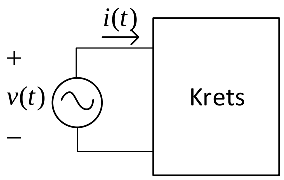
1 CIRCUIT ANALYSIS AND POWER CALCULATIONS

Consider the voltage and current waveforms taken on a single-phase circuit:



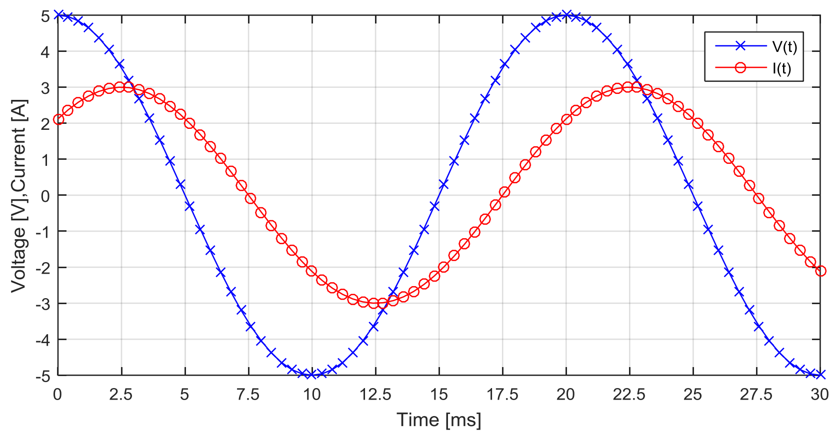


Figure 1: Time-domain waveforms of voltage and current in the circuit

a) Show that the phase angle difference between the voltage and current waveform is $\gamma = 45^o$

Solution: We observe in the figure that the time between peak voltage and peak current equals $\Delta t = 2.5\,ms$. Alternatively, we could take the time between two zero-crossings, or between two minimum values, Δt is the same in these cases.

We also note that the total period for the sinusoid is $T = 20\,ms$, which corresponds to a frequency of $f = \frac{1}{T} = 50\,Hz$.

We can then find the angle as the ratio between Δt and T multiplied with a full period of 360 degrees:

$$\gamma = \frac{2.5}{20} \cdot 360^o = 45^o$$

In the following problems we will also use radians: $\gamma = 45^o = \frac{\pi}{4}$ rad

b) Find an expression for the instantaneous power $p(t)$ consumed by the circuit. You don't need to simplify the expression

Solution: We first write the time-domain functions for voltage and current:

$$v(t) = 5\cos\left(2\pi \cdot 50t + 0\right) \qquad i(t) = 3\cos\left(2\pi \cdot 50t - \frac{\pi}{4}\right)$$

(Note the negative sign in the current phase angle, many students will do the mistake and use '+')
We can then find the instantaneous power directly as:

$$p(t) = v(t)i(t) = 5\cos\left(2\pi \cdot 50t + 0\right) \cdot 3\cos\left(2\pi \cdot 50t - \frac{\pi}{4}\right) = 15\cos\left(2\pi \cdot 50t\right)\cos\left(2\pi \cdot 50t - \frac{\pi}{4}\right)$$

We can probably simplify this expression by using trigonometric identities, but this was not asked for.

Students that write ω instead of $2\pi \cdot 50$ will not get full score.

c) Find the active power *P* and the reactive power *Q* **consumed** by the circuit *(as usual, both P and Q should be average values).* Is the circuit capacitive or inductive? Explain why.

Solution: This problem is recommended to be solved by phasor analysis. The current and voltage phasors are:

$$V = \frac{5}{\sqrt{2}} e^{j0} \quad I = \frac{3}{\sqrt{2}} e^{-j45}$$

NB: Here we have used RMS-quantities. Students that use peak values for the current and voltage phasors gets full score **<u>IF</u>** they divide by two in the power calculation. Otherwise they get point reduction.

The complex power is:

$$S = VI^* = \frac{15}{2} e^{j0} \left(e^{-j45}\right)^* = \frac{15}{2} e^{j45}$$

Finds the active and reactive power by splitting into real and imaginary parts:

$$P = \frac{15}{2}\cos(45) = \frac{15}{2\sqrt{2}} = \frac{15\sqrt{2}}{4} \approx 5.3033\, W$$

$$Q = \frac{15}{2}\sin(45) = \frac{15}{2\sqrt{2}} = \frac{15\sqrt{2}}{4} \approx 5.3033\, VAr$$

The circuit is ***<u>inductive</u>***, and we can reach this conclusion by observing either:

- The reactive power is positive
- The current waveform is *lagging* the voltage waveform

d) Find the equivalent impedance Z_{eq} of the circuit.

Solution: We can find the impedance by dividing the voltage phasor by the current phasor:

$$Z = \frac{V}{I} = \frac{\frac{5}{\sqrt{2}} e^{j0}}{\frac{3}{\sqrt{2}} e^{-j45}} = \frac{5}{3} e^{j45}$$

Here we would get the same answer if we used peak values instead of RMS-values.

2 Step responses of RC-circuit

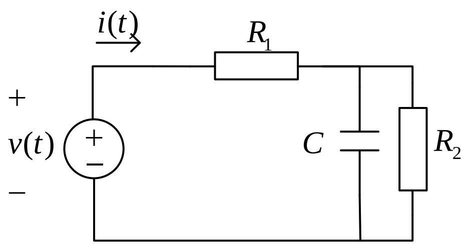
Consider the following circuit with numerical values:

$R_1 = 1\,\Omega$
$R_2 = 2\,\Omega$
$C = 1\,F$

(The capacitance value of *2 F* is unrealistically large, and is used for the purpose of simple expressions)

In problem a) and b) we assume that the voltage across the capacitor zero at $t = 0$

a) Show that the transfer function between current and voltage is $H(s) = \frac{V(s)}{I(s)} = \frac{2s+3}{2s+1}$



Solution: We have that:

$$V(s) = I(s)\cdot\left(R_1 + \frac{\frac{R_2}{sC}}{R_2 + \frac{1}{sC}}\right) = I(s)\cdot\left(R_1 + \frac{R_2}{1+sR_2C}\right)$$

$$H(s) = \left(R_1 + \frac{R_2}{1+sR_2C}\right) = 1 + \frac{2}{1+2s} = \frac{2s+3}{2s+1}$$

In problem b), we apply a voltage step at $t=0$, i.e.:

$$v(t) = \begin{cases} 0\,V & , \quad t<0 \\ 1\,V & , \quad t\geq 0 \end{cases}$$

b) Based on the transfer function from a), find $i(t)$ for $t \geq 0$

Solution: Want to solve this problem by Laplace analysis. The voltage $v(t)$ can be transformed into the Laplace-domain as a unit step:

$$V(s) = \frac{1}{s}$$

We can then find the Laplace transformed current based on the transfer function:

$$I(s) = \frac{V(s)}{H(s)} = \frac{1}{s}\cdot\frac{1}{\frac{2s+3}{2s+1}} = \frac{1}{s}\cdot\frac{1+2s}{3+2s}$$

We need to split this by partial fraction expansion:

$$I(s) = \frac{1}{s}\cdot\frac{1+2s}{3+2s} = \frac{A}{s} + \frac{B}{3+2s}$$

This can be done in many ways, for example by setting right hand side on common denominator:

$$\frac{1}{s}\cdot\frac{1+2s}{3+2s} = \frac{(3+2s)A + Bs}{s(3+2s)}$$

The solution is then given by:

$$2A + B = 2 \quad , \quad 3A = 1$$

$$\Rightarrow A = \frac{1}{3} \ , \ B = 2 - \frac{2}{3} = \frac{4}{3}$$

We then have:

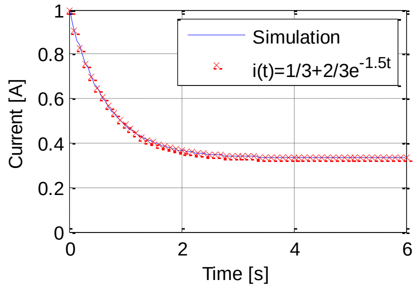
$$I(s)=\frac{1}{3}\cdot\frac{1}{s}+\frac{4}{3}\frac{1}{3+2s}=\frac{1}{3}\cdot\frac{1}{s}+\frac{2}{3}\cdot\frac{1}{\frac{3}{2}+s}$$

We can then write $i(t)$ as the inverse Laplace transform of $I(s)$:

$$i(t)=\frac{1}{3}u(t)+\frac{2}{3}e^{-\frac{3}{2}t}$$, where $u(t)$ is the unit-step function

NB: It is also equally correct to write simply $i(t)=\frac{1}{3}+\frac{2}{3}e^{-\frac{3}{2}t}$ since we were asked to find the expression for $t\geq 0$.

The waveform is plotted and validated against a simulation:



In problem c) we assume that $v(t)=100\cos(3t)$, and that the circuit has reached stationary/steady-state conditions.

c) Since the circuit operates in stationary conditions it is possible to write the current on the form $i(t)=A\cos(3t+\varphi)$. Find the numerical values for *A [Ampere]* and φ *[radians]*.

Solution:

This is really a phasor analysis problem since we know the voltage phasor and need to find the current phasor. The transfer function *H* represents the impedance.

Inserting numerical values for the transfer function, with $\omega=3$ rad/s:

$$H(j3)=1+\frac{2}{1+2\cdot j3}=1+\frac{2}{6.0828e^{j80.54}}=1+0.3288e^{-j80.54}=1.1028e^{-17.1}$$ after some complex arithmetic.

(Here we used the equivalent transfer function expression $H(s)=1+\frac{2}{1+2s}$)

We also have that the current in the frequency domain is equal to:

$$I(s)=\frac{V(s)}{H(s)}\Rightarrow I(j3)=\frac{V(j3)}{H(j3)}$$

Then, $I(j3) = \dfrac{100e^{j0}}{1.1028e^{-j17.1}} = 90.678e^{j17.1}$

The angle in radians is $17.1^o \cdot \dfrac{\pi}{180} = 0.299\ rad$

We can then write the time-domain current as

$i(t) = 90.678\cos(3t + 0.299)$

Hence, $A = 90.678$, $\varphi = 0.299$

3 Motors and Drive Systems

a) Figure 2 shows a typical inductor motor curve relating the mechanical speed ω_{mek} to the mechanical torque T_{mek}. Disregard mechanical gear and magnetic poles. Redraw the curve into your answer, and indicate the following items in the figure:

   I. Synchronous speed ω_s
   II. The speed at which the slip *s* equals 0.
   III. The speed at which the slip *s* equals 1.
   IV. The normal operating range
   V. The starting torque T_{start}

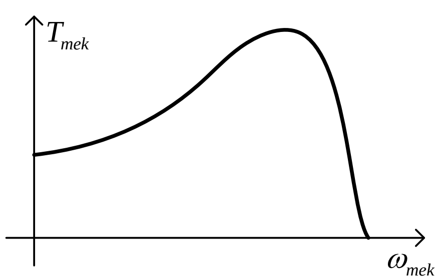
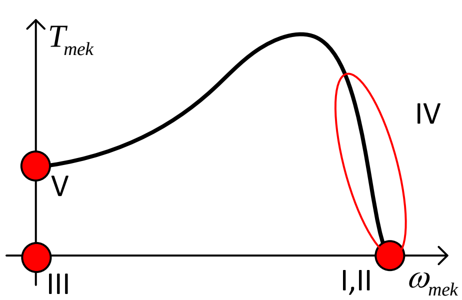


Figure 2: Induction motor mechanical torque versus mechanical speed

Solution: See the following figure. The synchronous speed equals the speed where the slip is 0, while the speed is zero when the slip is 1. Starting torque is the torque right after start-up of the machine, i.e. when the speed is 0. The nominal operating range includes approximately the line segment illustrated by the ellipse.



b) What is the torque of the machine when the mechanical speed equals synchronous speed?

Solution: The induction motor does not develop any torque at synchronous speed since the induced rotor field will rotate with the same speed as the stator field. The induced torque in an induction motor is always zero in this case.

Part 2: Wind turbine system

A wind turbine generator is directly connected to a power grid with fixed frequency $f_{grid} = 50\ Hz$ as shown in Figure 3. The system does <u>not</u> have any mechanical gear, but the generator is equipped with an unknown number of poles. ω_{mek} is the mechanical speed of the turbine and generator.

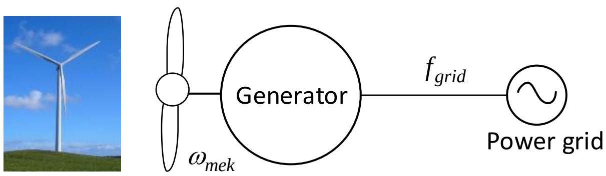


Figure 3: Wind turbine generator connected to a power grid

c) Assume $\omega_{mek} = 7.85$ rad/s . How many poles is the generator equipped with?

Solution: The number of poles can be found from the following relation:

$$f_{grid} = \frac{p}{2} f_{mek}$$

$$f_{grid} = \frac{p}{2} f_{mek}$$

$$p = 2\frac{f_{grid}}{f_{mek}} = 2\frac{f_{grid}}{\frac{\omega_{grid}}{2\pi}} = 4\pi\frac{f_{grid}}{\omega_{grid}} = 4\pi \cdot \frac{50}{7.85} = 80$$

Hence, the number of poles is 80.

In Figure 4 the relation between mechanical speed ω_{mek} and the produced power P_{wind} is presented for three different wind speeds $v = 6,8,10$ m/s .

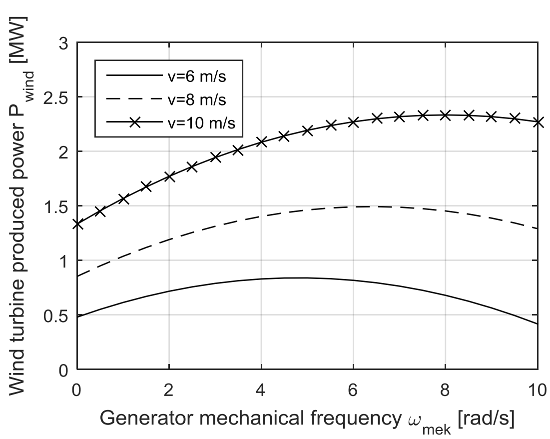


Figure 4: Relation between wind turbine mechanical speed ω_{mek} and produced power P_{wind}

d) Assume $\omega_{mek} = 7.85 \text{ rad/s}$ and wind speed $v = 6 \text{ m/s}$. Explain why the wind turbine configuration in Figure 3 is not optimal if the goal is to produce as much power as possible.

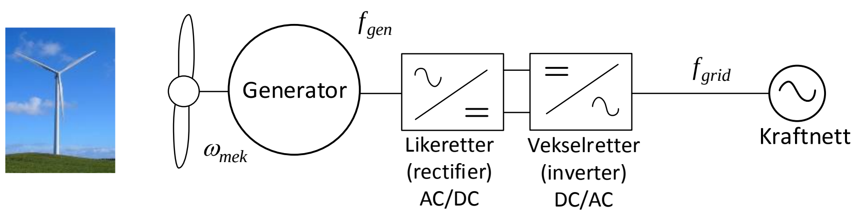
Solution: The system is not optimal since the generator is operating at fixed speed determined by the grid frequency. We can see from the figure that we can extract more power at $v = 6 \text{ m/s}$ if we reduce the mechanical speed to ~5 rad/s". We can also see that the optimal value for ω_{mek} depends on the wind speed. Therefore, a fixed-speed generator is not optimal.

NB: Using a different number of poles can bring us to the optimal point for $v = 6 \text{ m/s}$, but not for every wind speed. Students that propose this as a solution will therefore not get full score.

e) Sketch a wind turbine configuration that is able to extract the maximum available power at every wind speed. What kind of converter configuration should be used?

Solution:

An example sketch is the following:



To extract maximum available power, a variable-speed generator is needed. A variable-speed generator is equivalent to using a frequency converter between grid and generator. The student should mention "variable speed operation" or "frequency converter" or "AC-DC-AC converter" or "rectifier/inverter" etc. to get full score on the problem.

The sketch should include a rectifier and an inverter. It is not necessary to use these exact words, but is should be clarified that we need two converters, one from AC-to-DC, and one from DC-to-AC. With this solution, we can choose the generator frequency independent of the grid frequency.

Transformer and gearbox is not relevant, but students will not get minus points if they include them.

4 DC-MOTOR CONTROLLED BY A DC-DC CONVERTER

Consider the DC-motor controlled by a DC-DC converter in **Error! Reference source not found.**. The field winding is not shown in the figure, but it is assumed constant field voltage, hence the flux φ is constant.

We can use the average voltage V_a when calculating torque and speed of the DC-motor by using the DC-motor formulas (appendix). The term *no-load* is defined as the condition where the machine is producing zero torque, i.e. $T = 0$

a) Assume the average terminal voltage $V_a = 7.5V$, and $R_a = 0.1\Omega$. What is the no-load speed of the DC-motor when the flux constant is equal to $K\varphi = 0.1V\frac{s}{rad}$?

Solution: In no-load, when the torque $T=0$, the average armature current $I_a=0$ from the relation $T=K\varphi I_a$. Then, $E_a=V_a$. We also have $E_a=K\varphi\omega=0.1\omega$.

Since the duty cycle is $k=0.5$, the average voltage $V_a=kV_{dc}=0.5\cdot 15=7.5V$. Then we can find the no-load speed as:

$$\omega=\frac{7.5V}{0.1\frac{Vs}{rad}}=75\text{ rad/s}$$

The DC-DC converter consists of an *ideal* switch *S* and an *ideal* diode *D*, and is connected to a DC voltage source. The resulting voltage waveform for v_a is shown in **Error! Reference source not found.**. The variable *k* is called *duty-cycle*, and measures the time the switch is conducting divided by the entire period T_s:

$k=\frac{t_{on}}{T_s}$, where t_{on} is the time period where the switch is closed (ON).

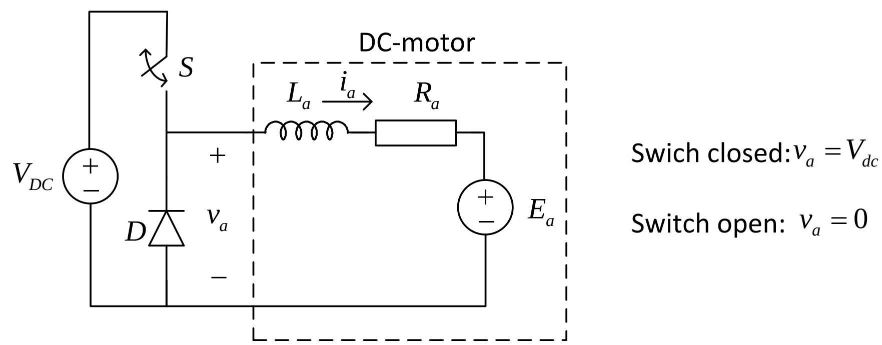


Figure 5: DC-motor controlled by a DC-DC converter

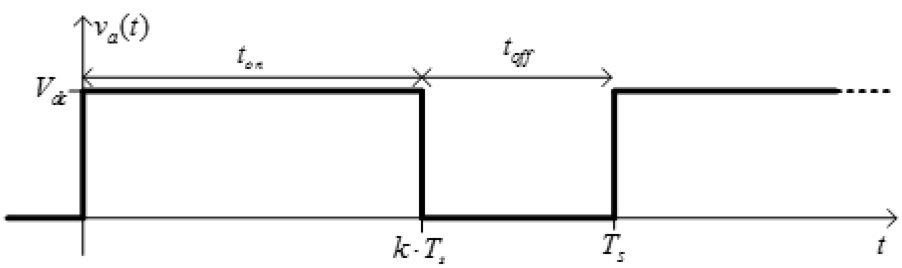


Figure 6: Waveform of the voltage applied to the DC-motor terminals (v_a)

b) Consider the waveform for $v_a(t)$ presented in **Error! Reference source not found.**. Find the expression of the average voltage $V_a=\frac{1}{T_s}\int_0^{T_s}v_a(t)dt$ as a function of the duty-cycle *k* and the DC-voltage V_{dc}

Solution: The integral is easy to solve if we view it as the area under the curve:

$$V_a = \frac{1}{T}\int_0^T v_a(t)dt = \frac{1}{T}\cdot\left(V_{dc}\cdot k\cdot T + 0\cdot(1-k)T\right) = k\cdot V_{dc}$$

In other words, the average DC-motor terminal voltage is proportional to the duty-cycle and to the DC-voltage.

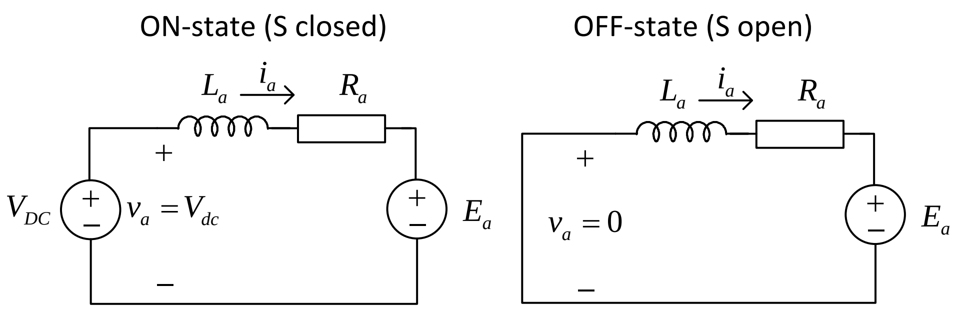
NB: *Note the difference between the duty cycle k and the flux constant K, and also the difference between torque T and period* T_s

c) Draw the two separate circuit diagrams corresponding to *closed switch* S and *open switch* S. Indicate the path of current flow in the two diagrams. Finally, show that the differential equation for i_a can be written as:

$$\frac{di_a}{dt} = \begin{cases} \frac{V_{dc} - R_a i_a - E_a}{L_a}, & \text{ON-state (S is closed)} \\ \frac{-R_a i_a - E_a}{L_a}, & \text{OFF-state (S is open)} \end{cases}$$

Hint: Assume i_a *is always positive, then the diode will always be on when the switch S is open.*

Solution: We first draw the circuit diagrams. With this way of drawing the circuit there is only one path for the current, indicated by the arrow. **NB:** Students can also draw other correct drawings where both the diode and switch is included, but must then include the correct current path.



One approach is to find an expression for the voltage across the inductor L_a in the two states.

<u>ON-state:</u> Since the switch is ON, the voltage $v_a = V_{dc}$. We can then write Kirchoffs voltage law for the DC-motor circuit:

$$-V_{dc} + L_a\frac{di_a}{dt} + R_a i_a + E_a = 0$$
$$\Rightarrow \frac{di_a}{dt} = \frac{V_{dc} - R_a i_a - E_a}{L_a}$$

<u>OFF-state:</u> When the switch is OFF, we have $v_a = 0$ since the diode is conducting the current. The Kirchoffs voltage law is then reduced to:

$$0 + L_a \frac{di_a}{dt} + R_a i_a + E_a = 0$$
$$\Rightarrow \frac{di_a}{dt} = \frac{-R_a i_a - E_a}{L_a}$$

Use the following numerical values in problem d):

$V_{dc} = 15\,V \quad R_a = 0\,\Omega \quad L_a = 1\,mH \quad T_s = 2\,ms \quad k = 0.5 \quad E_a = 7.5\,V$

d) Assume $i_a(0) = 7.5\,A$. Calculate $i_a(t)$ for $0 \le t \le T_s$. Make a sketch of the waveform.

NB: *Note that we have assumed* $R_a = 0$ *in order to simplify the calculations.*

Solution: We will solve this problem by solving the differential equations from c). They become considerably easier to solve when we assume $R_a = 0$:

ON-state ($0 < t \le kT$):

$$\frac{di_a}{dt} = \frac{V_{dc} - E_a}{L_a}$$
$$i_a(t) - i_a(0) = \int_0^t \frac{V_{dc} - E_a}{L_a} = \left[\frac{15 - 7.5}{0.001}\right]_0^t = 7500t$$
$$i_a(t) = 7500t + i_a(0) = 7500t + 7.5\,A \quad , \quad 0 < t \le kT$$

Note in particular the current at the end of the period: $i_a(kT) = 7.5 + 7500 \cdot 0.5 \cdot 0.002 = 15\,A$

OFF-state ($kT < t \le T$):

$$\frac{di_a}{dt} = \frac{-E_a}{L_a}$$
$$i_a(t) - i_a(kT) = \int_{kT}^{T} \frac{-E_a}{L_a} = \left[\frac{-7.5}{0.001}\right]_{kT}^{t} = -7500\left(t - kT\right)$$
$$i_a(t) = -7500\left(t - kT\right) + i_a(kT) = -7500\left(t - kT\right) + 15\,A \quad , \quad kT < t \le T$$

We can sketch the current as follows: