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Department of Mathematical Sciences

Examination paper for **TMA4110 Matematikk 3**

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Problem 1

a) Find the polar coordinates of the complex numbers z satisfying $iz = \bar{z}$.

b) Find all the solutions to $z^4 = (z-1)^4$.

Solution a) For $z = re^{i\theta}$. The polar coordinates of $\bar{z}$ are then $\bar{z} = re^{-i\theta}$. We also know $i = e^{i\pi/2}$. Thus we need to determine r and θ such that

$$iz = e^{i\pi/2}re^{i\theta} = re^{i(\theta+\pi/2)} = re^{-i\theta} = \bar{z}.$$

This equation holds if $r = 0$, i.e. $z = 0$, and if $\theta + \pi/2 = -\theta + 2\pi k$ for $k \in \mathbb{Z}$. The latter equation is satisfied if $\theta = -\pi/4 + \pi k$ for any $k \in \mathbb{Z}$. Hence $iz = \bar{z}$ is satisfied by all complex numbers of the form

$$z = re^{i(-\pi/4+\pi k)} \text{ for all } r \geq 0 \text{ and all } k \in \mathbb{Z}.$$

Alternatively, if we assume $\theta \in (-\pi, \pi]$, $iz = \bar{z}$ is satisfied by all complex numbers of the form

$$z = re^{-i\pi/4} \text{ or } z = re^{i3\pi/4} \text{ for all } r \geq 0.$$

b) The equation $z^4 = (z-1)^4$ is not satisfied by $z = 0$. In fact, the equation implies $|z| = |z-1|$ (recall that $|z|$ is a real non-negative number) and we know that all solutions must lie on the line $z = \frac{1}{2} + iy$ of complex numbers with real part $\frac{1}{2}$. Moreover, we see immediately that $z = \frac{1}{2}$ is a solution. To find the remaining two solutions (note that $z^4 = (z-1)^4$ is equivalent to $0 = -4z^3 + 6z^2 - 4z + 1$ and we have in total three solutions), we may divide by z^4 and get

$$z^4 = (z-1)^4 \iff \frac{(z-1)^4}{z^4} = 1 \iff \left(\frac{z-1}{z}\right)^4 = 1 \iff w^4 = 1 \text{ for } w := \frac{z-1}{z}.$$

For w we have the four solutions $w_k = e^{(2\pi i/4)k}$ for $k = 0, 1, 2, 3$. We check the four cases:
$k = 0$, $w = 1$: this would imply $z - 1 = z$ which is not possible. Thus this case does not yield a solution for $z^4 = (z-1)^4$.
$k = 1$, $w = i$: then $z - 1 = iz$, i.e. $z = \frac{1}{2}(1+i)$. Since we know that $\bar{z} = \frac{1}{2}(1-i)$ is then also a solution and since there are only three solutions, we could stop here. But let us check the other options anyway...
$k = 2$, $w = -1$: then $z - 1 = -z$, i.e. $z = \frac{1}{2}$ (the solution we had found immediately).
$k = 3$, $w = -i$: then $z - 1 = -iz$, i.e. $z = \frac{1}{2}(1-i)$ (as expected).
Thus the solutions are $z = \frac{1}{2}$, $z = \frac{1}{2}(1+i)$, and $z = \frac{1}{2}(1-i)$.

$$\begin{aligned}\text{Alternatively: } & z^4 = (z-1)^4 \iff (z-1)^4 - z^4 = 0 \iff ((z-1)^2 + z^2)((z-1)^2 - z^2) = 0 \\ & \iff (2z^2 - 2z + 1)(-2z + 1) = 0 \iff z = \tfrac{1}{2} \text{ or } z = \tfrac{1}{2}(1 \pm i).\end{aligned}$$

Problem 2

Consider the equation
$$y'' + 4y = q(t).$$

a) Find the general solution of the equation when $q(t) = 0$.

b) Find the general solution of the equation when $q(t) = \cos 3t$.

c) For $q(t) = e^{2t}$, find a solution satisfying the initial conditions $y(0) = \frac{1}{4}$ and $y'(0) = \frac{1}{2}$.

Solution

a) The general solution for $y'' + 4y = 0$ is
$$y_h(t) = c_1 \cos 2t + c_2 \sin 2t$$
for real numbers c_1 and c_2.

b) We need to find a particular solution of $y'' + 4y = \cos 3t$. We try the function
$$y(t) = A \cos 3t + B \sin 3t.$$
The second derivative is
$$y''(t) = -9A \cos 3t - 9B \sin 3t$$
and should satisfy
$$\cos 3t = y''(t) + 4y(t) = -9A \cos 3t - 9B \sin 3 + 4(A \cos 3t + B \sin 3t) = -5A \cos 3t - 5B \sin 3t.$$
To make both sides equal we need to choose $A = -\frac{1}{5}$ and $B = 0$. Thus a particular solution is given by
$$y_p(t) = -\frac{1}{5} \cos 3t.$$
The general solution is
$$y_h(t) + y_p(t) = c_1 \cos 2t + c_2 \sin 2t - \frac{1}{5} \cos 3t$$
for real numbers c_1 and c_2.

c) Again, we need to find a particular solution. Starting with $y(t) = Ce^{2t}$, we get $y_p(t) = \frac{1}{8}e^{2t}$. Thus general solution is

$$y(t) = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{8}e^{2t}.$$

To satisfy the initial condition we need

$$\frac{1}{4} = y(0) = c_1 + \frac{1}{8} \text{ and } \frac{1}{2} = y'(0) = -2c_2 + 2\frac{1}{8}.$$

Hence we need to choose $c_1 = \frac{1}{8}$ and $c_2 \frac{1}{8}$, and the solution is

$$y(t) = \frac{1}{8}\cos 2t + \frac{1}{8}\sin 2t + \frac{1}{8}e^{2t}.$$

Problem 3

Let

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Find a fundamental system of solutions for the system $x' = Ax$ of first order differential equations.

Solution

The matrix A has eigenvalue $\lambda = 0$ with eigenvector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and eigenvalue $\lambda = 2$ with eigenvector

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

This means

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}.$$

A fundamental system for

$$y' = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} y$$

is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \exp(2t) \end{bmatrix}.$$

Therefore, a fundamental system for $x' = Ax$ is is

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \exp(2t) \end{bmatrix} = \begin{bmatrix} \exp(2t) \\ -\exp(2t) \end{bmatrix}$$

Problem 4

Let z be a solution of $z^2+z+1=0$. Find a solution of the equation

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 1 & 1 & -1 \\ 1 & z & z^2 & 0 \\ 1 & z^2 & z & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

Solution

The equation $z^2+z+1=0$ ensures that every vector of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} a \\ a \\ a \\ b \end{bmatrix}$$

solves the two last equations. It therefore suffices to find such a and b that the first two equations are also satisfied. These read $3a+3b=9$ and $3a-b=1$, so that this forces $a=1$ and $b=2$, and

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$$

is indeed a solution.

Problem 5

Find the determinant and the inverse of the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix}.$$

Solution

Since the matrix is (lower) triangular, the determinant is

$$1 \cdot 2 \cdot 3 = 6.$$

The inverse is

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1/2 & 0 \\ 0 & -1/2 & 1/3 \end{bmatrix}.$$

Problem 6

Let

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \qquad v = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \qquad w = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}.$$

Find a non-zero linear combination of u and v that is orthogonal to w.

Solution

We need to find λ and μ such that $\lambda u + \mu v \neq 0$ and

$$(\lambda u + \mu v) \cdot w = 0.$$

Since u and v are (obviously) linearly independent, the first condition just means that not both λ and μ can be zero. Because of

$$u \cdot w = -1$$

and

$$v \cdot w = -2,$$

the second condition means

$$-\lambda - 2\mu = 0.$$

We can take $\lambda = 2$ and $\mu = -1$, for example, and

$$2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

is indeed orthogonal to w. Of course, every non-zero multiple of it will also do the job.

Problem 7

Let A be the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}.$$

a) Find a basis for the spaces $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$.

b) Determine the eigenvalues and eigenvectors of A.

c) Determine matrices P and D such that $A = PDP^{-1}$.

Solution

a) By adding the first row to the second and (-1) times the first row to the third row we see that A is row equivalent to

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence $\mathrm{Nul}(A)$ consists of all vectors in $\mathbb{R}^3$ of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

The set of vectors $\{v_1, v_2\}$ with

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form a basis of $\mathrm{Nul}(A)$. We can also read off from B that the first column in A is a pivot column. This shows that the vector

$$v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

forms a basis of $\mathrm{Col}(A)$.

b) We know from a) that 0 is an eigenvalue of A with multiplicity 2 (because $\dim \mathrm{Nul}(A) = 2$ and $A \neq 0$). The vectors v_1 and v_2 are linearly independent eigenvectors of A and span the eigenspace corresponding to the eigenvalue 0.
Since A is symmetric, we know that it is diagonalizable. This implies that v_3 is another eigenvector of A, since it is nonzero and orthogonal to v_1 and v_2. To determine the corresponding eigenvalue we just need to calculate

$$Av_3 = 3v_3.$$

Thus 3 is the remaining eigenvalue with eigenspace $\mathrm{Col}(A)$.

c) We know from b) that we can choose D as

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(we could have put the 3 also in the top left corner). To form P we just need to collect eigenvectors to the eigenvalues.

$$P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}.$$

The inverse is

$$P^{-1} = \frac{1}{3} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix}.$$

Now we can check

$$PDP^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & -1 & 1 \end{bmatrix} = A$$

Problem 8

The team of FC Troll can either win, draw or lose a game in their league. Even though Askeladden is not a fan of that team, he had followed FC Troll's results very closely for a while. He observed that the results show the following pattern:

- If they won a game, there is a 50% chance that they win and a 30% chance that they lose the next game.
- If they lost a game, there is a 80% chance that they lose and a 20% chance that they win the next game.
- If the last game was a draw, there is a 40% chance that the next game is again a draw and a 30% chance that they lose the next game.

After not watching any game for a while, Askeladden goes again in the stadium of FC Troll. What is the most likely outcome of the game? Give the probabilities for observing the three possible outcomes.

Solution The stochastic matrix which describes the probability of a win, loss or draw is

$$A = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}.$$

We want to find the stationary vector, it is a probability vector that satisfies $Av = v$. Thus we need to solve the system of linear equations with matrix $A - I$. After multiplying by 10, the Gauss elimination gives

$$10(A-I) = \begin{bmatrix} -5 & 2 & 3 \\ 3 & -2 & 3 \\ 2 & 0 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 3 & -2 & 3 \\ -5 & 2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & -2 & 12 \\ 0 & 2 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -3 \\ 0 & -2 & 12 \\ 0 & 0 & 0 \end{bmatrix}$$

The solutions satisfy $x_1 = 3x_3$ and $2x_2 = 12x_3$. Choosing $x_3 = 1$ yields $x_1 = 3$ and $x_2 = 6$. Then the stationary probability vector is

$$v = \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix}.$$

Thus the most likely outcome of the game is a loss with a 60% chance. The probability for a win is 30% and the probability for a draw is 10%.

Problem 9

Find the equation $y = mx + c$ of the line that best fits the data points $(0,1)$, $(1,-2)$, $(2,3)$ and $(3,6)$.

Solution

We are looking for the least square solution of the system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} m \\ c \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 6 \end{bmatrix}.$$

To find the solution we solve the system $A^T A\mathbf{x} = A^T\mathbf{b}$ which is

$$\begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} \begin{bmatrix} 22 \\ 8 \end{bmatrix}.$$

Gauss elimination gives

$$\begin{bmatrix} 14 & 6 & 22 \\ 6 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 7 & 3 & 11 \\ 3 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 3 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 \\ 0 & 5 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}.$$

The solution is $m = 2$ and $c = -1$. Thus the best fitting line is $y = 2x - 1$.

Problem 10

Let A be an $n \times n$ matrix such that $A = A \cdot A$. Let $\{x_1, \dots, x_k\}$ be a basis of $\mathrm{Nul}(A)$, and let $\{b_1, \dots, b_l\}$ be a basis of $\mathrm{Col}(A)$. Show that $\{x_1, \dots, x_k, b_1, \dots, b_l\}$ is a basis of $\mathbb{R}^n$.

Solution

The formula

$$n = \dim \mathrm{Nul}(A) + \dim \mathrm{Col}(A)$$

shows that $n = k + l$, so that it is sufficient to see that $\{x_1, \dots, x_k, b_1, \dots, b_l\}$ spans $\mathbb{R}^n$ *or* that it is linearly independent. We show that it spans $\mathbb{R}^n$:

Given any vector $v \in \mathbb{R}^n$, we can write

$$v = (v - Av) + Av.$$

The hypothesis $A^2 = A$ implies

$$A(v - Av) = Av - A^2 v = 0,$$

so that the first summand $v - Av$ lies in $\mathrm{Nul}(A)$. The second summand Av lies obviously in $\mathrm{Col}(A)$. Therefore $\mathbb{R}^n$ is spanned by the union of $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$, and then also by the set $\{x_1, \dots, x_k, b_1, \dots, b_l\}$.