# SIE2010 INFORMASJONS- OG SIGNALTEORI

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## Oppgave 1

**1)**

**2)** We first consider the case when $N = 2$. Initial conditions:
$h(-2) = 0, h(-1) = 0$. We then have

$$h(0) = \alpha h(-2) + \beta\delta(0) = \beta$$

$$h(1) = \alpha h(-1) + \beta\delta(1) = 0$$

$$h(2) = \alpha h(0) + \beta\delta(2) = \alpha\beta$$

$$h(3) = \alpha h(1) + \beta\delta(3) = \alpha \cdot 0 = 0$$

$$h(4) = \alpha h(2) + \beta\delta(4) = \alpha\alpha\beta = \alpha^2\beta$$

$$\vdots$$

$$h(n) = \begin{cases} \alpha^{n/2}\beta & n = 0, 2, 4, 6... \\ & n = 1, 3, 5... \end{cases}$$

For general case of N:

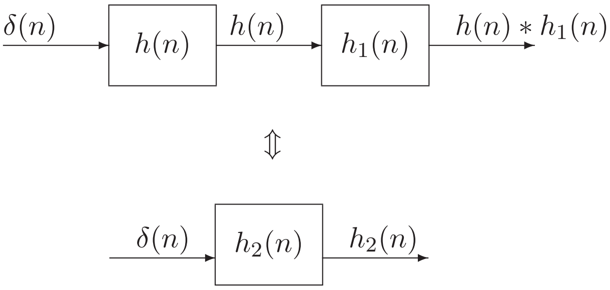
$$h(n) = \begin{cases} \alpha^{n/N}\beta & n = N \cdot (0, 1, 2, 3)... \\ 0 & \text{otherwise} \end{cases}$$

**3)** BIBO stability requires the magnitude of the roots of the characteristic equation to be less than one.
In case of $N = 2$, we have $-1 < \sqrt{\alpha} < 1$.
For general case of $N$, we need all $N$ roots of the polynomial $x^N = \alpha$ have magnitude less than one.

**4)** Frequency response of the filter has the form:

$$H(e^{j\omega}) = \frac{\beta}{1 + \alpha e^{-j\omega N}}$$

**5)** To obtain the unit sample response of the new filter, we cascade the original filter with unit sample response $h(n)$ with a simple FIR filter which has unit sample response $h_1(n)$. This filter contains a delay element D and one coefficient $-\gamma$. When feeding in a unit sample signal, the output of the cascade system will be the unit sample response of the new filter $h_2(n)$. The configuration is shown below




## Oppgave 2

**1)** Linear Independence:
Given a set of functions, if one function can *not* expressed as a linear combination of other functions, this function is then linearly independent of the other functions in the set.

**2)** By using orthogonal basis functions, we can avoid solving the linear equation system all over again each time we add a new term to the series expansion.
Orthogonal basis functions are defined as:

$$\int_{T_1}^{T_2} \varphi_k(t)\varphi_i^*(t)dt = \begin{cases} A_{kk} & \text{for} \quad i = k \\ 0 & \quad \text{otherwise} \end{cases}$$

In case of orthonormal basis functions, the magnitude of $A_{kk}$ becomes 1.

**3)** By Parseval's Theorm we have:

$$\int_{-1}^{1} |x(t)|^2 dt = \sum_{k=1}^{\infty} |\alpha_k|^2$$

Therefore the distortion can be rewritten as:

$$\begin{aligned} < \epsilon^2 >_N &= \sum_{k=1}^{\infty} |\alpha_k|^2 - \sum_{k=1}^{N} |\alpha_k|^2 \\ &= \sum_{k=N+1}^{\infty} |\alpha_k|^2 \end{aligned}$$

**4)**

$$\alpha_1 = \int_{-1}^{1} x(t)\Phi_1(t)dt = \int_{-1}^{1} \frac{1}{2}e^{rt}dt = \frac{1}{2r}(e^r - e^{-r})$$

$$\alpha_2 = \int_{-1}^{1} x(t)\Phi_2(t)dt = \int_{-1}^{1} e^{rt}\frac{2}{3}tdt = \frac{2}{3}(\frac{r-1}{r^2}e^r + \frac{r+1}{r^2}e^{-r})$$

x(t) can then be expressed as:

$$x(t) \approx \frac{1}{4r}(e^r - e^{-r}) + \frac{2}{3}t\frac{2}{3}[\frac{r-1}{r^2}e^r + \frac{r+1}{r^2}e^{-r}]$$

**5)** From Taylor series expansion of order 2 is given by:

$$x(t) \approx x_T(t) = \sum_{n=0}^{2} \frac{1}{n!}\left[\frac{d^n x(t)}{dt}\right]_{t=a}(t-a)^n$$

When $a = 0$, we have:

$$x_T(t) = 1 + \frac{1}{r}t + \frac{1}{2r^2}t^2$$

## Oppgave 3

**1)** Area of the pdf function should be 1. Therefore:

$$\begin{aligned} 0.5A/3 + 0.5A + 0.5A + 0.5A/3 &= 1 \\ \frac{4A}{3} &= 1 \\ A &= \frac{3}{4} \end{aligned}$$

**2)** For a uniform quantizer, the quantization noise $\sigma_Q^2$ can be approximated by $\Delta^2/12$, where $\Delta$ is the quantization step. Hence:

$$\sigma_Q^2 = \frac{\Delta^2}{12} = \frac{0.5^2}{12} = \frac{1}{48}$$

The exact formula for calculating $\sigma_Q^2$ is:

$$\sigma_Q^2 = \sum_{i=0}^{L-1} \int_{x_i}^{x_{i+1}} (x - y_i)^2 f_X(x)dx,$$

where $f_X(x)$ is the source pdf. The approximation made to achieve the simple form of $\Delta^2/12$ is by assuming the pdf is flat for the quantization interval. This actually is the case for this particular pdf, hence the exact formula and approximation will give the same results.

**3)**

$$H = \frac{1}{8}\log_2\frac{1}{8} + \frac{3}{8}\log_2\frac{3}{8} + \frac{3}{8}\log_2\frac{3}{8} + \frac{1}{8}\log_2\frac{1}{8} = 1.81$$

**4)** Channel capacity gives the maximum possible resolution in bits per symbol that can be transmitted error free. To transmit the quantized version of the signal exact, we have four quantization levels and therefore need 2 bits:

$$
\begin{aligned}
2 &= \frac{1}{2}\log_2(1 + \frac{P}{\sigma_N^2}) \\
4 &= \log_2(1 + \frac{P}{\sigma_N^2}) \\
1 + \frac{P}{\sigma_N^2} &= 16 \\
\frac{P}{\sigma_N^2} &= 15 = 11.76 \ \ [\mathrm{dB}]
\end{aligned}
$$

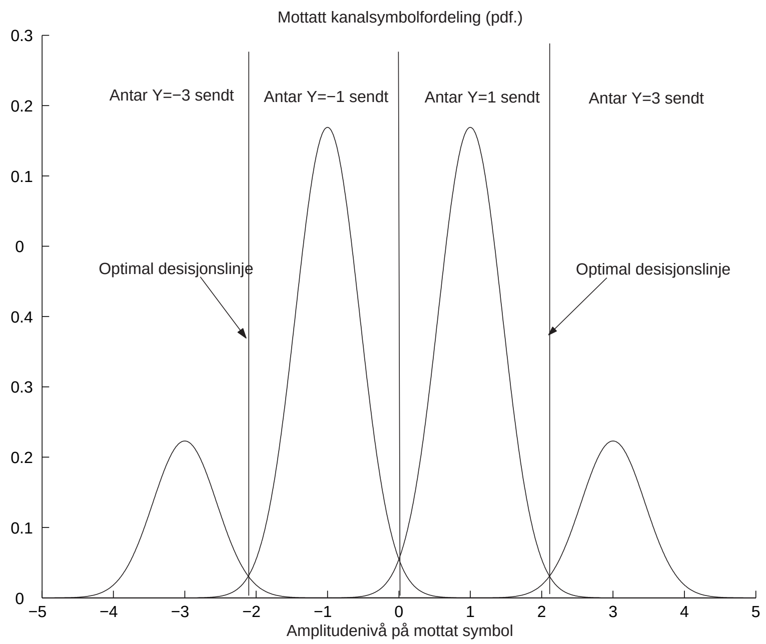
**5)** Signal power can be calculated by:

$$
\begin{aligned}
\int_{-\infty}^{\infty} x^2 f_X(x)dx &= 2\int_{-1}^{-1/2} x^2 \frac{1}{4}dx + 2\int_{-1/2}^{0} x^2 \frac{3}{4}dx \\
&= \frac{1}{6}\cdot\frac{9}{8} + \frac{1}{16} \\
&= \frac{1}{4}
\end{aligned}
$$

The noise power is then:

$$\sigma_N^2 = \frac{\frac{1}{4}}{15} = \frac{1}{60}$$

The figure shown below is taken from a similar exercises (Problem 11.3 in the text book). The concept is essentially the same. In practice the signal to noise ration has to be greater than 15. The channel noise makes the ‘eye-opening’ at the detection points smaller. Hence better signal to noise ration is needed for guaranteed bit error rate.




Figur 1: Skisse av mottatt kanalsymbolfordeling.