NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4125/30/35 Matematikk 4N/D**

Solution

Academic contact during examination:

Phone:

Examination date: 7. August, 2023

Examination time (from–to): 09:00–13:00

Permitted examination support material: C.
One sheet A4 paper, approved by the department (yellow sheet, “gul ark”) with own handwritten notes.
Certain simple calculators.

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- Good Luck! | Lykke til!

Language: English

Number of pages: 14

Number of pages enclosed: 0

Checked by:

Date Signature

Problem 1. (Fixed-point iterations)

Consider the nonlinear equation $x - \sqrt{\sin x} = 0$, with $x \in [\pi/6, \pi/2]$, for which we can write the fixed-point iteration as

$$x_{k+1} = \sqrt{\sin x_k}\,.$$

a) Show that the nonlinear equation has a unique solution r in the interval $[\pi/6, \pi/2]$, and that the fixed-point iterations above will converge to r for any initial guess x_0 in that interval.

b) Starting from $x_0 = \pi/2$, find an upper bound for the error $|x_{k+1} - r|$ after $k = 60$ iterations.
Important: you are *not* being asked to perform these iterations!

Solution.

a) We have the fixed-point equation $x = g(x)$, where $g(x) = \sqrt{\sin x}$. The fixed-point theorem, which guarantees the existence of the unique root r and also the convergence of the fixed-point iterations, depend on properties of the function $g(x)$. We need to verify the following conditions:

1) There exists a positive constant $L < 1$ so that $|g(x)'| \leq L$ for all $x \in [\pi/6, \pi/2]$
2) The function $g(x)$ stays within the interval of interest, that is: $g(x) \in [\pi/6, \pi/2]$ for all $x \in [\pi/6, \pi/2]$.

To check the first one, we must differentiate $g(x)$:

$$g(x) = (\sin x)^{\frac{1}{2}} \quad \Rightarrow g'(x) = \frac{1}{2}(\sin x)^{\frac{1}{2}-1}(\sin x)' = \frac{\cos x}{2\sqrt{\sin x}}\,.$$

Since $\cos x > 0$ for all $x \in [\pi/6, \pi/2]$, we have simply

$$|g'(x)| = g'(x) = \frac{\cos x}{2\sqrt{\sin x}}\,,$$

which is a **decreasing** function for $x \in [\pi/6, \pi/2]$, since the numerator is decreasing and the denominator is increasing (in the interval considered). Because $|g'(x)|$

is decreasing, we know that its maximum value in the interval $x \in [\pi/6, \pi/2]$ is simply $|g'(\pi/6)|$. Hence:

$$|g'(x)| \le |g'(\pi/6)| = \frac{\sqrt{3}/2}{2\sqrt{1/2}} = \frac{\sqrt{6}}{4} < 1\,.$$

The first condition of the theorem is therefore met, with $L = \sqrt{6}/4 \approx 0.612$.

Then, since $g(x)$ is clearly an **increasing function**, we know that its minimum and maximum values within the interval happen for $x = \pi/6$ and $x = \pi/2$, respectively:

$$g(\pi/6) \le g(x) \le g(\pi/2) \quad \Rightarrow \sqrt{2}/2 \le g(x) \le 1\,.$$

Having $g(x) \in [\sqrt{2}/2, 1]$ implies, in particular, $g(x) \in [\pi/6, \pi/2]$, since the interval $[\pi/6, \pi/2]$ contains $[\sqrt{2}/2, 1]$. The last condition is thus fulfilled, which shows that the fixed-point iterations will convergence to the root r, for any initial guess $x_0 \in [\pi/6, \pi/2]$.

b) As a consequence of the fixed-point theorem, we have the a-priori error estimate

$$|x_{k+1} - r| \le \frac{L^{k+1}}{1-L}|g(x_0) - x_0|.$$

Therefore, for $k = 60$ we have

$$|x_{61} - r| \le \frac{(\sqrt{6}/4)^{60+1}}{1-\sqrt{6}/4}\left|\sqrt{\sin \pi/2} - \pi/2\right| \approx 1.5 \times 10^{-13}\,.$$

If the student found the wrong constant L in item a), used this wrong value in b) but proceeded correctly, that is OK for item b). It is also OK if the student did the calculations considering $k + 1 = 60$ instead of $k = 60$.

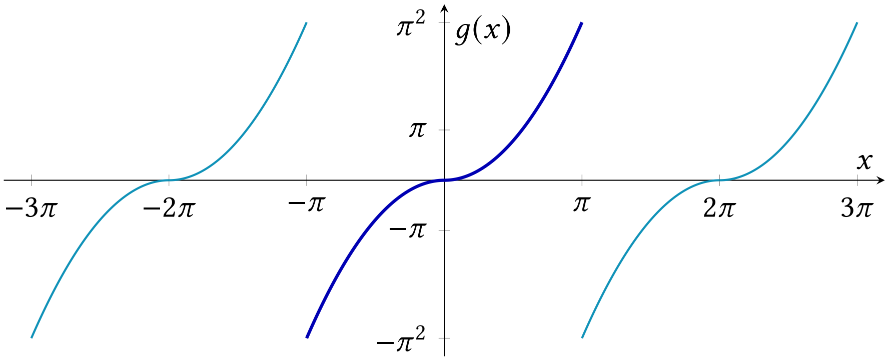
Problem 2. (Fourier Series)

The function $g(x) = x^2$ defined on the interval $[0, \pi]$ is to be extended to an odd function f with period 2π.

Sketch the function f on at least 3 periods and compute the coefficients of the real Fourier series of f.

Solution.

The sketch looks like (3 P.)



We can use that the odd extension is an odd function. Hence $a_n = 0$ for $n = 0, 1, 2, \ldots$. (1 P.)
For the b_n we use, that integrating over half an interval and multiplying that by 2 yields the result. Hence We get

$$b_n = \frac{2}{\pi}\int_0^\pi g(x)\sin(nx)\mathrm{d}x = \frac{2}{\pi}\int_0^\pi x^2\sin(nx)\mathrm{d}x$$

We apply integration by parts. We have in $\int fg'\mathrm{d}x = fg - \int f'g\mathrm{d}x$ here with $f(x) = x^2$ and $g'(x) = \sin(nx)$ so $f'(x) = 2x$ and $g(x) = -\frac{1}{n}\cos(nx)$ We obtain (2 P.)

$$b_n = \frac{2}{\pi}\Big(-\frac{x^2}{n}\cos(nx)\Big|_0^\pi + \int_0^\pi \frac{2x}{n}\cos(nx)\mathrm{d}x.\Big)$$

We perform another integration by parts on the second term with $f(x) = x$ and $g'(x) = \frac{1}{n}\cos(nx)$ and hence $f'(x) = 1$ and $g(x) = \frac{2}{n^2}\sin(nx)$. We obtain (2 P.)

$$b_n = \frac{2}{\pi}\Big(-\frac{\pi^2}{n}\cos(n\pi) + 0 + \frac{x}{n^2}\sin(nx)\Big|_0^\pi - \int_0^\pi \frac{2}{n^2}\sin(nx)\mathrm{d}x\Big)$$

We keep the first term, the central term vanishes since $\sin(n\pi) = \sin(0) = 0$ and we can compute the anti-derivative of the last integral (2 P.)

$$\begin{aligned} b_n = \frac{2}{\pi}\Big(-\frac{\pi^2}{n}\cos(n\pi) + \frac{2}{n^3}\cos(nx)\Big|_0^\pi\Big) &= \frac{2}{\pi}\Big(-\frac{\pi^2}{n}\cos(n\pi) + \frac{2}{n^3}\cos(n\pi) - \frac{2}{n^3}\Big) \\ &= \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}(-1)^n - \frac{4}{\pi n^3} \end{aligned}$$

The last simplification is not necessary to get the points for this computation.

Problem 3. (Discrete Fourier Transform)

In this task we consider the discrete Fourier Transform (DFT) for signals of length $n = 8$. We denote by $x_j = 2\pi j/8$, $j = 0, \ldots, 7$, corresponding sampling points on $[0, 2\pi)$.

a) We consider the function $f(x) = e^{3ix}$, $x \in [0, 2\pi)$, and its sampling values $f_j = f(x_j)$ at the points x_j from above.
Compute the discrete Fourier transform $\hat{\mathbf{f}}$ of the vector $\mathbf{f} = (f_0, \ldots, f_7)$.

b) Let $\hat{\mathbf{g}} = (0, i, 0, 0, 0, 0, 0, -i)$ be a result of a DFT. Is the original signal $\mathbf{g} = \mathcal{F}_8^{-1}\hat{\mathbf{g}}$ real-valued?

Solution.

a) We can easily read off the Fourier coefficients $c_k(f) = \begin{cases} 1 & \text{if } k = 3, \\ 0 & \text{else.} \end{cases}$ (1 P.)

Since f is bandlimited with $N = 3 < 4 = \frac{8}{2}$ the Fourier transform of length $n = 8$ is exact, i.e. $\hat{\mathbf{f}} = (c_0(f), c_1(f), c_2(f), c_3(f), c_{-4}(f), c_{-3}(f), c_{-2}(f), c_{-1}(f)) = (0, 0, 0, 1, 0, 0, 0, 0)$.

Alternatively the same argument can be done mentioning the Aliasing Lemma.(4 P.)

b) Similarly to a) we can associate the coefficients to the Fourier coefficients of a bandlimited function $g(x)$ with $c_k(g) = 0$ for $k \neq \pm 1$ and $c_{\pm 1}(g) = \pm\frac{1}{8}i$, where the factor $\frac{1}{8}$ already anticipates the inverse DFT.

This in turn yields that g is (up to a constant scaling) a sine function. To be precise the vector $\mathbf{g}$ is obtained by sampling $g(x) = \frac{1}{4}\sin(x)$.

Alternatively one can also compute all 8 terms (consisting of 2 summands each) by hand and argue, without even computing these exactly, that the exponentials cancel out w.r.t. their complex components. (5 P.)

The exact values of $\mathbf{g}$ are $\mathbf{g} = \frac{1}{4}\big(0, \frac{1}{\sqrt{2}}, 1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}\big)$

If the students are exact with their Fourier transform and use the unitary one, a factor of $\frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}}$ *is also ok. With the physical definition (the* $1/8$ *upfront the DFT instead of its inverse) even the factor* 2.

Problem 4. (Laplace transform)

Using the Laplace transform, solve the third-order ordinary differential equation

$$y''' - y' = 1,$$

with initial conditions $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 1$.

Solution.

Applying the Laplace transform to the ODE, we get

$$s^3 Y(s) - y''(0) - sY(s) = \frac{1}{s},$$

that is,

$$(s^3 - s)Y(s) = 1 + \frac{1}{s} = \frac{s+1}{s},$$

which gives us

$$Y(s) = \frac{s+1}{s^2(s^2-1)} = \frac{s+1}{s^2(s-1)(s+1)} = \frac{1}{s^2(s-1)}.$$

Decomposition of $Y(s)$ into partial fractions is done via

$$Y(s) = \frac{1}{s^2(s-1)} = \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s^2}.$$

To find the coefficients, we have to satisfy

$$1 = As^2 + Bs(s-1) + C(s-1) \quad \text{for all} \quad s.$$

In particular, setting $s = 1$ gives immediately $A = 1$, while using $s = 0$ results in $C = -1$, so that $B = -1$ (alternatively, one can solve a 3×3 linear system to find A, B, C, which of course will give the same values). Hence:

$$Y(s) = \frac{1}{s-1} - \frac{1}{s} - \frac{1}{s^2} \quad \Rightarrow \quad y(t) = e^t - 1 - t.$$

Problem 5. (Convolution)

Using the Laplace transform, solve the integro-differential equation

$$y'(t) - 5\int_0^t y(t-\tau)\cos\tau \, d\tau = 8\sin t,$$

with the initial condition $y(0) = 0$.

Solution.

This equation can be written as $y'(t) - 5y(t) * \cos t = 8\sin t$. Applying the Laplace transform to both sides, we get

$$[sY(s) - y(0)] - 5Y(s)\frac{s}{s^2+1} = \frac{8}{s^2+1}\,.$$

We therefore have

$$\left[s\left(1 - \frac{5}{s^2+1}\right)\right] Y(s) = \frac{8}{s^2+1} \quad \Rightarrow \quad s\left[\frac{s^2-4}{s^2+1}\right] Y(s) = \frac{8}{s^2+1}\,,$$

that is,

$$Y(s) = \frac{8}{s(s^2-4)} = \frac{8}{s(s-2)(s+2)}\,.$$

Decomposition of $Y(s)$ into partial fractions is done via

$$Y(s) = \frac{8}{(s-2)(s+2)s} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{s}\,.$$

To find the coefficients, we have to satisfy

$$8 = As(s+2) + Bs(s-2) + C(s-2)(s+2) \quad \text{for all} \quad s\,.$$

In particular, setting $s = 0$ gives immediately $C = -2$, while using $s = \pm 2$ results in $A = B = 1$ (alternatively, one can solve a 3×3 linear system to find A, B, C, which of course will give the same values). Hence:

$$Y(s) = \frac{1}{s-2} + \frac{1}{s+2} - \frac{2}{s} \quad \Rightarrow \quad y(t) = \mathrm{e}^{2t} + \mathrm{e}^{-2t} - 2 = 2(\cosh 2t - 1)\,.$$

Problem 6. (Understanding Code)

Consider the following Python code for a certain numerical method:

```python
import numpy as np

def f(x):
    return 2-2*x

def Method(f,a,b,N):
    x = np.linspace(a,b,N+1)
    S = 0
    for i in range(N):
        S = 0.5*( x[i+1] - x[i] )*( f(x[i+1]) + f(x[i]) )

    return S
```

If the method had been implemented correctly, running `Method(f,0,1,N)` should return an output equal to 1.0 regardless of the input `N`. However, there is a **mistake on one line of the code** that prevents this. In fact, running `Method(f,0,1,2)`, `Method(f,0,1,4)` and `Method(f,0,1,10)`, for example, will return 0.25, 0.0625 and 0.01, respectively.

a) Find the mistake and rewrite the incorrect line so as to have the correct implementation.

b) Once the mistake is fixed, what numerical method will be effectively implemented?

Solution.

a) The composite trapezoidal rule consists in doing

$$S = \sum_{i=0}^{N} \frac{f(x_{i+1}) + f(x_i)}{2}(x_{i+1} - x_i)\,,$$

but the implementation presented ignores the sum and only delivers the integral over the last sub-interval (x_i, x_{i+1}). We therefore have to correct line 10 to `S = S + .5*(x[i+1]-x[i])*(f(x[i+1])+f(x[i]))` or, even shorter, to `S += .5*(x[i+1]-x[i])*(f(x[i+1])+f(x[i]))`.

b) The (composite) trapezoidal rule.

Problem 7. (Separation of Variables)

Find all non-trivial solutions of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u \qquad \text{where } 0 < x < \pi \quad \text{and } t > 0$$

that are of the form $u(x,t) = F(x)G(t)$ and that satisfy the boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(\pi,t) = 0 \qquad \text{for } t > 0.$$

Solution.

We insert the equation $u(x,t) = F(x)G(t)$ into the PDE and obtain the equation

$$F(x)\dot{G}(t) = F''(x)G(t) + F(x)G(t).$$

Dividing by $G(t)$ and $F(x)$ yields the equation

$$\frac{\dot{G}(t)}{G(t)} = \frac{F''(x) + F(x)}{F(x)} = k,$$

where k is some constant. From this we obtain the two ODEs

$$\left.\begin{aligned} F'' &= (k-1)F \\ \dot{G} &= kG \end{aligned}\right\}$$

We consider now the possible solutions of the equation for F. Thus we have three possibilities:

$k > 1$: Denote $p = \sqrt{k-1} > 0$. Then we have the solution

$$F(x) = Ae^{px} + Be^{-px}.$$

From the boundary condition $F(0) = 0$ we get the condition $A = -B$, which implies that $F(x) = A(e^{px} - e^{-px})$. From the boundary condition $F(\pi) = 0$, we now obtain that

$$A(e^{p\pi} - e^{-p\pi}) = 0,$$

which is only possible if $A = 0$, as $e^{p\pi} > 1$ and $e^{-p\pi} < 1$. Thus we only end up with the trivial solution.

$k = 1$: Here we have the ODE $F'' = 0$, which has the general solution

$$F(x) = A + Bx.$$

From the boundary condition $F(0) = 0$ we get that $A = 0$ and thus $F(x) = Bx$. Now the condition $F(\pi) = 0$ implies that also $B = 0$. Thus we obtain, again, only trivial solutions.

$k < 1$: Denote $p = \sqrt{1-k} > 0$. Then we have the solution

$$F(x) = A\cos(px) + B\sin(px).$$

From the boundary condition $F(0) = 0$ we obtain that $A = 0$ and thus $F(x) = B\sin(px)$. Now the boundary condition $F(\pi) = 0$ implies that either $B = 0$ (which gives the trivial solution) or $\sin(p\pi) = 0$. The latter is satisfied if $p = n$ for some $n = 1, 2, \ldots$

We thus obtain the non-trivial solutions

$$F(x) = B\sin(nx) \qquad \text{for } n = 1, 2, \ldots$$

Since

$$k = 1 - p^2 = 1 - n^2,$$

the corresponding solution for G is

$$G(t) = Ce^{kt} = Ce^{(1-n^2)t}.$$

In total, we have the non-trivial solutions

$$u_n(x,t) = Ce^{(1-n^2)t}\sin(nx) \qquad \text{for } n = 1, 2, \ldots$$

Problem 8. (Solution to a PDE)

The equation

$$\frac{\partial^2 u}{\partial t^2} + 7u = \frac{\partial^2 u}{\partial x^2} \qquad \text{for } 0 < x < \pi \text{ and } t > 0$$

with boundary conditions

$$u(0,t) = 0 \qquad \text{and} \qquad u(\pi,t) = 0 \qquad \text{for } t > 0$$

has the general solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin(nx)\Big(A_n\cos\big(t\sqrt{7+n^2}\big) + B_n\sin\big(t\sqrt{7+n^2}\big)\Big).$$

(You don't have to show this!)

Use this information to find the solution that additionally satisfies the initial conditions

$$u(x,0) = \sin(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 2\sin(3x) \qquad \text{for } 0 < x < \pi.$$

Solution.

Inserting $t = 0$ into the general solution, we obtain that

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(nx).$$

Now the initial condition states that this should be equal to

$$u(x,0) = \sin(x).$$

By comparing the coefficients in front of the sine functions, we see that

$$A_1 = 1, \qquad \text{and } A_n = 0 \text{ else.}$$

Next we differentiate the general solution with respect to t, which gives

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} \sin(nx)\Big(-A_n\sqrt{7+n^2}\sin\big(t\sqrt{7+n^2}\big) + B_n\sqrt{7+n^2}\cos\big(t\sqrt{7+n^2}\big)\Big)$$

In particular, we obtain for $t = 0$ that

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} B_n\sqrt{7+n^2}\sin(nx).$$

The second initial condition states that this should be equal to

$$\frac{\partial u}{\partial t}(x,0) = 2\sin(3x).$$

Again, we have to compare the coefficients in front of the sine functions. For $n = 3$ we obtain the equation

$$B_3\sqrt{7+3^2} = 2,$$

which simplifies to

$$B_3 = \frac{1}{2},$$

and all other coefficients are equal to 0.

Thus the solution with these initial conditions reads

$$u(x,t) = \sin(x)\cos(\sqrt{8}t) + \frac{1}{2}\sin(3x)\sin(4t).$$

Problem 9. (Numerical Solution of PDEs)

Consider the heat equation

$$u_t(x,t) = \frac{1}{4}u_{xx}(x,t) \qquad \text{for } 0 \le x \le 1 \text{ and } t \ge 0,$$

with boundary conditions

$$u(0,t) = \cos(t), \quad u(1,t) = 0, \qquad \text{for } t \ge 0,$$

and initial condition

$$u(x,0) = 1 - x \qquad \text{for } 0 \le x \le 1.$$

Set up an explicit finite difference scheme for this equation. Use the step length $h = 0.25$ in the spatial direction, and the step length $k = 0.1$ in the temporal direction.

Use your finite difference scheme in order to find an approximation to $u(0.25, 0.2)$.

Solution.

- *Discretisation of the domain and equation:*

  We start by defining the grid points $x_i = ih = 0.25i$, $i = 0, \ldots, 4$, and $t_n = nk = 0.1n$, $n = 0, 1, 2, \ldots$

  Next we use that

$$u_t(x_i,t_n) = \frac{u(x_i,t_{n+1}) - u(x_i,t_n)}{k} + O(k),$$
$$u_{xx}(x_i,t_n) = \frac{u(x_{i+1},t_n) - 2u(x_i,t_n) + u(x_{i-1},t_n)}{h^2} + O(h^2).$$

  We now ignore the error terms, approximate $U_i^n \approx u(x_i,t_n)$, and insert the finite difference approximations in the PDE. Then we obtain the equation

$$\frac{U_i^{n+1} - U_i^n}{k} = \frac{1}{4}\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{h^2}.$$

  Solving this for U_i^{n+1}, we obtain the expressions

$$U_i^{n+1} = U_i^n + \frac{k}{4h^2}\left(U_{i-1}^n - 2U_i^n + U_{i+1}^n\right) = 0.2U_i^n + 0.4(U_{i-1}^n + U_{i+1}^n) \tag{1}$$

  for $i = 1, 2, 3$, and $n = 1, 2, 3, \ldots$, where we have used that $k/4h^2 = 0.1/0.25 = 0.4$.

- *Boundary conditions:*

  For $i = 1$ and $i = 3$, the expressions above depend on U_0^n and U_4^n, which we need to define using the boundary conditions $u(0,t) = \cos(t)$ and $u(1,t) = 0$. From these we obtain that

  $$U_0^n = \cos(t_n) = \cos(0.1n) \qquad \text{and} \qquad U_4^n = 0,$$

  which we use in (1)

- *Initial conditions:*

  For $n = 0$ we make use of the initial condition $u(x,0) = 1 - x$, which yields

  $$U_i^0 = 1 - x_i = 1 - 0.25i \qquad \text{for } i = 0, \dots, 4.$$

- *Complete algorithm:*

  Summarising everything above, we obtain the following method:

  Initialise: Define

  $$U_i^0 = 1 - 0.25i \qquad \text{for } i = 1, 2, 3.$$

  Iteration: For $n = 1, 2, \dots$ define

  $$U_0^n = \cos(0.1n) \qquad \text{and} \qquad U_4^n = 0,$$

  and compute

  $$U_i^{n+1} = 0.2U_i^n + 0.4(U_{i-1}^n + U_{i+1}^n) \qquad \text{for } i = 1, 2, 3.$$

Finally, we use this method for approximating $u(0.25, 0.2)$. We are using a step length $h = 0.25$ in spatial direction and a step length $k = 0.1$ in temporal direction, and thus $(0.25, 0.2) = (h, 2k) = (x_1, t_2)$. That is, we have the approximation $u(0.25, 0.2) \approx U_1^2$, which means that we have to compute U_1^2 with the algorithm defined above. We initialise

$$U_0^0 = 1, \quad U_1^0 = 0.75, \quad U_2^0 = 0.5, \quad U_3^0 = 0.25, \quad U_4^0 = 0.$$

Next we compute the values U_i^1 for $i = 1, 2, 3$ (we actually do not need U_3^1 for the computation of U_1^2):

$$\begin{aligned}
U_1^1 &= 0.2U_1^0 + 0.4(U_0^0 + U_2^0) = 0.75, \\
U_2^1 &= 0.2U_2^0 + 0.4(U_1^0 + U_3^0) = 0.5, \\
U_3^1 &= 0.2U_3^0 + 0.4(U_2^0 + U_4^0) = 0.25.
\end{aligned}$$

Finally, we set $U_0^1 = \cos(0.1)$ and compute

$$U_1^2 = 0.2U_1^1 + 0.4(U_0^1 + U_2^1) = 0.15 + 0.4\cos(0.1) + 0.2 \approx 0.748.$$

Thus

$$u(0.25, 0.2) \approx U_1^2 \approx 0.748.$$

Problem 10. (Numerical Methods for ODEs)

a) Rewrite the second order differential equation

$$u'' + 8u' + 7u = 0, \qquad u(0) = 1, \quad u'(0) = 1/2$$

as a system of first order differential equations.

b) Apply Euler's method to the system, and perform one step with step size $h = 0.1$.

c) What is the maximum step size h for which we can get a stable solution when Euler's method is applied to the system of ODEs from point a).

Solution.

a) Let $y_1 = u$ and $y_2 = u'$. The system becomes

$$\begin{aligned} y_1' &= y_2, & y_1(0) &= 1, \\ y_2' &= -7y_1 - 8y_2, & y_2(0) &= 1/2. \end{aligned}$$

b) Euler's method is given by

$$y_{n+1} = y_n + hf(t_n, y_n),$$

which in our case becomes

$$\begin{aligned} y_{1,1} &= 1 + 0.1 \cdot 0.5 = 1.05, \\ y_{2,1} &= 0.5 + 0.1 \cdot (-7 \cdot 1.0 - 8 \cdot 0.5) = -0.6. \end{aligned}$$

c) This is an issue of linear stability analysis.

The systems of equations can be written as

$$y' = Ay, \text{ with } A = \begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix}.$$

The matrix A has eigenvalues -1 and -7.

If the method is applied to the linear scalar test equation $y' = \lambda y$, where λ represents one of the eigenvalues of A, we get

$$y_{n+1} = R(z)y_n, \qquad R(z) = 1 + z \text{ with } z = \lambda h.$$

The numerical solution is stable if the step size h is chosen such that $|R(z)| \leq 1$, that is $-2 \leq z \leq 0$. For $\lambda = -1$ this means $h \leq 2$, for $\lambda = -7$, it gives $h \leq 2/7 = 0.2857\cdots$. Thus, stepsizes has to be chosen in the interval $0 < h \leq 2/7$ for the solution to be stable.