Problem 1 [15 points]

There are four versions of this exercise with different constants $c = -1, -2, 2, 3$.

The function f is given as

$$f(t) = \begin{cases} 1, & 0 \le t \le 1, \\ c, & t > 1. \end{cases} \tag{1}$$

a) Compute the Laplace transform of f.

Solution: By the definition of the step function $u(t-a)$, we can write

$$f = 1 + (c-1)u(t-1)$$

thus the Laplace transform formula for the step function gives

$$F(s) = 1/s + (c-1)e^{-s}/s.$$

Input $c = -1, -2, 2, 3$ gives the final results respectively.

b) Show that

$$\mathcal{L}\left(\int_0^t e^{-x} y(x)dx\right) = \frac{Y(s+1)}{s},$$

where $\mathcal{L}$ denotes the Laplace transform and $Y := \mathcal{L}(y)$.

Solution: Notice that

$$\int_0^t e^{-x} y(x)dx = e^{-t} \int_0^t e^{t-x} y(x)dx = e^{-t}(e^t * y),$$

where $e^t * y$ denotes the Laplace transform. Using the s-shifting theorem, we get

$$\mathcal{L}(e^{-t} e^t * y) = \mathcal{L}(e^t * y)(s+1) = \frac{Y(s+1)}{s}$$

By the Laplace convolution theorem, we have

$$\mathcal{L}(e^t * y) = \mathcal{L}(e^t) \cdot Y = Y(s)/(s-1),$$

which gives

$$\mathcal{L}(e^t * y)(s+1) = \frac{Y(s+1)}{s}.$$

The proof is complete.

An alternative approach is to note that the function

$$z(t) := \int_0^t e^{-x} y(x)\, dx$$

solves the ODE

$$z'(t) = e^{-t} y(t), \qquad z(0) = 0.$$

By the s-shift theorem, the Laplace transform of $e^{-x}y(x)$ is $Y(s+1)$. Computing the Laplace transform of the ODE yields

$$s\mathcal{L}(z)(t) = Y(s+1).$$

Now we may divide by s and arrive at the desired result.

c) Use the formula in part **b)** in order to find the solution $y(x)$ of

$$\int_0^t e^{-x} y(x)\, dx = f(t).$$

Here f is the function defined in (1).

Solution:

Apply the Laplace transform to both sides, by a) and b), we get

$$Y(s+1)/s = 1/s + (c-1)e^{-s}/s$$

hence

$$Y(s+1) = 1 + (c-1)e^{-s}.$$

By a change of variable, we get

$$Y(s) = 1 + (c-1)e^{-(s-1)}$$

which gives $y(x) = \delta(x) + (c-1)e\,\delta(x-1)$. Input $c = -1, -2, 2, 3$ gives the final results respectively.

Problem 2 [5 points]

There are four versions of this exercise with different constants $\alpha = -1, -2, 2, 3$**.**

Find the complex Fourier series of the function

$$f(x) = \begin{cases} e^{ix} + \alpha, & 0 \le x \le \pi, \\ e^{ix} - 1, & -\pi \le x < 0. \end{cases}$$

Solution:

Notice that we can write f as $e^{ix} + g$, where

$$g(x) = \begin{cases} \alpha, & 0 \le x \le \pi, \\ -1, & -\pi \le x < 0. \end{cases}$$

Hence it suffices to compute the Fourier series for g and add e^{ix} to the final result.

For c_0 we get

$$c_0 = \frac{1}{2\pi}\Big(\int_{-\pi}^{0} -1dx + \int_0^{\pi} \alpha dx\Big) = (\alpha - 1)/2.$$

For other c_n, we have

$$c_n = \frac{1}{2\pi}\Big(\int_{-\pi}^{0} -e^{-inx}dx + \alpha \int_0^{\pi} e^{-inx}dx\Big).$$

Since $\int_{-\pi}^{\pi} e^{-inx}dx = 0$, we get $\int_{-\pi}^{0} -e^{-inx}dx = \int_0^{\pi} e^{-inx}dx$, which gives

$$c_n = \frac{\alpha+1}{2\pi}\int_0^{\pi} e^{-inx}dx = \frac{\alpha+1}{2\pi} \cdot \frac{1-(-1)^n}{in}.$$

Notice that $1 - (-1)^n = 0$ when n is even and $1 - (-1)^n = 2$ when n is odd, thus

$$f(x) = e^{ix} + \frac{\alpha - 1}{2} + \sum_{n \text{ odd}} \frac{1+\alpha}{i\pi n} e^{inx}.$$

Input $\alpha = -1, -2, 2, 3$ gives the final results respectively.

Problem 3 [10 points]

Use the convolution theorem for the Fourier transform in order to find the function f that solves the equation

$$\int_{-\infty}^{+\infty} e^{-(x-t)^2} f(t)\,dt = \sqrt{2\pi}\, x e^{-x^2/2}.$$

Solution: Take the Fourier transform. The Fourier convolution theorem gives

$$\sqrt{2\pi}\hat{f}(w) \cdot \frac{1}{\sqrt{2}} e^{-w^2/4} = \sqrt{2\pi}\widehat{xe^{-x^2/2}}.$$

Now we use that

$$xe^{-x^2/2} = -(e^{-x^2/2})',$$

which implies that

$$\widehat{xe^{-x^2/2}} = -iw\widehat{e^{-x^2/2}} = -iwe^{-x^2/2}.$$

Now the first identity reduces to

$$\sqrt{2\pi}\hat{f}(w) \cdot \frac{1}{\sqrt{2}} e^{-w^2/4} = \sqrt{2\pi}(-iwe^{-w^2/2}),$$

which gives

$$\hat{f}(w) = -iw\sqrt{2}e^{-w^2/4} = -2iw\widehat{e^{-t^2}} = (-2)\widehat{(e^{-t^2})'},$$

(the final identity uses again that $\widehat{g'} = iw\hat{g}$) from which we know that the Fourier transform of f is equal to the Fourier transform of $(-2)(e^{-t^2})'$. Now we know that f must be equal to $(-2)(e^{-t^2})'$ (you can use, for example, the Fourier inversion formula to prove this), hence we get

$$f(t) = (-2)(e^{-t^2})' = 4te^{-t^2}.$$

Problem 4 *TMA4135 Mathematics 4D:* [5 points]

Let f, g be two smooth functions and let $c > 0$ be a constant. Show that the function

$$u(x,t) := f(cx+t) + g(2cx-2t) + \sin cx \cos t$$

satisfies the wave equation $u_{xx} = c^2 u_{tt}$.

Solution:

Both u_{xx} and $c^2 u_{tt}$ equal

$$c^2 f''(cx+t) + 4c^2 g''(2cx-2t) - c^2 \sin cx \cos t.$$

Problem 4 *TMA4130 Mathematics 4N:* [5 points]

Show that the Fourier transform of

$$f(x) = \begin{cases} x^2, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

is the function

$$\hat{f}(w) = \begin{cases} \frac{1}{3} \cdot \sqrt{\frac{2}{\pi}}, & w = 0, \\ \sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w} + \frac{2\cos w}{w^2} - \frac{2 \sin w}{w^3} \right), & w \neq 0, \end{cases}$$

Solution:

By definition

$$\hat{f}(0) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x^2 = \frac{1}{3} \cdot \sqrt{\frac{2}{\pi}},$$

and when $w \neq 0$ we have

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} x^2 e^{-ixw}.$$

Integration by parts gives

$$\int_{-1}^{1} x^2 e^{-ixw} = x^2 \frac{e^{-ixw}}{-iw} |_{-1}^{1} - \int_{-1}^{1} 2x \frac{e^{-ixw}}{-iw} = 2\frac{\sin w}{w} + \int_{-1}^{1} 2x \frac{e^{-ixw}}{iw}$$

and

$$\int_{-1}^{1} 2x \frac{e^{-ixw}}{iw} = 2x \frac{e^{-ixw}}{w^2} |_{-1}^{1} - \int_{-1}^{1} 2 \frac{e^{-ixw}}{w^2} = 4\frac{\cos w}{w^2} - 4 \frac{\sin w}{w^3}$$

Hence

$$\hat{f} = \sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w} + \frac{2\cos w}{w^2} - \frac{2\sin w}{w^3} \right).$$

Problem 5 [20 points]

Consider the heat equation

$$u_t = c^2 u_{xx} + \alpha \tag{2}$$

where $c > 0$ and $\alpha \in \mathbb{R}$ are given constants.

a) Show that the function

$$w(x,t) = \frac{\alpha}{2c^2} x(\pi - x)$$

satisfies the equation $w_t = c^2 w_{xx} + \alpha$.

b) Find the solution of equation (2) for $x \in [0, \pi]$ and $t > 0$ with the boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = \frac{\alpha}{2c^2} x(\pi - x) + \begin{cases} 0, & 0 \le x < \frac{\pi}{2}, \\ x - \pi, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

c) Find $\lim_{t\to\infty} u(x,t)$.

Solution:

a) We have $w_t = 0$, and $w_{xx} = -\alpha/c^2$, which implies that $w_t - c^2 w_{xx} = \alpha$.

b) Consider the function $v = u - w$. Linearity (or superposition) implies that v satisfies

$$v_t = c^2 v_{xx}, \quad v(0,t) = v(\pi,t) = 0, \quad t > 0,$$

(note that w vanishes at $x = 0$ and π), with initial data

$$v(x,0) = u(x,0) - w(x,0) = \begin{cases} 0, & 0 \le x < \frac{\pi}{2}, \\ x - \pi, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

Standard separation of variables (Kreyszig, pp. 558ff) gives that

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-(cn)^2 t}$$

where B_n are given as the Fourier coefficients of the initial data, thus

$$v(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) = \begin{cases} 0, & 0 \le x < \frac{\pi}{2}, \\ x - \pi, & \frac{\pi}{2} < x \le \pi. \end{cases}$$

Standard formulas for Fourier series yield

$$\begin{aligned}
B_n &= \frac{2}{\pi}\int_0^{\pi} \sin(nx) w(x,0) dx \\
&= \frac{2}{\pi}\int_{\pi/2}^{\pi} \sin(nx)(x-\pi) dx \\
&= \frac{2}{\pi}\Big[- \Big|_{\pi/2}^{\pi} \frac{1}{n}\cos(nx)(x-\pi) + \frac{1}{n}\int_{\pi/2}^{\pi} \cos(nx) dx \Big] \\
&= -\frac{1}{n}\cos\big(\frac{n\pi}{2}\big) - \frac{2}{\pi n^2}\sin\big(\frac{n\pi}{2}\big).
\end{aligned}$$

Thus the answer reads

$$\begin{aligned}
u(x,t) &= w(x,t) + v(x,t) \\
&= \frac{\alpha}{2c^2}x(\pi - x) - \sum_{n=1}^{\infty}\Big(\frac{1}{n}\cos\big(\frac{n\pi}{2}\big) + \frac{2}{\pi n^2}\sin\big(\frac{n\pi}{2}\big)\Big)\sin(nx)e^{-(cn)^2 t}.
\end{aligned}$$

c) We see that each term in the infinite sum contains the exponentially decaying factor $e^{-(cn)^2 t}$. Thus these terms will all vanish in the limit when $t \to \infty$. Hence

$$u(x,t) \to \frac{\alpha}{2c^2}x(\pi - x), \quad t \to \infty.$$

Problem 6 [10 points]

There are four versions of this exercise with slightly different assumptions.

We are given a continuously differentiable function $g\colon [0,1] \to \mathbb{R}$ with the following properties:

Version I:

- $g(0) = 0.2$ and $g(1) = 0.7$.
- $0.1 \le g'(x) \le 0.9$ for all $0 \le x \le 1$.

Version II:

- $g(0) = 0.7$ and $g(1) = 0.2$.
- $-0.9 \le g'(x) \le -0.1$ for all $0 \le x \le 1$.

Version III:

- $g(0) = 0.4$ and $g(1) = 0.8$.
- $0.2 \le g'(x) \le 0.8$ for all $0 \le x \le 1$.

Version IV:

- $g(0) = 0.8$ and $g(1) = 0.4$.
- $-0.8 \le g'(x) \le -0.2$ for all $0 \le x \le 1$.

The remaining part of the exercise is the same for all versions.

We consider now the fixed point iteration

$$x_{k+1} = g(x_k)$$

with $x_0 = 0$.

a) Show that the function g has a unique fixed point r in the interval $[0,1]$ and that the fixed point iteration converges to r.

b) Provide an upper bound for the number of iterations that are required until $|x_k - r| \le 10^{-6}$.

Solution:

Part a), Version I: Since $g'(x) \geq 0.1$, the function g is (strictly) increasing. Since moreover $g(0) \geq 0$ and $g(1) \leq 1$, it follows that $0 \leq g(x) \leq 1$ for all $x \in [0,1]$. That is, the function g maps the interval $[0,1]$ to itself. Moreover, the derivative of g satisfies the condition $|g'(x)| \leq L := 0.9 < 1$ for all $x \in [0,1]$. Thus all the conditions of the fixed point theorem are satisfied, and therefore g has a unique fixed point r in the interval $[0,1]$ and the fixed point iteration converges to r.

Version II: Here the function is strictly decreasing, and $g(0) \leq 1$, $g(1) \geq 0$. Else the argumentation is the same as for version I.

Version III: Here we have that $|g(x)| \leq L := 0.8 < 1$. Else the argumentation is the same as for version I.

Version IV: Here the function is strictly decreasing, and $g(0) \leq 1$, $g(1) \geq 0$. Moreover, $|g(x)| \leq L := 0.8 < 1$. Else the argumentation is the same as for version I.

Part b) For the number of iterations, we have the a–priori error estimate

$$|x_k - r| \leq \frac{L^k}{1-L}|g(x_0) - x_0|.$$

Thus we need that

$$\frac{L^k}{1-L}|g(x_0) - x_0| \leq 10^{-6}$$

or, as $x_0 = 0$,

$$L^k \leq 10^{-6}\frac{(1-L)}{|g(x_0)|}.$$

Taking logarithms on each side and dividing by $\log(L)$ (note that $\log(L) < 0$, which explains why the inequality is reversed!), we obtain the condition

$$k \geq \frac{1}{\log(L)} \log\Big(10^{-6}\frac{(1-L)}{|g(x_0)|}\Big).$$

Version I: Here we have $L = 0.9$ and $g(x_0) = 0.2$. This results in the estimate

$$k \geq \frac{\log 0.5 \cdot 10^{-6}}{\log 0.9} \approx 137.7.$$

That is, we need at most 138 iterations.

Version II: Here we have $L = 0.9$ and $g(x_0) = 0.7$. This results in the estimate

$$k \geq \frac{\log((1/7) \cdot 10^{-6})}{\log 0.9} \approx 149.6.$$

That is, we need at most 150 iterations.

Version III: Here we have $L = 0.8$ and $g(x_0) = 0.4$. This results in the estimate

$$k \geq \frac{\log 0.5 \cdot 10^{-6}}{\log 0.8} \approx 65.01.$$

That is, we need at most 66 iterations.

Version IV: Here we have $L = 0.8$ and $g(x_0) = 0.8$. This results in the estimate

$$k \geq \frac{\log 0.25 \cdot 10^{-6}}{\log 0.8} \approx 68.1.$$

That is, we need at most 69 iterations.

Alternatives:

Replacing $|g(x_0) - x_0|$ by 1 (which is the length of the interval) in the a–priori error estimate yields also a valid solution, though the estimate is weaker. For versions I and II, this results in 153 iterations; for versions III and IV in 70 iterations.

Finally, an alternative is the estimate

$$|x_k - r| \leq L^k |x_0 - r| \leq L^k,$$

which yields the estimate

$$k \geq \log(10^(- 6))/\log(L).$$

For versions I and II, this yields at most 132 iterations; for versions III and IV, at most 62 iterations.

Problem 7 [10 points]

Consider the data points

x_i	-2	-1	1	2
$f(x_i)$	-5	0	1	4

a) Use Lagrange interpolation to find the polynomial of minimal degree interpolating these points. Express the polynomial in the form

$$p_n(x) = a_n x^n + \cdots + a_1 x + a_0.$$

b) Determine the Newton form of the interpolating polynomial.

c) Verify that the solutions in (a) and (b) are the same.

d) Use your result to find an approximation to $f(0)$.

Solution:

a) With $x_0 = -2$, $x_1 = -1$, $x_2 = 1$, $x_3 = 2$, we get the following cardinal functions:

$$\ell_0(x) = \prod_{j=1}^{3} \frac{x - x_j}{x_0 - x_j} = \frac{(x+1)(x-1)(x-2)}{(-2-(-1))(-2-1)(-2-2)} = \frac{x^3 - 2x^2 - x + 2}{-12}$$

$$\ell_2(x) = \prod_{j \neq 2} \frac{x - x_j}{x_2 - x_j} = \frac{(x+2)(x+1)(x-2)}{(1+2)(1+1)(1-2)} = \frac{x^3 + x^2 - 4x - 4}{-6}$$

$$\ell_3(x) = \prod_{j \neq 3} \frac{x - x_j}{x_3 - x_j} = \frac{(x+2)(x+1)(x-1)}{(2+2)(2+1)(2-1)} = \frac{x^3 + 2x^2 - x - 2}{12}$$

There is no need to compute $\ell_1(x)$ because the function value is zero.

The interpolating polynomial in Lagrange form is

$$\begin{aligned} p(x) &= -5\ell_0(x) + \ell_2(x) + 4\ell_3(x) \\ &= \left(\tfrac{5}{12}x^3 - \tfrac{10}{12}x^2 - \tfrac{5}{12}x + \tfrac{10}{12}\right) + \left(\frac{x^3 + x^2 - 4x - 4}{-6}\right) + \left(\tfrac{4}{12}x^3 + \tfrac{8}{12}x^2 - \tfrac{4}{12}x - \tfrac{8}{12}\right) \\ &= \frac{7}{12}x^3 - \frac{1}{3}x^2 - \frac{1}{12}x + \frac{10}{12} \end{aligned}$$

b) Newton form:

$$\begin{array}{ccccc}
-2 & \boxed{-5} & & & \\
 & & \boxed{5} & & \\
-1 & 0 & & \boxed{-\frac{3}{2}} & \\
 & & \frac{1}{2} & & \boxed{\frac{7}{12}} \\
1 & 1 & & \frac{5}{6} & \\
 & & 3 & & \\
2 & 4 & & &
\end{array}$$

Polynomial:

$$p(x) = -5 + (x-(-2))\left(5 + (x-(-1))\left(-\tfrac{3}{2} + \tfrac{7}{12}(x-1)\right)\right)$$

c) Simplify one of the forms, for example:

$$\begin{aligned}
p(x) &= -5 + (x-(-2))\left(5 + (x-(-1))\left(-\tfrac{3}{2} + \tfrac{7}{12}(x-1)\right)\right) \\
&= -5 + (x-(-2))\left(5 + (x-(-1))\left(-\tfrac{3}{2} + \tfrac{7}{12}x - \tfrac{7}{12}\right)\right) \\
&= -5 + (x-(-2))\left(5 + (x-(-1))\left(-\tfrac{25}{12} + \tfrac{7}{12}x\right)\right) \\
&= -5 + (x-(-2))\left(5 + (x+1)\left(-\tfrac{25}{12} + \tfrac{7}{12}x\right)\right) \\
&= -5 + (x-(-2))\left(5 + \left(-\tfrac{25}{12}x + \tfrac{7}{12}x^2 - \tfrac{25}{12} + \tfrac{7}{12}x\right)\right) \\
&= -5 + (x-(-2))\left(5 + \left(\tfrac{7}{12}x^2 - \tfrac{18}{12}x - \tfrac{25}{12}\right)\right) \\
&= -5 + (x-(-2))\left(\tfrac{60}{12} + \left(\tfrac{7}{12}x^2 - \tfrac{18}{12}x - \tfrac{25}{12}\right)\right) \\
&= -5 + (x-(-2))\left(\tfrac{7}{12}x^2 - \tfrac{18}{12}x + \tfrac{35}{12}\right) \\
&= -5 + (x+2)\left(\tfrac{7}{12}x^2 - \tfrac{18}{12}x + \tfrac{35}{12}\right) \\
&= -5 + (\tfrac{7}{12}x^3 - \tfrac{18}{12}x^2 + \tfrac{35}{12}x + \tfrac{14}{12}x^2 - \tfrac{36}{12}x + \tfrac{70}{12}) \\
&= -\tfrac{60}{12} + (\tfrac{7}{12}x^3 - \tfrac{1}{3}x^2 - \tfrac{1}{12}x + \tfrac{70}{12}) \\
&= \tfrac{7}{12}x^3 - \tfrac{1}{3}x^2 - \tfrac{1}{12}x + \tfrac{10}{12}
\end{aligned}$$

d) Use the expanded form to get $f(0) \approx p(0) = \frac{7}{12}\cdot 0^3 - \frac{1}{3}\cdot 0^2 - \frac{1}{12}\cdot 0 + \frac{10}{12} = \frac{10}{12}$.

Problem 8 [5 points]

Let

$$f(x) = \begin{cases} 1/(x+1)^2, & \text{if } x > 0, \\ x+1, & \text{if } x \le 0. \end{cases}$$

Find an approximation to $\int_{-1}^{1} f(x)dx$ using Simpson's rule, and compute the error.

Solution:

Approximative solution:

$$\begin{aligned} \int_{-1}^{1} f(t)dt &\approx \frac{1}{3}\left(f(-1) + 4f(0) + f(1)\right) \\ &= \frac{1}{3}\left(0 + 4 \cdot 1 + \frac{1}{4}\right) \\ &= \frac{4}{3} + \frac{1}{12} \\ &= \frac{17}{12} \end{aligned}$$

Exact solution:

$$\begin{aligned} \int_{-1}^{1} f(t)dt &= \int_{-1}^{0} f(t)dt + \int_{0}^{1} f(t)dt \\ &= \int_{-1}^{0} (x+1)dt + \int_{0}^{1} 1/(x+1)^2 dt \\ &= \left[\frac{1}{2}x^2 + x\right]_{-1}^{0} + \left[-\frac{1}{x+1}\right]_{0}^{1} \\ &= \left(0 - \left(-\frac{1}{2}\right)\right) + \left(-\frac{1}{2} - (-1)\right) \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1 \end{aligned}$$

Error: $e = \frac{17}{12} - 1 = \frac{5}{12}$

Problem 9 [8 points]

There are four versions of this exercise, each with a different RK-method.

We are given the following python code, in which one step of a Runge–Kutta method is implemented.

Version I:

```
def onestep(f, x, y, h):
    k1 = f(x, y)
    k2 = f(x+h/4, y+h*k1/4)
    k3 = f(x+h, y+h*(k1+k2)/2)
    y_next = y + h*(2*k2/3+k3/3)
    x_next = x + h
    return x_next, y_next
```

Version II:

```
def onestep(f, x, y, h):
    k1 = f(x, y)
    k2 = f(x+h/2, y+h*k1/2)
    k3 = f(x+h, y+h*(k1+k2)/2)
    y_next = y + h*(k1/3+k2/3+k3/3)
    x_next = x + h
    return x_next, y_next
```

Version III:

```
def onestep(f, x, y, h):
    k1 = f(x, y)
    k2 = f(x+2*h/3, y+2*h*k1/3)
    k3 = f(x+h, y+h*(k1+k2)/2)
    y_next = y + h*(5*k1/12+k2/4+k3/3)
    x_next = x + h
    return x_next, y_next
```

Version IV:

```
def onestep(f, x, y, h):
    k1 = f(x, y)
    k2 = f(x+3*h/4, y+3*h*k1/4)
    k3 = f(x+h, y+h*(k1+k2)/2)
    y_next = y + h*(4*k1/9+2*k2/9+k3/3)
    x_next = x + h
    return x_next, y_next
```

Write down the Butcher tableau of the method, and determine the method's order.

Solution:

The Butcher tableaux are:

Version I

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/4 & 1/4 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 \\ \hline & 0 & 2/3 & 1/3 \end{array}$$

Version II

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 \\ \hline & 1/3 & 1/3 & 1/3 \end{array}$$

Version III

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 2/3 & 2/3 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 \\ \hline & 5/12 & 1/4 & 1/3 \end{array}$$

Version IV

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 3/4 & 3/4 & 0 & 0 \\ 1 & 1/2 & 1/2 & 0 \\ \hline & 4/9 & 2/9 & 1/3 \end{array}$$

The order conditions are:

Version I:

- $p = 1$: The condition $\sum_i b_i = 1$ yields: $0 + 2/3 + 1/3 = 1$, which is satisfied.
- $p = 2$: The condition $\sum_i b_i c_i = 1/2$ yields: $0 + \frac{2}{3}\frac{1}{4} + \frac{1}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$, which is satisfied.
- $p = 3$: The condition $\sum_i b_i c_i^2 = 1/3$ yields: $0 + \frac{2}{3}\frac{1}{16} + \frac{1}{3} \neq \frac{1}{3}$.

Version II:

- $p = 1$: The condition $\sum_i b_i = 1$ yields: $1/3 + 1/3 + 1/3 = 1$, which is satisfied.
- $p = 2$: The condition $\sum_i b_i c_i = 1/2$ yields: $0 + \frac{1}{3}\frac{1}{2} + \frac{1}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$, which is satisfied.
- $p = 3$: The condition $\sum_i b_i c_i^2 = 1/3$ yields: $0 + \frac{1}{3}\frac{1}{4} + \frac{1}{3} \neq \frac{1}{3}$.

Version III:

- $p = 1$: The condition $\sum_i b_i = 1$ yields: $5/12 + 1/4 + 1/3 = 1$, which is satisfied.
- $p = 2$: The condition $\sum_i b_i c_i = 1/2$ yields: $0 + \frac{1}{4}\frac{2}{3} + \frac{1}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$, which is satisfied.
- $p = 3$: The condition $\sum_i b_i c_i^2 = 1/3$ yields: $0 + \frac{1}{4}\frac{4}{9} + \frac{1}{3} \neq \frac{1}{3}$.

Version IV:

- $p = 1$: The condition $\sum_i b_i = 1$ yields: $4/9 + 2/9 + 1/3 = 1$, which is satisfied.
- $p = 2$: The condition $\sum_i b_i c_i = 1/2$ yields: $0 + \frac{2}{9}\frac{3}{4} + \frac{1}{3} = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$, which is satisfied.
- $p = 3$: The condition $\sum_i b_i c_i^2 = 1/3$ yields: $0 + \frac{2}{9}\frac{9}{16} + \frac{1}{3} \neq \frac{1}{3}$.

Thus all the methods are of order 2.

Problem 10 [12 points]

We consider the time-dependent PDE

$$u_t = u_{xx} + xu_x$$

with initial conditions

$$u(x,0) = x^2 \qquad \text{for } 0 < x < 1$$

and boundary conditions

$$u(0,t) = t \quad \text{and } u(1,t) = 1 \qquad \text{for } t > 0.$$

a) Perform a semi-discretisation of the PDE using central differences for the approximations of the x-derivatives. Use equidistant grid points $x_i = i\Delta x$ with a grid size $\Delta x = 1/M$.

b) We now want to use the trapezoidal rule for ODEs in order to compute a numerical solution of the system obtained in part **a)**. Set up the linear system that has to be solved in each step for an arbitrary time step $\Delta t > 0$.

Set up specifically the system for $M = 2$ and $\Delta t = 1/2$, and compute a numerical approximation of $u(1/2, 1)$.

Solution:

a) We start with choosing $M \in \mathbb{N}$ and setting $\Delta x = 1/M$ and $x_i = i\Delta x$ for $i = 0, \dots, M$. Using central differences for the approximation of the derivatives on the right hand side, we then obtain at the interior grid points the equations

$$\frac{\partial u}{\partial t}(x_i,t) = \frac{u(x_i - \Delta x, t) - 2u(x_i,t) + u(x_i + \Delta x, t)}{\Delta x^2} + x_i \frac{u(x_i + \Delta x, t) - u(x_i - \Delta x, t)}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

Approximating $U_i(t) \approx u(x_i,t)$ and ignoring the error term yields

$$U_i'(t) = \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{\Delta x^2} + x_i \frac{U_{i+1}(t) - U_{i-1}(t)}{2\Delta x}$$

for $i = 1, \dots, M-1$. In addition, we have the initial condition

$$U_i(0) = x_i^2 \qquad \text{for } i = 1, \dots, M-1,$$

and the boundary values

$$U_0(t) = t \qquad \text{and} \qquad U_M(t) = 1.$$

b) We now choose a time step Δt and approximate $U_i^n \approx u(x_i, t_n)$ with $t_n = n\Delta t$. Then the trapezoidal rule (or the Crank-Nicolson method) reads as

$$U_i^{n+1} = U_i^n + \frac{\Delta t}{2}\Big(\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{\Delta x^2} + x_i \frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x} + \frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{\Delta x^2} + x_i \frac{U_{i+1}^{n+1} - U_{i-1}^{n+1}}{2\Delta x}\Big)$$

for $i = 1, \dots, M-1$ and $n \ge 0$. In addition, we have

$$U_0^n = t_n = n\Delta t \qquad \text{and} \qquad U_M^n = 1,$$

and

$$U_i^0 = x_i^2 \qquad \text{for } i = 1, \dots, M-1.$$

For the specific case of $M = 2$ and $\Delta x = 1/2$, the only unknown is U_1^{n+1}. If we insert the boundary values $U_0^n = t_n$ and $U_2^n = 1$, as well as $x_1 = 1/2$, we end up with the equation

$$U_1^{n+1} = U_1^n + \frac{\Delta t}{2}\Big(4(t_n - 2U_1^n + 1) + \frac{1}{2}(1 - t_n) + 4(t_{n+1} - 2U_1^{n+1} + 1) + \frac{1}{2}(1 - t_{n+1})\Big).$$

This can be solved explicitly for U_1^{n+1} and we obtain

$$U_1^{n+1} = \frac{1}{1 + 4\Delta t}\Big((1 - 4\Delta t)U_1^n + \frac{9\Delta t}{2} + \frac{7\Delta t}{4}(t_n + t_{n+1})\Big)$$

For the specific case $\Delta t = 1/2$ and $t_n = n/2$, this results in

$$U_1^{n+1} = \frac{1}{3}\Big(-U_1^n + \frac{9}{4} + \frac{7}{8}\Big(n + \frac{1}{2}\Big)\Big) = \frac{1}{3}\Big(-U_1^n + \frac{43}{16} + \frac{7n}{8}\Big).$$

With the initial value $U_1^0 = 1/4$, we thus obtain that

$$U_1^1 = \frac{1}{3}\Big(-\frac{1}{4} + \frac{43}{16}\Big) = \frac{13}{16}$$

and

$$U_1^2 = \frac{1}{3}\Big(-\frac{13}{16} + \frac{43}{16} + \frac{7}{8}\Big) = \frac{7}{8}.$$