} y_2^2}$$

We can thus sketch the set $\{20y_1^2 - 5y_2^2 = 20\} \subseteq \mathbb{R}^2$ with respect to the coordinates y_1, y_2:



Problem 9

Let A be a $m \times m$ square matrix. Let λ be an eigen value of A. Show that the set

$$\{\vec{x} \in \mathbb{R}^m : A\vec{x} = \lambda\vec{x}\}$$

is a subspace of $\mathbb{R}^m$.

Solution:

By definition we have

$$E_\lambda = \{\vec{x} \in \mathbb{R}^m : A\vec{x} = \lambda\vec{x}\} \subseteq \mathbb{R}^m$$

so E_λ is a <u>subset</u> of $\mathbb{R}^m$.

Let $\vec{0} \in \mathbb{R}^m$ be the zero-vector.

The rules for matrix multiplication yield

$$A\vec{0} = \vec{0} = \lambda \cdot \vec{0}$$

and thus $\vec{0} \in E_\lambda$

Assume that $\vec{x}, \vec{y} \in E_\lambda$ we then have to check that for all $r \in \mathbb{R}$ the linear combination

$\vec{x} + r\vec{y}$ is also contained in E_λ

With the rules of matrix-vector multiplication we derive

$$A \cdot (\vec{x} + r\vec{y}) = A\vec{x} + A(r\vec{y}) = \underbrace{\underbrace{A\vec{x}}_{=\lambda\vec{x}} + r \underbrace{A\vec{y}}_{=\lambda\vec{y}}}_{\text{since } \vec{x}, \vec{y} \in E_\lambda}$$

$$= \lambda\vec{x} + r\lambda\vec{y}$$

$$= \lambda(\vec{x} + r\vec{y})$$

Comparing the beginning and the end of the equation we see: $(\vec{x} + r\vec{y}) \in E_\lambda$ for all $r \in \mathbb{R}$

From $E_\lambda \subseteq \mathbb{R}^m$, $\vec{0} \in {}_\lambda$ and $,\vec{y} \in E_\lambda, r - \mathbb{R} \Rightarrow \vec{x} + r\vec{y} \in E$

we conclu : E_λ is a subspace of $\mathbb{R}^m$.