NORGES TEKNISK-NATURVITSKAPLEGE UNIVERSITET
INSTITUTT FOR TELETEKNIKK
Signalbehandling

Example solution
for SIE2010 Informasjons- og signalteori,
May 2003

Problem I

a. The *unit sample response* is the output signal of a linear, shift invariant system when a unit sample is input. It can be therefore be found experimentally by first inputting a signal component of unit size, and thereafter only zeros.

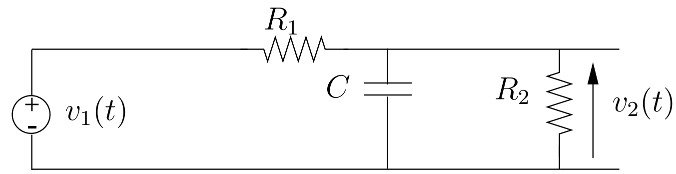
b. *BIBO stability* means *bounded input bounded output*. That is, if a signal is amplitude limited: $|x(n)| < S_1$, then the output signal is also bounded: $|y(n)| < S_2$. The two bounds S_1 and S_2 can be different.

The following implications apply (Not necessary for a complete answer):

1. $\sum_{n=0}^{\infty} |h(n)| < \infty$. (Causality assumed.)
2. The roots of the characteristic polynomial of its difference equation must satisfy $|\alpha_k| < 1$.

For an unstable system the unit sample response will increase in the sense that its envelope will be an increasing function. An example can be generated by the difference equation $y(n) = ay(n-1) + x(n)$, where $a > 1$. Then the unit sample response is given by $h(n) = a^n u(n)$. This will then be an exponentially increasing function.

An analog filter is given in the figure.



c. To find the relation between the input signal $v_1(t)$ and the output signal $v_2(t)$, we use Kirchhoffs voltage and current relations. Define by $i(t)$ the current through the resistor R_1. The voltage balance can be stated as:

1. $v_1(t) = R_1 i(t) + v_2(t)$.

The current $i(t)$ is split between the capacitor and the resistor R_2, and can therefore be calculated as:

2. $i(t) = C\frac{dv_2(t)}{dt} + \frac{v_2(t)}{R_2}$.

We insert this relation into the previous equation and thus obtain an equation with just the input- and output signals and the known components. After a small rearrangement we obtain:

$$R_1 C\frac{dv_2(t)}{dt} + \left(1 + \frac{R_1}{R_2}\right) v_2(t) = v_1(t).$$

d. When Fourier transforming the left hand sides of the differential equation, we have to know, or derive (see pages 107-108) that

$$\mathcal{F}\left\{\frac{dv(t)}{dt}\right\} = j\Omega V(j\Omega) \text{ when } \mathcal{F}\left\{v(t)\right\} = V(j\Omega).$$

Then by transforming term by term we obtain

$$R_1 C j\Omega V_2(j\Omega) + \left(1 + \frac{R_1}{R_2}\right) V_2(j\Omega) = V_1(j\Omega).$$

Solving for the frequency response we get

$$H(j\Omega) = \frac{V_2(j\Omega)}{V_1(j\Omega)} = \frac{1}{1 + \frac{R_1}{R_2} + j\Omega R_1 C}.$$

e. Given the differential equation we could also find the frequency response by

1. Solving for the impulse response (solution with zero initial conditions and a Dirac delta function as input), and then Fourier transforming the impulse response.

2. Applying a complex exponential function of the form $v_1(t) = e^{j\Omega t}$ and assuming that the output signal is of the same form, that is $v_2(t) = He^{j\Omega t}$. Inserting this into the differential equation and solving for H, we get the same result.

3. Direct use of frequency domain relations that can be derived directly from the circuit diagram:

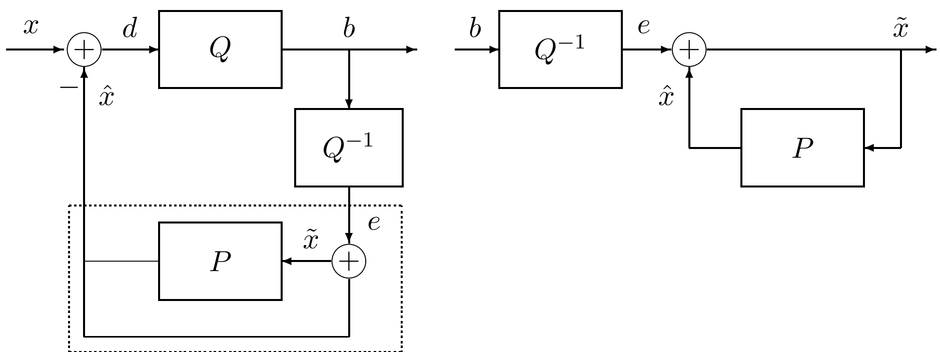
$$V_1(j\Omega) = R_1 I(j\Omega) + V_2(j\Omega).$$

and

$$I(j\Omega) = j\Omega C V_2(j\Omega) + \frac{V_2(j\Omega)}{R_2}.$$

Combining these equation and solving with respect to $H(j\Omega) = V_2(j\Omega)/V_1(j\Omega)$, we obtain the same result as above.

Problem II



a. There are two reasons why signals can be compressed:
First, the signals that can be interpreted by human observers have to be redundant. This means that signal samples are statistically dependent. Signal decomposition can be used to remove redundancy for obtaining more efficient representations.
Second, the human perception system is not perfect. Certain types of noise can not be detected while the signal is present and the noise level is low enough. In addition, different modes in communication require different quality standards. Totally distortionless transmission might not be necessary at all times. For a more detailed description, see page 180 to 181 of the textbook.

b. Differential Pulse Coded Modulation (DPCM) operates as the following:
In the encoder, predictor P is used to predict the current value of x based on **previous** values. The quantizer then quantizes the prediction error d

which is the difference between x and $\tilde{x}$. The quantization index b is being coded and transmitted (with entropy coding for example). Through the inverse quantization block Q^{-1}, we obtain the quantized prediction error e. e is then combined with predicted value of x, $\tilde{x}$, and fed through the predictor to produce a new predicted value $\hat{x}$. The predictor P must contain at least one delay.
The decoder is actually embedded in the encoder. The inverse quantization process recovers the quantized prediction error e, which is combined with the predicted value $\hat{x}$ to form the decoded $\tilde{x}$. The same predictor P is used. The main advantage of DPCM is that the predictor is able to remove/reduce redundancy in the input signal so the much lower dynamic range is needed for the quantizer and fewer bits are required for transmission. Hence we have compression.

c. When the input signal is an AR(1) process, the optimal predictor then contains only one prediction coefficient ρ, which is the same as the one that forms the AR(1) process, and one delay. The prediction error variance σ_d^2 is minimized and can be calculated by $(1-\rho^2)\sigma_x^2$. The prediction error sequence is also white when an optimal predictor is used.

d. Since we are quantizing at 3 bits, we can use the approximation formula for calculating the quantization noise variance for a uniform quantizer. With 8 levels and range of 4, we have quantization step $\Delta = 4/8 = 1/2$. This will also be the noise variance at the output because the noise is unchanged in the receiver.

$$\sigma_q^2 = \frac{\Delta^2}{12} = \frac{1/4}{12} = 1/48.$$

e. To avoid overloading of the quantizer, the uniformly distributed quantizer input has to have range from -2 to 2. This means the quantizer input variance

$$\sigma_d^2 = \int_{-A}^{A} \frac{1}{2A} x^2 dx = \int_{-2}^{2} \frac{1}{4} x^2 dx = \frac{4}{3}$$

The maximum allowed input signal variance σ_x^2 can be calculated by

$$\sigma_x^2 = \frac{\sigma_d^2}{1-\rho^2} = \frac{4/3}{1-.9^2} = 7.0175.$$

Problem III

a. Two continuous functions $r(t)$ and $s(t)$ defined over the interval $t \in [T_1, T_2]$ are *orthogonal* if

$$\int_{T_1}^{T_2} r(t)s^*(t)dt = 0.$$

ii. Two discrete time functions $r(n)$ and $s(n)$ defined over the interval $n \in [N_1, N_2]$ are *orthogonal* if

$$\sum_{N_1}^{N_2} r(n)s^*(n) = 0.$$

b. The basis function

$$\phi_0(t) = \text{sinc}(t/T) = \frac{1}{T}\sin(\pi(t/T))/(\pi(t/T)).$$

has the Fourier transform

$$\Phi_0(j\Omega) = \begin{cases} 1 \text{ for } -\pi/T \leq \Omega \leq \pi/T, \\ 0 \text{ otherwise }. \end{cases}$$

Using the shift property of the Fourier transform (given in the enclosure of the exam), we obtain

$$\Phi_n(j\Omega) = e^{-j\Omega nT}\Phi_0(j\Omega).$$

By using Parseval's relation we obtain

$$\int_{-\infty}^{\infty} \phi_n(t)\phi_k(t)dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_n(j\Omega)\Phi_k^*(j\Omega)d\Omega = \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} e^{-j\Omega nT}e^{j\Omega kT}d\Omega = \frac{1}{T}\delta_{k,n},$$

that is, the integral is equal to zero whenever $k \neq n$. This proves the orthogonality.

c. Using Parseval's relation, the integrand in the frequency domain is going to be zero everywhere as the two functions do not overlap.

d. A receiver structure for a system which transmits Ksimultaneous symbols is shown in the figure.

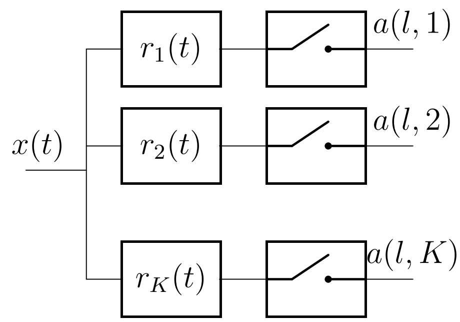


Figure 1:

The filters are matched to the received symbols, that is the impulse response of channel k is given by $r_k(t) = h_k(T_d - t)$, where $h_k(t)$ is the received signal in that channel when one symbol has been sent at $t = 0$ and T_d is the length of that symbol. The matched filters are followed by samplers which measure the signal at the correct detection points.

e. Finite length signals with infinite bandwidth can be orthogonal even if their Fourier transforms overlap. This means that the signals overlap both in the time and frequency domains. This is, of course the case in part b. In general, for two real functions that overlap to be orthogonal, their product have to have positive and negative parts. This mechanism is easy to understand if one function is symmetric and the other is antisymmetric.

For the complex case, it is more involved. Then the product of the two functions must have components in all different directions in the complex plain that cancel when they are "summed".