Problem 1 [15 points]

There are four versions of this exercise with different constants $c = -1, -2, 2, 3$.

The function f is given as

$$f(t) = \begin{cases} 1, & 0 \leq t \leq 1, \\ c, & t > 1. \end{cases} \tag{1}$$

a) Compute the Laplace transform of f.

b) Show that

$$\mathcal{L}\left(\int_0^t e^{-x} y(x) dx\right) = \frac{Y(s+1)}{s},$$

where $\mathcal{L}$ denotes the Laplace transform and $Y := \mathcal{L}(y)$.

c) Use the formula in part **b)** in order to find the solution $y(x)$ of

$$\int_0^t e^{-x} y(x)\, dx = f(t).$$

Here f is the function defined in (1).

Problem 2 [5 points]

There are four versions of this exercise with different constants $\alpha = -1, -2, 2, 3$.

Find the complex Fourier series of the function

$$f(x) = \begin{cases} e^{ix} + \alpha, & 0 \leq x \leq \pi, \\ e^{ix} - 1, & -\pi \leq x < 0. \end{cases}$$

Problem 3 [10 points]

Use the convolution theorem for the Fourier transform in order to find the function f that solves the equation

$$\int_{-\infty}^{+\infty} e^{-(x-t)^2} f(t)\, dt = \sqrt{2\pi}\, x e^{-x^2/2}.$$

Problem 4 *TMA4135 Mathematics 4D:* [5 points]

Let f, g be two smooth functions and let $c > 0$ be a constant. Show that the function

$$u(x,t) := f(cx+t) + g(2cx-2t) + \sin cx \cos t$$

satisfies the wave equation $u_{xx} = c^2 u_{tt}$.

Problem 4 *TMA4130 Mathematics 4N:* [5 points]

Show that the Fourier transform of

$$f(x) = \begin{cases} x^2, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

is the function

$$\hat{f}(w) = \begin{cases} \frac{1}{3} \cdot \sqrt{\frac{2}{\pi}}, & w = 0, \\ \sqrt{\frac{2}{\pi}} \left(\frac{\sin w}{w} + \frac{2\cos w}{w^2} - \frac{2\sin w}{w^3} \right), & w \neq 0, \end{cases}$$

Problem 5 [20 points]

Consider the heat equation

$$u_t = c^2 u_{xx} + \alpha \tag{2}$$

where $c > 0$ and $\alpha \in \mathbb{R}$ are given constants.

a) Show that the function

$$w(x,t) = \frac{\alpha}{2c^2} x(\pi - x)$$

satisfies the equation $w_t = c^2 w_{xx} + \alpha$.

b) Find the solution of equation (2) for $x \in [0, \pi]$ and $t > 0$ with the boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad t > 0,$$

and the initial condition

$$u(x,0) = \frac{\alpha}{2c^2} x(\pi - x) + \begin{cases} 0, & 0 \leq x < \frac{\pi}{2}, \\ x - \pi, & \frac{\pi}{2} < x \leq \pi. \end{cases}$$

c) Find $\lim_{t\to\infty} u(x,t)$.

Problem 6 [10 points]

There are four versions of this exercise with slightly different assumptions.

We are given a continuously differentiable function $g\colon [0,1] \to \mathbb{R}$ with the following properties:

Version I:

- $g(0) = 0.2$ and $g(1) = 0.7$.
- $0.1 \le g'(x) \le 0.9$ for all $0 \le x \le 1$.

Version II:

- $g(0) = 0.7$ and $g(1) = 0.2$.
- $-0.9 \le g'(x) \le -0.1$ for all $0 \le x \le 1$.

Version III:

- $g(0) = 0.4$ and $g(1) = 0.8$.
- $0.2 \le g'(x) \le 0.8$ for all $0 \le x \le 1$.

Version IV:

- $g(0) = 0.8$ and $g(1) = 0.4$.
- $-0.8 \le g'(x) \le -0.2$ for all $0 \le x \le 1$.

The remaining part of the exercise is the same for all versions.

We consider now the fixed point iteration

$$x_{k+1} = g(x_k)$$

with $x_0 = 0$.

a) Show that the function g has a unique fixed point r in the interval $[0,1]$ and that the fixed point iteration converges to r.

b) Provide an upper bound for the number of iterations that are required until $|x_k - r| \le 10^{-6}$.

Problem 7 [10 points]

Consider the data points

x_i	-2	-1	1	2
$f(x_i)$	-5	0	1	4

a) Use Lagrange interpolation to find the polynomial of minimal degree interpolating these points. Express the polynomial in the form

$$p_n(x) = a_n x^n + \cdots + a_1 x + a_0.$$

b) Determine the Newton form of the interpolating polynomial.

c) Verify that the solutions in (a) and (b) are the same.

d) Use your result to find an approximation to $f(0)$.

Problem 8 [5 points]

Let

$$f(x) = \begin{cases} 1/(x+1)^2, & \text{if } x > 0, \\ x+1, & \text{if } x \leq 0. \end{cases}$$

Find an approximation to $\int_{-1}^{1} f(x)dx$ using Simpson's rule, and compute the error.

Problem 9 [8 points]

There are four versions of this exercise, each with a different RK-method.

We are given the following python code, in which one step of a Runge–Kutta method is implemented.

Version I:

```
def onestep(f, x, y, h):
    k1 = f(x, y)
    k2 = f(x+h/4, y+h*k1/4)
    k3 = f(x+h, y+h*(k1+k2)/2)
    y_next = y + h*(2*k2/3+k3/3)
    x_next = x + h
    return x_next, y_next
```

Version II:

```
def onestep(f, x, y, h):
    k1 = f(x, y)
    k2 = f(x+h/2, y+h*k1/2)
    k3 = f(x+h, y+h*(k1+k2)/2)
    y_next = y + h*(k1/3+k2/3+k3/3)
    x_next = x + h
    return x_next, y_next
```

Version III:

```
def onestep(f, x, y, h):
    k1 = f(x, y)
    k2 = f(x+2*h/3, y+2*h*k1/3)
    k3 = f(x+h, y+h*(k1+k2)/2)
    y_next = y + h*(5*k1/12+k2/4+k3/3)
    x_next = x + h
    return x_next, y_next
```

Version IV:

```
def onestep(f, x, y, h):
    k1 = f(x, y)
    k2 = f(x+3*h/4, y+3*h*k1/4)
    k3 = f(x+h, y+h*(k1+k2)/2)
    y_next = y + h*(4*k1/9+2*k2/9+k3/3)
    x_next = x + h
    return x_next, y_next
```

Write down the Butcher tableau of the method, and determine the method's order.

Problem 10 [12 points]

We consider the time-dependent PDE

$$u_t = u_{xx} + x u_x$$

with initial conditions

$$u(x,0) = x^2 \qquad \text{for } 0 < x < 1$$

and boundary conditions

$$u(0,t) = t \quad \text{and } u(1,t) = 1 \qquad \text{for } t > 0.$$

a) Perform a semi-discretisation of the PDE using central differences for the approximations of the x-derivatives. Use equidistant grid points $x_i = i\Delta x$ with a grid size $\Delta x = 1/M$.

b) We now want to use the trapezoidal rule for ODEs in order to compute a numerical solution of the system obtained in part **a)**. Set up the linear system that has to be solved in each step for an arbitrary time step $\Delta t > 0$.

Set up specifically the system for $M = 2$ and $\Delta t = 1/2$, and compute a numerical approximation of $u(1/2, 1)$.