# TTT4110 Information and Signal Theory Solution to continuation exam 

## Problem I

DI-structure:


DII-structure:

(a)

The direct form I and direct form II stuctures are given above. The direct form II structure has an advantage over direct form I with respect to the number of memory blocks used. Direct form II uses two memory blocks, while direct form I uses three memory blocks.
(b) The unit sample response can be found by applying a unit sample at the input of the filter and writing out the output in an iterative manner. We assume zero initial conditions, which means that $h(n)=0$ for $n<0$. The unit sample response is given by difference equation:

$$
h(n)=\delta(n)-0.5 h(n-2)
$$

The output written in an iterative manner:

$$
\begin{aligned}
& h(0)=1 \\
& h(1)=0 \\
& h(2)=-0.5 \\
& h(3)=0 \\
& h(4)=0.25
\end{aligned}
$$

From these outputs we see that the unit sample response can be expressed by:

$$
h(n)= \begin{cases}(-0.5)^{n / 2} & \text { if } n \geq 0 \text { and } \mathrm{n} \text { even } \\ 0 & \text { otherwise }\end{cases}
$$

(c) To show that this causal system is BIBO stable, we need to show that:

$$
\sum_{n=0}^{\infty}|h(n)|<\infty
$$

We proceed with the unit sample response from (b):

$$
\sum_{n=0}^{\infty}|h(n)|<\infty=\sum_{n=0, n \text { even }}^{\infty}\left|(-0.5)^{n / 2}\right|=\sum_{m=0}^{\infty}\left|(-0.5)^{m}\right|=\sum_{m=0}^{\infty} 0.5^{m}=\frac{1}{1-0.5}=2<\infty
$$

By this we have shown that $h(n)$ is BIBO stable.
(d) Now we want to find the unit sample response of the first system. We go back to the difference equation that describes the system, and apply a unit sample as an input to the system. The difference equation becomes:

$$
h_{t o t}(n)+0.5 h_{t o t}(n-2)=\delta(n)+3 \delta(n-1)
$$

We see from the difference equation that we have two signals at the input. By exploiting linearity we know that the output signal can written as a superposition of the two different outputs that would result from applying each of the signals alone as an input. Therefore solving first:

$$
h(n)+0.5 h(n-2)=\delta(n)
$$

And then solving:

$$
h(n)+0.5 h(n-2)=3 \delta(n-1)
$$

The sum of the solution of each of the two difference equations would give the unit sample response of the complete system. The first difference equation is the same equation as in (b). The second difference equation has the solution $3 h(n-1)$. The reason for this is that the difference equation given in (b) is both linear and shiftinvariant. The unit sample response of the complete system is therefore:

$$
h_{\text {tot }}(n)=h(n)+3 h(n-1)== \begin{cases}(-0.5)^{n / 2} & \text { for } n \geq 0 \text { and } n \text { even } \\ 3(-0.5)^{\frac{n-1}{2}} & \text { for } n \geq 1 \text { and } n \text { odd } \\ 0 & \mathrm{n}<0\end{cases}
$$

Note that the solution was found by noting that the difference equation in (b) was already solved, and by using this solution to find the unit sample response of the complete system. There is also an alternative way of finding the unit sample response. If we go
back to the direct form II structure of the first system, we can rewrite the difference equation and express the system with two difference equations:

$$
\begin{aligned}
& v(n)=x(n)-0.5 v(n-2) \\
& y(n)=v(n)+3 v(n-1)
\end{aligned}
$$

Where $v(n)$ is an intermediate signal. Each of the two difference equations represents a filter. Such that the complete system consist of a cascade of two filters. We recognize that the first difference equation is the same difference equation as in (b), and therefore we know the unit sample response of this filter. We denote this filter by $h_{1}(n)$. So now we need to find the unit sample response of the second filter from the second difference equation. We apply a unit sample at the input and the unit sample response becomes:

$$
h_{2}(n)=\delta(n)+3 \delta(n-1)
$$

Since the complete system consist of a cascade of two filters we find the unit sample response of the complete system by folding the two filters:

$$
\begin{aligned}
h(n) & =h_{1}(n) * h_{2}(n)=h_{1}(n) *(\delta(n)+3 \delta(n-1))=h_{1}(n)+3 h_{1}(n-1) \\
& = \begin{cases}(-0.5)^{n / 2} & \text { for } n \geq 0 \text { and } n \text { even } \\
3(-0.5)^{\frac{n-1}{2}} & \text { for } n \geq 1 \text { and } n \text { odd } \\
0 & \mathrm{n}<0\end{cases}
\end{aligned}
$$

Now we can argue that $h(n)$ is BIBO-stable by using the fact that $h_{1}(n)$ is BIBO-stable. It was shown in (c) that $h_{1}(n)$ was BIBO-stable. The criteria for BIBO-stability:

$$
\sum_{n=0}^{\infty}|h(n)|<\infty
$$

And then we check if $h(n)$ fulfills this criteria:

$$
\sum_{n=0}^{\infty}|h(n)|=\sum_{n=0}^{\infty}\left|h_{1}(n)+3 h_{1}(n-1)\right| \leq \sum_{n=0}^{\infty}\left|h_{1}(n)\right|+3 \sum_{n=0}^{\infty}\left|h_{1}(n-1)\right|<\infty
$$

In the last step we used the fact that the sum of two numbers less than $\infty$, is less than $\infty$. Therefore we have shown that $h(n)$ is BIBO-stable.

## Problem II

(a) DTFT of $x(n)$ :

$$
X\left(e^{j \omega}\right)=\sum_{n=-\infty}^{\infty} x(n) e^{-j \omega n}=\sum_{n=0, \text { even }}^{\infty} \alpha^{n} e^{-j \omega n}=\sum_{m=0}^{\infty} \alpha^{2 m} e^{-j 2 \omega m}=\sum_{m=0}^{\infty}\left(\alpha^{2} e^{-j 2 \omega}\right)^{m}=\frac{1}{1-\alpha^{2} e^{-j 2 \omega}}
$$

The expression for the discrete time Fourier transform is found by summing the terms of a geometric series. For the sum of the geometric series to converge a certain restriction
has to be put on $\alpha$ :

$$
\begin{aligned}
& \left|\alpha^{2} e^{-j 2 \omega}\right|<1 \\
& \left|\alpha e^{-j \omega}\right|^{2}<1 \\
& \left|\alpha e^{-j \omega}\right|<1 \\
& |\alpha|<1
\end{aligned}
$$

For the sum of the geometric series to converge which means that the expression is valid, the absolute value of $\alpha$ need to be less than one.
(b) The Fourier transform of a shortened sequence of finite duration is given by:

$$
X\left(e^{j \omega}\right)=\sum_{n=0}^{N-1} x(n) e^{-j \omega n}=\sum_{m=0}^{(N-1) / 2} \alpha^{2 m} e^{-j 2 \omega m}=\frac{1-\left(\alpha^{2} e^{-j 2 \omega}\right)^{(N-1) / 2+1}}{1-\alpha^{2} e^{-j 2 \omega}}
$$

We have assumed that $N-1$ is an even number. The only restriction on $\alpha$ now is:

$$
|\alpha|<\infty
$$

(c) By performing DFT we obtain discrete frequency components, which means we obtain components only at discrete frequencies. Let us denote the DTFT of $x(n)$ by $X_{D T F T}\left(e^{j \omega}\right)$, and the DFT of $x(n)$ by $X_{D F T}(k)$. The relationship between $X_{D T F T}\left(e^{j \omega}\right)$ and $X_{D F T}(k)$ is that:

$$
X_{D F T}(k)=X_{D T F T}\left(e^{j \frac{2 \pi k}{N}}\right) \text { for } k=0, \ldots ., N-1
$$

This means that $X_{D F T}(k)$ is obtained by sampling $X_{D T F T}\left(e^{j \omega}\right)$, with samples taken at a distance $2 \pi / N$ from each other.
(d) By using DFT we obtain frequency components at discrete frequencies only. Another thing is that the infinite sequence is approximated by a finite sequence. Let us denote the infinite sequence by $x(n)$. The shortened sequence $x^{\prime}(n)$ can be described mathematically by:

$$
x^{\prime}(n)=x(n) \cdot w(n)
$$

Here $w(n)$ is window function which is zero for $n \geq N$. In the frequency domain the relationship becomes:

$$
X^{\prime}\left(e^{j \omega}\right)=X\left(e^{j \omega}\right) * W\left(e^{j \omega}\right)
$$

Which shows that $X^{\prime}\left(e^{j \omega}\right)$ results from folding $X\left(e^{j \omega}\right)$ with $W\left(e^{j \omega}\right)$. $W\left(e^{j \omega}\right)$ becomes a sinc function in the frequency domain, since it was a window function in the time domain. A wide window function $w(n)$ in the time domain gives a narrow sinc function $W\left(e^{j \omega}\right)$ in the frequency domain. This means that $X^{\prime}\left(e^{j \omega}\right)$ gets more similar to $X\left(e^{j \omega}\right)$. In the limit of an infinite wide window function $w(n)$, then we get $W\left(e^{j \omega}\right)=\delta(\omega)$ and therefore $X^{\prime}\left(e^{j \omega}\right)=X\left(e^{j \omega}\right) * \delta(\omega)=X\left(e^{j \omega}\right)$. When the length of $w(n)$ is short, then $W\left(e^{j \omega}\right)$ becomes a broad sinc function. And $X^{\prime}\left(e^{j \omega}\right)$ will not be equal to $X\left(e^{j \omega}\right)$, because $X^{\prime}\left(e^{j \omega}\right)$ is the result of folding $X\left(e^{j \omega}\right)$ with a broad sinc function.

## Problem III

(a) The quantization noise variance from a uniform quantizer is given by:

$$
\sigma_{Q}^{2}=\Delta^{2} / 12
$$

For this expression to be accurate, the signal dynamic range need to be split into a sufficient high number of quantization intervals. If this is the case, then there are two approximations that are made in order to arrive at the above expression. The first approximation is that the probability density function (pdf) is flat inside a quantization interval. Which means that the pdf inside the interval $x \in\left[x_{i}, x_{i+1}\right]$ is equal to $f_{X}\left(y_{i}\right)$, where $y_{i}$ is the midpoint inside the quantization interval. The other approximation made is :

$$
\sum_{i=0}^{L-1} f_{X} y_{i} \cdot \Delta=1
$$

Where $L$ is the number of representation points. This means that the sum is an approximation to the total probability mass.
(b) The reconstructed signal is given by:

$$
y(n)=x(n)+q(n)
$$

The power spectral density is therefore given by:

$$
S_{Y Y}(\omega)=S_{X X}(\omega)+S_{Q Q}(\omega)=S_{X X}(\omega)+\sigma_{Q}^{2}=S_{X X}(\omega)+\Delta^{2} / 12
$$

We have used the fact that the quantization noise $q(n)$ is white and uncorrelated with the signal $x(n)$.
(c) The reconstructed signal $y(n)$ is put through a FIR filter given by:

$$
w(n)=y(n)+b y(n-1)
$$

To find the power spectral density we first start by taking the Fourier transform of the difference equation:

$$
W(\omega)=Y(\omega)+b Y(\omega) e^{-j \omega}
$$

From this we can find the power spectral density:

$$
\begin{aligned}
& S_{W W}(\omega)=E\left\{W(\omega) W^{*}(\omega)\right\}=E\left\{\left(Y(\omega)+b Y(\omega) e^{-j \omega}\right)\left(Y^{*}(\omega)+b Y^{*}(\omega) e^{j \omega}\right)\right\} \\
& =S_{Y Y}(\omega)+2 b S_{Y Y}(\omega) \cos \omega+b^{2} S_{Y Y}(\omega)=\left(1+2 b \cos \omega+b^{2}\right) S_{Y Y}(\omega) \\
& =\left(1+2 b \cos \omega+b^{2}\right)\left(S_{X X}(\omega)+\Delta^{2} / 12\right)
\end{aligned}
$$

Now we insert the expression given for $S_{X X}(\omega)$ :

$$
S_{W W}(\omega)=\left(1+2 b \cos \omega+b^{2}\right)\left(\frac{1-\rho^{2}}{1+\rho^{2}-2 \rho \cos \omega} \sigma_{X}^{2}+\Delta^{2} / 12\right)
$$

In order for the signal component to be white, which means flat spectrum, the filter parameter $b$ has to be $b=-\rho$. This gives:

$$
S_{W W}(\omega)=\left(1-\rho^{2}\right) \sigma_{X}^{2}+\left(1+\rho^{2}-2 \rho \cos \omega\right) \Delta^{2} / 12
$$

The power spectral density of the noise is then given by:

$$
\left(1+\rho^{2}-2 \rho \cos \omega\right) \Delta^{2} / 12
$$

(d) The lowest possible bit-rate that the quantized signal can be represented by, is given by the entropy.

$$
H(X)=h(X)-\log _{2}(\Delta)
$$

The signal is assumed to be Gaussian and the quantizer interval is given by $\Delta=\sigma_{X} / 4$. The entropy of the quantized signal is then:

$$
\begin{aligned}
& H(X)=1 / 2 \log _{2}\left(2 \pi e \sigma_{X}^{2}\right)-\log _{2}\left(\sigma_{X} / 4\right)=1 / 2 \log _{2}\left(2 \pi e \sigma_{X}^{2}\right)-1 / 2 \log _{2}\left(\frac{\sigma_{X}^{2}}{16}\right) \\
& =1 / 2 \log _{2}(32 \pi e)=4.05 \mathrm{bits}
\end{aligned}
$$

