Norwegian University of Science and Technology
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TTT4110 Information and Signal Theory
Solution to the Exam of June 4, 2005

Problem 1 [17 points]

(a) [*6 points*] The digital filter's frequency response $H(\omega)$ is given in polar form, i.e. $H(\omega) = |H(\omega)|\, e^{j\phi(\omega)}$, where $|H(\omega)| = 0.8\cos\omega + 0.9$ is the filter's amplitude response, and $\phi(\omega) = -\omega$ is the filter's phase response (this being because $0.8\cos\omega + 0.9$ is a positive real number for all ω).

We determine the filter type by looking at the behaviour of $|H(\omega)|$. $|H(\omega)|$ is a monotonically decreasing function for $\omega \in [0, \pi]$, satisfying $|H(0)| = 1.7$ and $|H(\pi)| = 0.1$. It is depicted in Fig. 1.

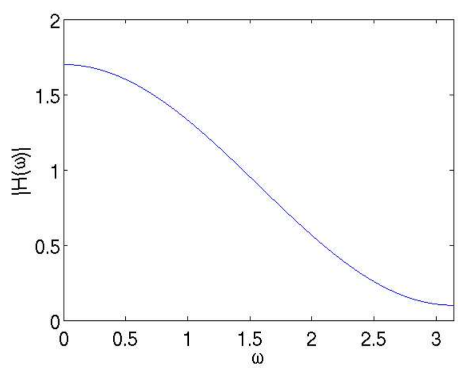


Figure 1: Amplitude response

Since the filter lets low frequencies through and attenuates high frequencies, we have a low-pass filter.

The group delay of the filter $\tau_g(\omega)$ is given by

$$\tau_g(\omega) = -\frac{d\phi(\omega)}{d\omega} = 1.$$

We now need to find the amplification and phase delay of the filter at the digital frequency $f = \frac{1}{8}$. We have $\omega = 2\pi f = \frac{\pi}{4}$. The filter amplification is thus given by

$$\left|H\left(\frac{\pi}{4}\right)\right| = 0.8\cos\frac{\pi}{4} + 0.9 = 0.8\frac{\sqrt{2}}{2} + 0.9 \approx 1.47 \text{ times},$$

and, since

$$\phi\left(\frac{\pi}{4}\right) = -\frac{\pi}{4},$$

we have a phase delay of $\pi/4$ radians.

(b) [*7 points*] The relation between the unit sample response $h(n)$ and the impulse response $H(\omega)$ of the filter is

$$H(\omega) = \mathrm{DTFT}\{h(n)\} = \sum_{n=-\infty}^{\infty} h(n)e^{-j\omega n}. \tag{1}$$

We also have

$$H(\omega) = e^{-j\omega}\left(0.8\,\frac{e^{j\omega}+e^{-j\omega}}{2} + 0.9\right) = 0.4 + 0.4\,e^{-j2\omega} + 0.9\,e^{-j\omega}. \tag{2}$$

We hence conclude, by comparing (1) and (2), that

$$h(n) = \begin{cases} 0.4 & \text{if } n = 0 \text{ or } n = 2, \\ 0.9 & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In order to find a difference equation describing the relation between the input and output signals of the filter in the time domain, we write

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = 0.4 + 0.4e^{-j2\omega} + 0.9e^{-j\omega},$$

where $X(\omega) = \mathrm{DTFT}\{x(n)\}$ and $Y(\omega) = \mathrm{DTFT}\{y(n)\}$ respectively denote the discrete-time Fourier transforms (DTFT) of the time domain signal $x(n)$ (filter input) and the time domain signal $y(n)$ (filter output). We thus have

$$Y(\omega) = 0.4X(\omega) + 0.4X(\omega)e^{-j2\omega} + 0.9X(\omega)e^{-j\omega},$$

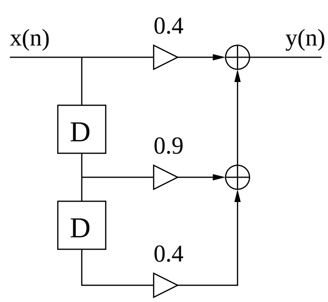
and by taking the inverse discrete-time Fourier transform (IDTFT) of the above equation, we obtain

$$y(n) = 0.4x(n) + 0.4x(n-2) + 0.9x(n-1),$$

where we have made use of the fact that $\mathrm{DTFT}\{x(n-k)\} = X(\omega)e^{-j\omega k}$.

The impulse response $h(n)$ having finite length (equal to 3), this is an FIR (finite impulse response) filter.

A possible filter structure for the filter is shown below.



(c) [*4 points*] We obtain

$$H_{tot}(\omega) = H_1(\omega)\cdot(H_1(\omega) + H_2(\omega))\cdot H_2(\omega),$$

and

$$h_{tot}(n) = h_1(n) * (h_1(n) + h_2(n)) * h_2(n).$$

Problem 2 [13 points]

(a) [*3 points*] Since the sampling period is $T_s = 0.125$ ms., the sampling frequency is

$$F_s = \frac{1}{T_s} = 8 \text{ kHz.}$$

If a signal $x(n)$ is obtained by sampling an analog signal $s(t)$ with sampling frequency F_s ($t = n/F_s$), the relation between the spectrum $X(f)$ of $x(n)$ and the spectrum $S(F)$ of $s(t)$ is given by

$$X\left(\frac{F}{F_s}\right) = F_s \sum_{k=-\infty}^{\infty} S(F - kF_s).$$

Using also the relation $f = \frac{F}{F_s}$ (where f is the digital frequency), it is easy to draw the figure shown below.

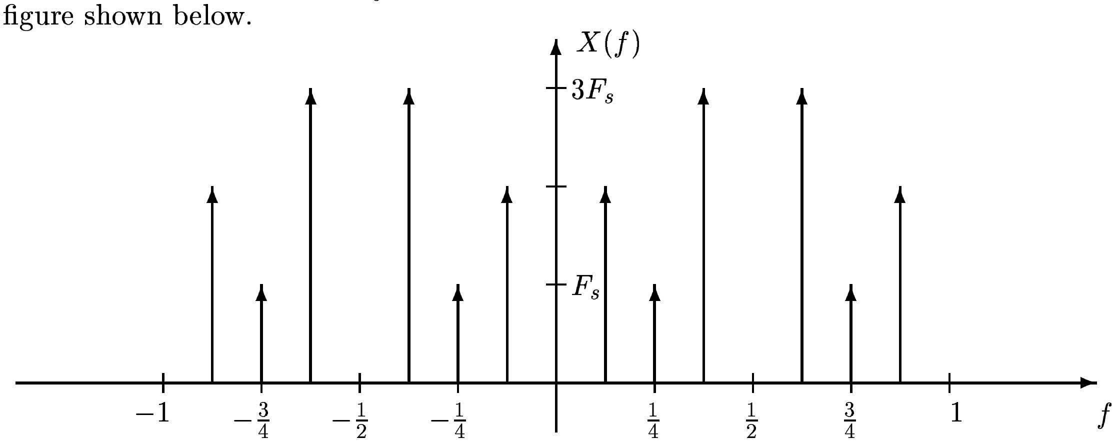


Figure 2: Spectrum $X(f)$ of the signal $x(n)$ for $f \in [-1, 1]$ ($F_s = 8$ kHz.)

(b) [*4 points*] We can avoid aliasing when sampling at a frequency F_s if the signal which is sampled is bandlimited to $F_s/2$. The antialiasing filter must therefore block all frequency components satisfying $F \geq F_s/2$ contained in the signal that is input to it, and leave unaffected all frequency components satisfying $F < F_s/2$ contained in the signal that is input to it.

For the given $s(t)$ and $F_s = 8$ kHz., this means that the antialiasing filter must leave the frequency components at $F = 1$ kHz. and $F = 3$ kHz. unaffected, whereas the frequency component at $F = 6$ kHz. must be blocked.

This can be for example realised by a filter having the amplitude response sketched in Fig. 3. The amplitude spectrum of $x(n)$ when this filter is used is shown in Fig. 4.

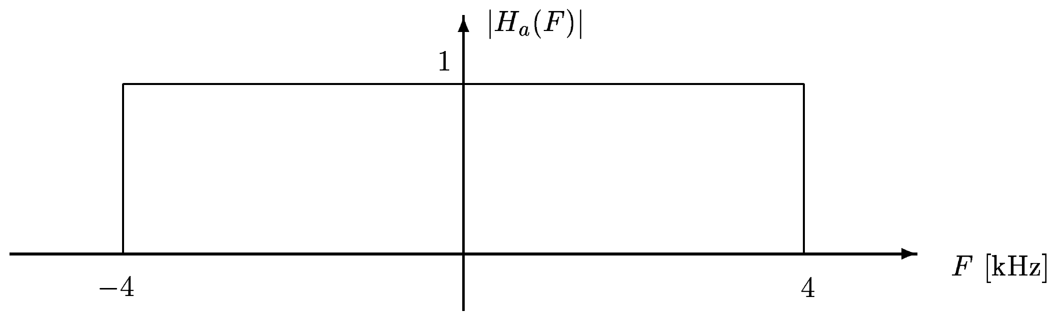


Figure 3: Amplitude response of a filter satisfying the criteria exposed in problem 2(b).

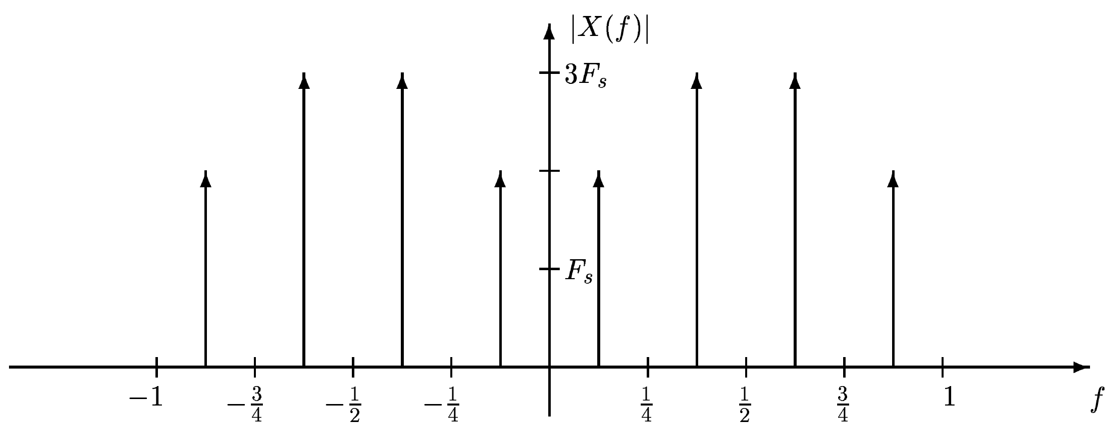


Figure 4: Amplitude spectrum $|X(f)|$ of the signal $x(n)$ for $f \in [-1, 1]$ when the antialiasing filter from Fig. 3 is used.

(c) [*6 points*] The signal $x_s(t)$ has the same spectrum as the signal $x(n)$. This is shown in the following:

$$
\begin{aligned}
X_s(F) &= \int_{-\infty}^{\infty} x_s(t)e^{-j2\pi Ft}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(n)\delta(t-nT_s)\ e^{-j2\pi Ft}dt \\
&= \sum_{n=-\infty}^{\infty} x(n) \int_{-\infty}^{\infty} \delta(t-nT_s)\ e^{-j2\pi Ft}dt = \sum_{n=-\infty}^{\infty} x(n)\ e^{-j2\pi FnT_s} \\
&= \sum_{n=-\infty}^{\infty} x(n)\ e^{-j2\pi fn} = X(f).
\end{aligned}
$$

The reconstruction filter must filter out all the spectral replicas introduced by the sampling process, and compensate for the scaling factor F_s (also introduced by the sampling process).

In this specific case, the reconstruction filter must thus have an amplitude response of $1/F_s$ at $F = 1$ kHz. and $F = 3$ kHz., and must filter out the frequency components with $F > F_s/2 = 4;$kHz. This can for example be realised by a filter having the amplitude response sketched in Fig. 5.

The amplitude spectrum $|X_r(F)|$ of the reconstructed signal $x_r(t)$ when this reconstruction filter is used is shown in Fig. 6.

We now have to find an expression for the reconstructed signal $x_r(t)$ as a function of $x(n)$ and the impulse response of the reconstruction filter $h_r(t)$. This can be achieved as follows:

$$
\begin{aligned}
x_r(t) &= x_s(t) * h_r(t) \\
&= \left(\sum_{n=-\infty}^{\infty} x(n)\delta(t-nT_s) \right) * h_r(t) \\
&= \sum_{n=-\infty}^{\infty} x(n)\left[\delta(t-nT_s) * h_r(t)\right] \\
&= \sum_{n=-\infty}^{\infty} x(n)h_r(t-nT_s).
\end{aligned}
$$

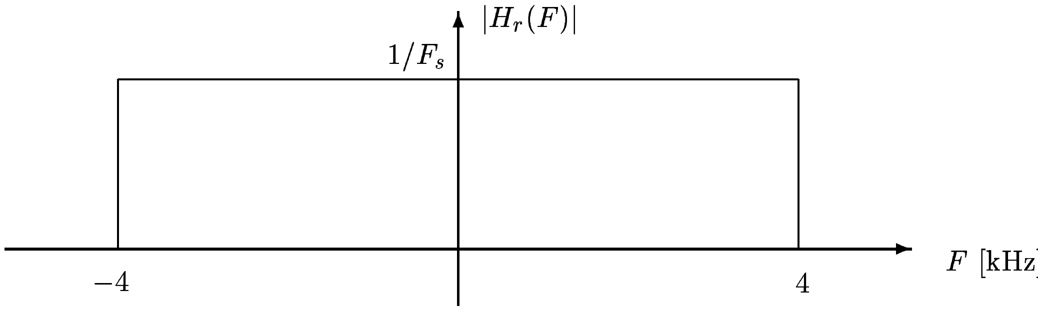


Figure 5: Amplitude response of a reconstruction filter satisfying the criteria exposed in problem 2(c).

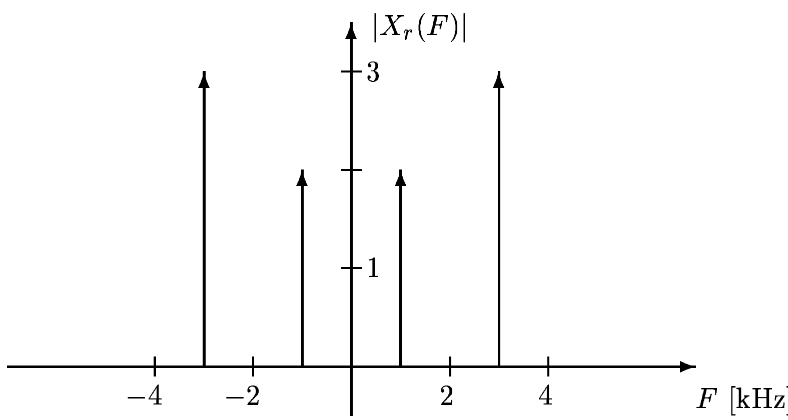


Figure 6: Amplitude spectrum $|X_r(F)|$ of the signal $x_r(t)$ when the antialiasing filter from Fig. 3 and the reconstruction filter from Fig. 5 are used.

Problem 3 [20 points]

(a) [*3 points*] The decision levels $\{x_i\}$ and representation levels $\{y_i\}$ of a uniform quantiser must fulfil the following requirements:

$$x_{i+1} - x_i = \Delta, \qquad \forall i,$$

and

$$y_i = \frac{x_i + x_{i+1}}{2}, \qquad \forall i.$$

Combining these two requirements with the fact that we wish to design a uniform quantiser with 6 quantisation intervals, the dynamic range of which must coincide with $[-1, 1]$ (the dynamic range of the signal to be quantised), we deduce that the decision levels are given by $-1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}$ and 1, and that the representation levels are given by $-\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{2}$ and $\frac{5}{6}$.

(b) [*5 points*] We now have to express the quantisation error variance σ_q^2 as a function of $\{x_i\}$, $\{y_i\}$ and $f_X(x)$ (defined in the problem statement). The mean value of the quantisation error μ_q is assumed to be equal to zero. Therefore,

$$\begin{aligned}
\sigma_q^2 &= E[q^2] = E\left[(x - Q[x])^2\right] = \int_{-\infty}^{\infty} (x - Q[x])^2 f_X(x)\, dx \\
&= \sum_i \int_{x_i}^{x_{i+1}} (x - y_i)^2 f_X(x)\, dx
\end{aligned}$$

Since $f_X(x)$ is constant at each quantisation interval, the exact value for σ_q^2 can be computed by

$$\sigma_q^2 = \frac{\Delta^2}{12} = \frac{(1/3)^2}{12} = \frac{1}{108}$$

Alternatively, we have that

$$\begin{aligned}
\sigma_q^2 &= \sum_i f_X(y_i) \int_{x_i}^{x_i+\Delta} \left(x - x_i - \frac{\Delta}{2}\right)^2 dx = \sum_i f_X(y_i) \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} u^2\, du \\
&= \left[\frac{u^3}{3}\right]_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} \sum_i f_X(y_i) = 2 \cdot \frac{\Delta^3}{8 \cdot 3} \cdot 2 \cdot \left(\frac{3}{4} + \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{108},
\end{aligned}$$

(c) [*2 points*] Yes, this can be achieved by reducing the size of the quantisation intervals in places where $f_X(x)$ is large, and increasing it where $f_X(x)$ is small. In this way, on average, values of x arising often will lead to smaller quantisation errors, whereas those which arise less frequently will lead to larger quantisation errors than the quantisation error that would be obtained if a uniform quantiser were used. This results in a reduced value for σ_q^2.

(d) [*3 points*] We have 6 representation levels. The necessary amount of bits per sample of the quantised signal $x_q(n)$, assuming that each codeword is associated with one representation level and that all codewords have the same length, is thus

$$b = \lceil \log_2 6 \rceil = 3.$$

Representation Level	Codeword
y_0	001
y_1	010
y_2	011
y_3	100
y_4	101
y_5	110

(With 2 bits, only $2^2 = 4$ different levels can be represented, whereas it is possible to represent up to $2^3 = 8$ levels with 3 bits.) A possible code would be

(e) [*3 points*] In entropy coding, symbols which arise very frequently are assigned short codewords, where as symbols which seldom arise are assigned longer codewords. An example of a uniquely decodable code designed according to this principle, which is more efficient than the code from 3(d), is shown in the following table:

Representation Level	Codeword
y_0	110
y_1	100
y_2	00
y_3	01
y_4	101
y_5	111

(f) [*4 points*] The average codeword length $\overline{L}$ when the code from 3(e) is used is given by

$$\overline{L} = \sum_{i=0}^{5} p_i l_i$$

where p_i denotes the probability that representation level y_i arise, and l_i is the length of the codeword assigned to representation level y_i. Now, since $p_i = \int_{x_i}^{x_{i+1}} f_X(x)dx$, we have $p_0 = p_5 = \frac{1}{12}$, $p_1 = p_4 = \frac{1}{6}$, and $p_2 = p_3 = \frac{1}{4}$. Hence,

$$\overline{L} = 2 \cdot \left(\frac{1}{12} \cdot 3 + \frac{1}{6} \cdot 3 + \frac{1}{4} \cdot 2 \right) = 2.5 \text{ bits/representation level.}$$

The lower limit for the average codeword length that is necessary to represent the signal $x_q(n)$ is given by the signal entropy

$$\begin{aligned} H &= \sum_i p_i \log_2 \frac{1}{p_i} \\ &= 2 \left(\frac{1}{12} \log_2 12 + \frac{1}{6} \log_2 6 + \frac{1}{4} \log_2 4 \right) \\ &\approx 2.46 \text{ bits/representation level.} \end{aligned}$$

Problem 4 [11 points]

(a) [*4 points*] We must have

$$g(lT + \Delta t) = \begin{cases} 1 & \text{for} \quad l = 0, \\ 0 & \text{for} \quad l \in \mathbb{Z} \setminus \{0\}, \end{cases}$$

for some T and Δt in order for intersymbol interference (ISI) free transmission to be possible.

For the impulse response in Fig. 6(a) (see problem statement), we have

$$g(t) = \begin{cases} 0 & \text{for} \quad t \in \mathbb{Z} \setminus \{2\}, \\ 1 & \text{for} \quad t = 2, \end{cases}$$

so this function meets the ISI-free transmission criterion if $\Delta t = 2$ ms. and $T \in \mathbb{N}$.

The maximum possible signalling speed $\frac{1}{T}$ is obtained when T is as small as possible, i.e. $T = 1$ ms. This yields a maximum signalling speed of $\frac{1}{T} = 1000$ channel symbols per second.

For the impulse response from Fig. 6(b) (see problem statement), this criterion is met for $\Delta t = 3$ ms. and $T \geq 2$ ms. The maximum possible signalling speed is thus $\frac{1}{T} = \frac{1}{2\,\text{ms.}} = 500$ channel symbols per second.

(b) [*2 points*] In the relation

$$C = \frac{1}{2} \log_2 \left(1 + \frac{P}{\sigma_N^2} \right),$$

C is the the maximum possible number of bits per channel symbol for which transmission with arbitrarily low probability of error is possible (channel capacity), and P and σ_N^2 respectively denote the average signal and noise power at the receiver of the communication channel.

(c) [*3 points*] We have shown in 4(a) that a maximum of 1000 channel symbols/second can be sent over the channel shown in Fig. 6(a) (see problem statement) without ISI. In order to be able to send 4000 bits/second over this channel, the channel capacity must hence be of at least 4 bits/channel symbol. The channel noise being Gaussian, we must have

$$C = \frac{1}{2} \log_2 \left(1 + \frac{P}{\sigma_N^2} \right) > 4,$$

where $P/\sigma_N^2 \triangleq \text{SNR}$ is the average signal to noise ratio at the receiver of the communication system. We thus conclude that we must have

$$\text{SNR} > 2^8 - 1 = 255.$$

(d) [*2 points*] In a digital system for information transfer, channel coding is used to introduce redundancy in the signal that is to be sent over the digital communication channel. The redundant information helps detect and correct errors which might arise because of the presence of channel noise. (If the information stream were sent over the communication channel without being previously encoded, it would be impossible to detect or correct errors arising due to channel noise.)