1 RLC CIRCUIT

The input voltage is $v_{in}(t)$, while the output voltage $v_o(t)$ is defined as the voltage across the LC-branch.

Assume $v_{in}(t) = 10\cos(\omega t)$ throughout the entire problem. $R = 1\,\Omega$ and $L = 1\,mH$, while *C* is unknown.

We will first assume this circuit is used as a filter with input $v_{in}(t)$ and output $v_o(t)$, and $H(s) = \dfrac{V_o(s)}{V_{in}(s)}$

a) What kind of filter does $H(s)$ represent? (lowpass, highpass, band-pass or band-stop)? Give reasons for your answer.

Answer: *This is a band-stop filter since* $|H(j\omega)|$ *is 1 at low frequencies (the capacitor is blocking current) and* $|H(j\omega)|$ *is 1 at high frequencies (the inductor is blocking current). At the LC resonance frequency* $\omega = \dfrac{1}{\sqrt{LC}}$, *the output is zero because the LC-branch is a short circuit.*

b) We want $v_o(t)$ to have as small amplitude as possible when the frequency is 250 Hz. Find the value of *C* that minimizes the amplitude of $v_o(t)$ at this frequency. What is the value of the minimum amplitude?

Answer*: We want the filter to have the resonance frequency at 250 Hz, e.g.*

$$\omega = \frac{1}{\sqrt{LC}} = 2\pi \cdot 250 \Rightarrow C = \frac{1}{L \cdot (2\pi \cdot 250)^2} = 405.3\,\mu\text{H}$$

We could also find this answer by looking at the transfer function $H(s) = \dfrac{sL + \frac{1}{sC}}{sL + \frac{1}{sC} + R}$ *and find*

$$H(\mathrm{j}\omega) = \frac{j\omega L - j\frac{1}{\omega C}}{j\omega L - j\frac{1}{\omega C} + R}.$$

The absolute values is $|H(\mathrm{j}\omega)| = \dfrac{\omega L - \frac{1}{\omega C}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$

From this expression we get minimum amplitude when the numerator is zero, e.g. $\omega L = \dfrac{1}{\omega C}$

Assume now the frequency is 50 Hz. If you did not solve *b),* use $C = 1\,mF$ in problems c) and d).

c) How much active and reactive power is supplied from the voltage source? Is the circuit inductive or capacitive (give reasons for your answer)?

Answer: *We find the total impedance seen from the source (at 50 Hz):*

$$Z_{tot} = 1 + j2\pi \cdot 50 \cdot 10^{-3} + \frac{1}{j2\pi \cdot 50 \cdot 405.3 \cdot 10^{-6}} = 1 - j7.539 = 7.606 e^{-j82.4^o}$$

The active and reactive power is then:

$$S = \frac{|V|^2}{Z_{tot}^*} = \frac{\left(\frac{10}{\sqrt{2}}\right)^2}{7.606 e^{j82.4^o}} = 0.864 - j6.517$$

$$P = 0.864 \text{ W}$$

$$Q = -6.517 \text{ VAr}$$

Positive reactive power means *<u>capacitive</u>. We can also claim the circuit is capacitive at this frequency since the impedance angle is negative.*

Note that we divided the voltage by $\sqrt{2}$ *to get the RMS value since 10 V is the peak magnitude of the cosine voltage source.*

Using C=1 mF:

$$Z_{tot} = 1 + j2\pi \cdot 50 \cdot 10^{-3} + \frac{1}{j2\pi \cdot 50 \cdot 1 \cdot 10^{-3}} = 1 - j2.8689 = 3.0382 e^{-j70.8^o}$$

The active and reactive power is then:

$$P = 5.417 \text{ W}$$

$$Q = -15.54 \text{ VAr}$$

Assume now we connect another resistor Z_{load} in parallel to the LC branch as shown in **Error! Reference source not found.**.

d) Find the value of Z_{load} that maximizes the active power consumption in Z_{load}.

Answer: This can be solved in several ways. The simplest (but still quite challenging) method is to apply the maximum power transfer theorem and set the impedance Z_{load} equal to the conjugate of the equivalent Thevenin impedance Z_{th} seen from the terminals of Z_{load}. Z_{th} is found as the parallel connection of R_1 with the series connection of L and C given by $X_{LC} = \omega L - \frac{1}{\omega C}$

$$Z_{th} = \frac{R_1 \cdot jX_{LC}}{R_1 + jX_{LC}} = \frac{R_1 X_{LC}^2 + jR_1^2 X_{LC}}{R_1^2 + X_{LC}^2} = \frac{R_1 X_{LC}^2}{R_1^2 + X_{LC}^2} + j\frac{R_1^2 X_{LC}}{R_1^2 + X_{LC}^2}$$

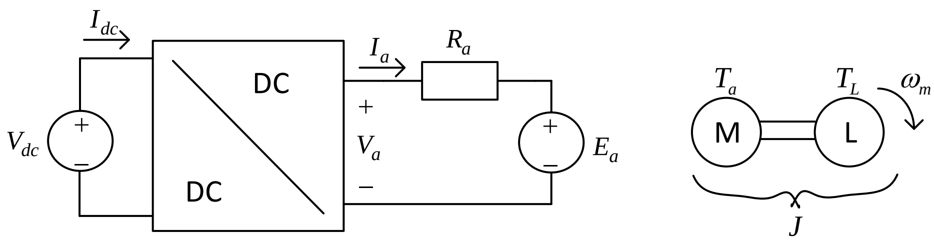
$$R_{th} = \frac{R_1 X_{LC}^2}{R_1^2 + X_{LC}^2}$$

$$X_{th} = \frac{R_1^2 X_{LC}}{R_1^2 + X_{LC}^2}$$

Consequently, $Z_{load} = Z_{th}^{*} = \frac{R_1 X_{LC}^2}{R_1^2 + X_{LC}^2} - j\frac{R_1^2 X_{LC}}{R_1^2 + X_{LC}^2}$ will maximize the active power consumption in Z_{load}.

With numerical values: $Z_{load} = 0.9827 - j0.1303$

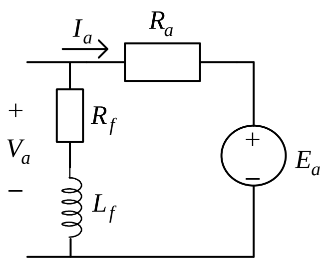
2 DC-MOTOR DRIVEN BY BATTERY AND DC/DC CONVERTER



The figure shows a battery with voltage V_{dc} connected to a DC-motor with resistance R_a and induced voltage E_a through a DC/DC-converter. The DC/DC-converter control system can set the required duty-cycle *D* defined as $D = \frac{V_a}{V_{dc}}$. The mechanical load has a load torque T_L .

a) The field winding is not shown in the figure. Make a complete schematic of a shunt-excited DC-machine, including the field winding resistance R_f and field winding inductance L_f .

Answer: Shunt excited means that the field winding is supplied from the same voltage as the armature winding, i.e. they are connected in parallel (shunt). The schematic is as follows:



Assume in b) and c) that $T_L = k_L \omega_m$, where ω_m is the rotational speed of the shaft. The system has no mechanical gearbox. In problems b) and c), assume the motor operates in steady-state, i.e. no acceleration. Some useful formulas for the DC-motor is included in appendix.

Numerical values: $R_a = 0.1\,\Omega$, $k\varphi = 0.3\,\frac{Vs}{rad}$, $k_L = 0.1\,\frac{Nms}{rad}$ and $V_{dc} = 100\,V$

b) What duty-cycle is needed in order to have a shaft speed of $\omega_m = 200$ rad/s ?

Answer: *As the motor operates in steady-state, we have* $T_L = T_a \Rightarrow k\varphi I_a = k_L \omega_m \Rightarrow I_a = \frac{0.1 \cdot 200}{0.3} = 66.67\ A$

The induced voltage is $E_a = 0.3 \cdot 200 = 60$ *V. The terminal voltage is then*

$V_t = E_a + R_a I_a = 60 + 0.1 \cdot 66.67 = 66.67\,V$. *Consequently, the duty cycle is* $D = \frac{66.67}{100} = 0.667$

c) Find an expression for the motor speed ω_m expressed by k_L, D, V_{dc}, k, R_a, as well as the motor constant *k* and the flux φ (these are the only variables in the desired expression)

Answer: *We have the following equations:*

$$V_a = DV_{dc} \quad , \quad V_a = R_a I_a + E_a \quad , \quad E_a = k\varphi\omega_m \quad , \quad T_a = k\varphi I_a \quad , \quad T_L = k_L\omega_m$$

We need to impose $T_a = T_L$

Solving yields: $R_a I_a + k\varphi\omega_m = DV_{dc} \quad , \quad I_a = \dfrac{k_L\omega_m}{k\varphi}$

$R_a \dfrac{k_L\omega_m}{k\varphi} + k\varphi\omega_m = DV_{dc}$ *and we get* $\omega_m = \dfrac{DV_{dc}}{k\varphi + \dfrac{R_a k_L}{k\varphi}}$

Assume now that the load torque is zero, $T_L = 0$*. Assume also the system has a total moment of inertia equal to* J*. At* $t = 0$*, the DC/DC-converter makes a step in duty cycle from* $D = 0$ *to* $D = D_0$*.*

d) Find the expression for the motor speed $\omega_m(t)$ for $t \geq 0$ (only variables, no numerical values).

Answer: *We want to solve this with Laplace transform (can also use differential equations)*

We need to add the equation for angular acceleration:

$sJ\omega_m = T_a - T_L$*, with* $T_L = 0$ *we get* $sJ\omega_m = k\varphi I_a$

Circuit equation: $V_a = R_a I_a + E_a \Rightarrow DV_{dc} = R_a \dfrac{sJ\omega_m}{k\varphi} + k\varphi\omega_m$

We find the transfer function from Duty cycle D to speed ω_m*:*

$$H(s) = \frac{\omega_m(s)}{D(s)} = \frac{V_{dc}}{s\dfrac{R_a J}{k\varphi} + k\varphi} = \frac{V_{dc}k\varphi}{R_a J}\frac{1}{s + \dfrac{(k\varphi)^2}{R_a J}} = K\frac{1}{s+a}$$

Where $K = \dfrac{V_{dc}k\varphi}{R_a J} \quad , \quad a = \dfrac{(k\varphi)^2}{R_a J}$ *is introduced for simpler expressions.*

We are making a step in duty cycle, hence $D(s) = \dfrac{D_0}{s}$ *. This gives the following expression for speed:*

$$\omega_m(s) = KD_0\frac{1}{s+a}\frac{1}{s}$$

Performing partial fraction expansion yields: $\omega_m(s) = KD_0\dfrac{1}{s+a}\dfrac{1}{s} = \dfrac{KD_0}{a}\left(\dfrac{1}{s} - \dfrac{1}{s+a}\right)$

And we can make the inverse transform:

$$\omega_m(t) = \frac{KD_0}{a}\left(1 - e^{-at}\right) = \frac{V_{dc}}{k\varphi}D_0\left(1 - e^{-\frac{(k\varphi)^2}{R_a J}t}\right)$$

3 CIRCUIT CALCULATIONS INCLUDING IRON CORE

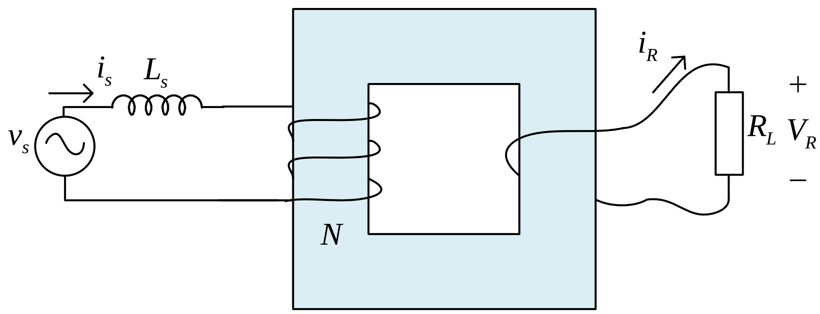


Figure 1 – Voltage source, iron core and load resistance

A resistor is connected to a voltage source through an iron core as presented in Figure 1. The core has $N = 3$ windings on the voltage source side, and a single winding at the load resistor side. We make the following assumptions:

- All induced magnetic field is kept within the core
- The iron is ideal, i.e. $\mu_r = \infty$
- The core will not saturate

a) Explain why we can treat the iron core as an ideal transformer with ratio *N*.

NB: This problem is withdrawn from the exam judgement as it is not clear how to answer it. In some way, the answer is given in the problem text.

b) If i_s is positive, is i_R positive or negative? Explain why.

NB: This problem is withdrawn from the exam judgement as it cannot be solved. It is the flux derivative that determines the direction of ir, and the derivative cannot be identified only based on the sign of the primary current.

If the frequency is $f = 50\ Hz$, $v_s = 10\ V\ (RMS)$ and $R_L = 0.5\ \Omega$, the current i_s has an RMS-value of $2\ \text{A}$.

c) Find the numerical value of the inductor *L*

Answer: *This can be solved in several ways. We choose to find the total impedance seen from the voltage source. This is given by* $Z_{tot} = j\omega L + \frac{R_L}{N^2}$ *. The fact that we need to divide by the turns number squared can be derived or remembered from the assignments.*

We can find an expression for the source current i_s *:*

$$i_s = \frac{v_s}{j\omega L + N^2 R_L}$$

The absolute value (RMS) is $\left|i_s\right| = \dfrac{v_s}{\sqrt{(\omega L)^2 + \left(N^2 R_L\right)^2}}$

Setting this equal to 2 A yields:

$$2 = \frac{10}{\sqrt{(\omega L)^2 + (9 \cdot 0.5)^2}} \Rightarrow 25 = (\omega L)^2 + 20.25 \Rightarrow L = \frac{\sqrt{4.75}}{2 \cdot \pi \cdot 50} = 6.94\, mH$$

d) Derive an expression for the power consumed by the resistor R_L. No numerical values, only with symbols (v_s, L_s, N, R_L, f)

Answer: We already have the expression for the source current found above. We can then easily find the current through the resistor as:

$$i_R = \frac{N v_s}{j\omega L + N^2 R_L}$$

The active power through the resistor can be found as $P_{load} = \left|I_R\right|^2 R_L$

This is equal to

$$P_{load} = \frac{(N v_s)^2 R_L}{(\omega L)^2 + \left(N^2 R_L\right)^2} = \frac{(N v_s)^2 R_L}{(2\pi f L)^2 + \left(N^2 R_L\right)^2}$$

This expression can be found also by other approaches, and all expressions that are equivalent will get full score.

4 DIODE RECTIFIER

Figure 1 shows a diode rectifier where the AC-voltage is a triangular wave. In problem a) and b), the DC-side is a constant current source I_{dc} , while in problem c) and d) it is represented by a resistor R.

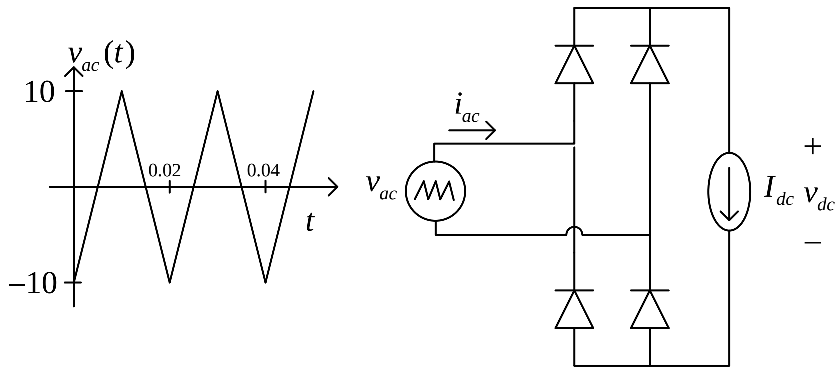


Figure 2: Diode rectifier supplied by a triangular voltage source. Constant DC-current

Assume $I_{dc} = 5A$

a) Find and sketch the DC-side voltage $v_{dc}(t)$ and the AC-side current $i_{ac}(t)$ for $0 \le t \le 0.04$ s.

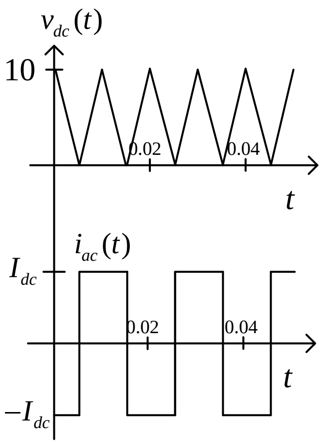
Answer: Similar to when a sinusoidal AC-voltage is applied (see the assignment), the DC-voltage is the absolute value of the AC-voltage $v_{dc}(t) = abs(v_{ac}(t))$

We can find the ac-current in several ways. One method is to set the instantaneous power equal on both sides: $P_{ac}(t) = v_{ac}(t)i_{ac}(t) = P_{dc}(t) = I_{dc}v_{dc}(t)$

This gives: $i_{ac} = I_{dc}\dfrac{v_{dc}(t)}{v_{ac}(t)}$.

The ratio between dc- and ac voltage is either +1 or -1, and is equal to the sign of the ac-voltage. We can express this mathematically as $i_{ac}(t) = I_{dc}\,\mathrm{sgn}(v_{ac}(t))$.

We can plot both functions as:



b) What is the average power delivered from the AC voltage source?

Answer: We can find this from either the dc- or the ac-side (power is equal on both sides). Here we choose the dc-side since the current is constant there. The average power is then equal to the dc-current times the average dc-voltage.

$$P = I_{dc} \cdot \frac{10}{2} = 5 \cdot 5 = 25\,W$$

Here we have used that the average of a triangular wave is the peak-to-peak value divided by two.

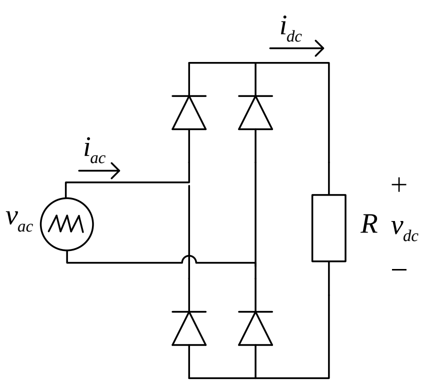


Figure 3 - Diode rectifier supplied by a triangular voltage source. Resistive DC-load

Now the constant current source is replaced with a resistor $R = 2\,\Omega$ as shown in Figure 2.

c) Find and sketch the DC-side current $i_{dc}(t)$ and AC-side current $i_{ac}(t)$ for $0 \leq t \leq 0.04\,s$.

The dc-voltage is the same as before: $v_{dc}(t) = abs(v_{ac}(t))$, hence the dc-current is found as

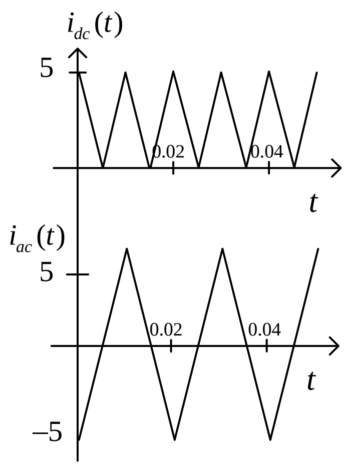
$$i_{dc}(t) = \frac{v_{dc}(t)}{R} = \frac{abs(v_{ac}(t))}{R}$$

We find the ac-current by setting dc- and ac-instantaneous power equal:

$$v_{ac}(t)i_{ac}(t) = v_{dc}(t)i_{dc}(t)$$

$$i_{ac}(t) = \frac{v_{dc}(t)i_{dc}(t)}{v_{ac}(t)} = i_{dc}(t) \cdot \mathrm{sgn}(v_{ac}(t))$$

Hence, the general expression is the same as before. We can now plot both functions:



d) What is the average power delivered from the AC voltage source?

Answer: We find the time-domain expression for instantaneous power as:

$$v_{ac}(t)i_{ac}(t) = v_{ac}(t)i_{dc}(t) \cdot \mathrm{sgn}(v_{ac}(t)) = abs(v_{ac}(t)) \cdot i_{dc}(t) = abs(v_{ac}(t)) \frac{abs(v_{ac}(t))}{R}$$

$$= \frac{abs(v_{ac}(t))^2}{R}$$

Since the instantaneous power is a triangular wave multiplied with itself, we could use the RMS value of a triangular wave and multiply it with itself to get the average value. By using the RMS definition we can show that the RMS-value of a triangular wave is $\frac{V_p}{\sqrt{3}}$, where V_p is the peak value.

Then, we can find that the average power is:

$$P = \frac{\left(\frac{10}{\sqrt{3}}\right)^2}{R} = \frac{100}{6} = 16.667\,W$$

We could also find this number in different ways, e.g. by finding the actual time-domain expression for $v_{ac}(t)$, multiplying it with itself, and integrate over a period to find the average.