Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4110 Information and Signal Theory
Solution to exam

Problem I

(a) The frequency response is found by taking the Fourier-transform:

$$H(j\Omega) = \int_0^\infty e^{\alpha t} e^{-j\Omega t} dt = \left[\frac{1}{\alpha - j\Omega} e^{(\alpha - j\Omega)t}\right]_0^\infty = \frac{1}{j\Omega - \alpha}$$

(b) The digital filter is obtained by sampling the analog filter:

$$h_d(n) = h(nT) = e^{\alpha n T} u(nT)$$

The frequency response of the digital filter is given by:

$$H_d(e^\omega) = \sum_{n=0}^{\infty} e^{\alpha n T} e^{-j\omega n} = \sum_{n=0}^{\infty} e^{(\alpha T - j\omega)\cdot n} = \frac{1}{1 - e^{\alpha T - j\omega}}$$

We have used the formula for a geometric series, and we have also assumed that $\mathrm{Re}(\alpha) < 0$.

(c) We sketch the magnitude of the frequency response when $\alpha = -1$ and $T = 1/2$.

$$H_d(e^\omega) = \frac{1}{1 - e^{-1/2 - j\omega}}$$

The magnitude squared is found by:

$$|H_d(e^\omega)|^2 = \frac{1}{1 - e^{-1/2 - j\omega}} \cdot \frac{1}{1 - e^{-1/2 + j\omega}} = \frac{1}{1 + e^{-1} - 2e^{-1/2}\cos(\omega)}$$

The magnitude is therefore:

$$|H_d(e^{j\omega})| = \frac{1}{\sqrt{1 + e^{-1} - 2e^{-1/2}\cos(\omega)}}$$

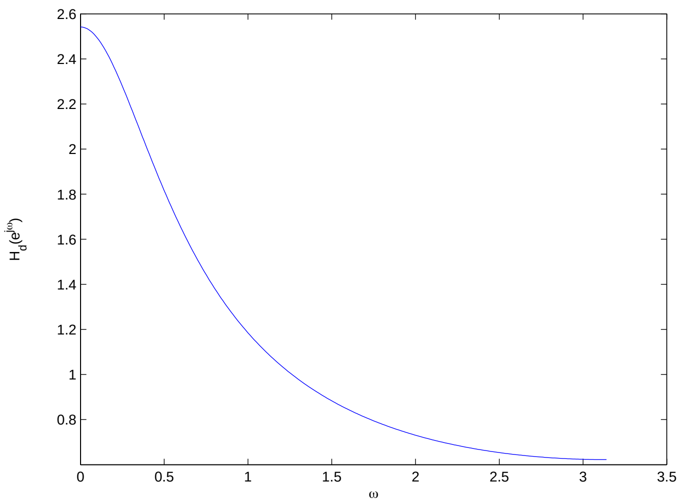
The magnitude of the analog filter is given by:

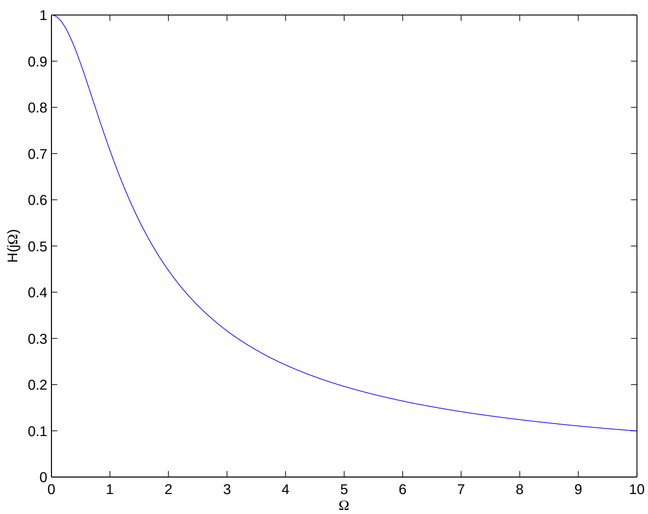
$$|H(j\Omega)| = \frac{1}{\sqrt{(j\Omega - \alpha)\cdot(-j\Omega - \alpha)}} = \frac{1}{\Omega^2 + \alpha^2}$$

By inserting $\alpha = -1$ we obtain:

$$|H(j\Omega)| = \frac{1}{\Omega^2 + 1}$$

The graphs of the frequency responses of the digital and analog filters are given below.





(d) We now truncate the impulse response to make it into a FIR filter. We choose the length of the filter to be N=4. The frequency response of the truncated filter is given by:

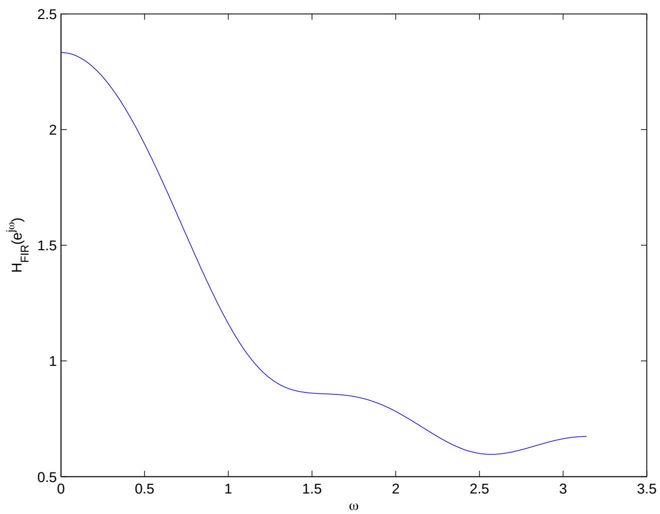
$$H_{FIR}(e^{j\omega}) = \sum_{n=0}^{4} e^{-1/2n} e^{-j\omega n} = \frac{1 - e^{(-1/2-j\omega)\cdot 5}}{1 - e^{-1/2-j\omega}}$$

The magnitude response squared is given by:

$$|H_{FIR}(e^{j\omega})|^2 = \frac{1 - e^{(-1/2-j\omega)\cdot 5}}{1 - e^{-1/2-j\omega}} \cdot \frac{1 - e^{(-1/2+j\omega)\cdot 5}}{1 - e^{-1/2+j\omega}} = \frac{1 - 2e^{-5/2}\cos(5\omega) + e^{-5}}{1 - 2e^{-1/2}\cos(\omega) + e^{-1}}$$

The magnitude response is given by:

$$|H_{FIR}(e^{j\omega})| = \frac{\sqrt{1 - 2e^{-5/2}\cos(5\omega) + e^{-5}}}{\sqrt{1 - 2e^{-1/2}\cos(\omega) + e^{-1}}}$$



By comparing the sketches of the frequency response of the IIR filter and the truncated FIR filter we see that they are somewhat similar, but not quite the same. The truncated filter has some ripples which is not present for the IIR filter. Note that truncating the filter can be obtained by multiplying the IIR filter by a rectangular pulse, with the duration of the rectangular pulse being equal to the length of the truncated filter. In the frequency domain this operation is equivalent to folding the frequency response of the IIR filter with a sinc function. This will result in a different frequency response of the truncated filter compared to the IIR filter. By increasing the length of the FIR filter, the frequency response gets more similar to the frequency response of the IIR-filter.

Problem II

(a) The unit sample response is found by applying a unit sample $\delta(n)$ on the input:

$$h(n) = \alpha h(n-1) + \beta\delta(n)$$

We see that the filter is causal since it is only dependent on previous input values. We try to find a couple of values of the unit sample response by applying the iterative method:

$$\begin{aligned} h(0) &= \beta \\ h(1) &= \alpha\beta \\ h(2) &= \alpha^2\beta \end{aligned}$$

We now see that the unit sample response is given by:

$$h(n) = \alpha^n \beta u(n)$$

We have used the unit step function $u(n)$ to indicate that $h(n) = 0$ for $n < 0$.

(b) We find the frequency response by taking the Fourier transform of $h(n)$.

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \beta\alpha^n e^{-j\omega n} = \beta \sum_{n=0}^{\infty} \left(\alpha e^{-j\omega}\right)^n = \frac{\beta}{1-\alpha e^{-j\omega}}$$

(c) $x(n)$ is an autoregressive process (AR) of first order $\left(AR(1)\right)$. The power spectral density $S_{XX}(\omega)$ is found by:

$$S_{XX}(\omega) = |H(e^{j\omega})|^2 S_{EE}(\omega) = \frac{\beta}{1-\alpha e^{-j\omega}} \cdot \frac{\beta}{1-\alpha e^{j\omega}} \sigma_E^2 = \frac{\sigma_E^2 \beta^2}{1-2\alpha\cos(\omega)+\alpha^2}$$

(d) The variance σ_X^2 can be found directly from the difference equation:

$$\sigma_X^2 = E\{x(n)x^*(n)\} = E\left\{\left(\alpha x(n-1)+\beta e(n)\right)\left(\alpha^* x^*(n-1)+\beta^* e^*(n)\right)\right\} = |\alpha|^2\sigma_X^2 + |\beta|^2\sigma_E^2$$

Since $x(n-1)$ cannot depend on $e(n)$. Solving for σ_X^2 we get:

$$\sigma_X^2 = \frac{\sigma_E^2|\beta|^2}{1-|\alpha|^2}$$

We have assumed the general case here that α and β might be complex.

(e) There are several ways to prove that the filter given by $y(n) = ax(n) + bx(n-1)$ is a whitening filter. One procedure is to solve directly from the difference equations. We insert the difference equation for $x(n)$ into the difference equation for $y(n)$:

$$y(n) = ax(n) + bx(n-1) = a\left(\alpha x(n-1) + \beta e(n)\right) + bx(n-1)$$

We see that in order for the output $y(n)$ to be equal to $e(n)$ these two equations need to be solved for the filter parameters:

$$a\beta = 1$$
$$a\alpha = -b$$

Which gives the solution $a = \frac{1}{\beta}$ and $b = -\frac{\alpha}{\beta}$. We have proved that for these filter parameters the output $y(n)$ is equal to $e(n)$, which is Gaussian, white noise.

(f) If we assume that $y(n)$ is quantized and then de-quantized, then we can write the de-quantized signal as:

$$y_q(n) = \frac{1}{\beta}x(n) - \frac{\alpha}{\beta}x(n-1) + q(n)$$

Where $q(n)$ is the quantization noise modeled as white noise. We have used the parameters $a = \frac{1}{\beta}$ and $b = -\frac{\alpha}{\beta}$ which was found in (e). $x(n)$ can be solved for:

$$x(n) = \beta y_q(n) + \alpha x(n-1) - q(n)$$

From this equation we see a reasonable reconstruction scheme:

$$\hat{x}(n) = \beta y_q(n) + \alpha\hat{x}(n-1)$$

$\hat{x}(n)$ is the estimated value of $x(n)$.

(g) We are going to choose the filter parameters such that the inputs to the quantizer have equal variances. The output of the whitening filter is written as before:

$$y(n) = ax(n) + bx(n-1)$$

We insert $x(n) = \alpha x(n-1) + \beta e(n)$ into the above equation to obtain:

$$y(n) = a\alpha x(n-1) + a\beta e(n) + bx(n-1)$$
$$y(n) = \Big(a\alpha + b\Big)x(n-1) + a\beta e(n)$$

Next we calculate the variance σ_Y^2. Note that $e(n)$ is zero mean, and because of this both $x(n)$ and $y(n)$ are also zero mean. This means that $\sigma_Y^2 = \mathcal{E}\{y(n)y^*(n)\}$. We further assume that all the signals are real signals, and therefore the variance is given by $\sigma_Y^2 = \mathcal{E}\{y(n)y(n)\}$:

$$\sigma_Y^2 = \mathcal{E}\Bigg\{\Big((a\alpha + b)x(n-1) + a\beta e(n)\Big)\Big((a\alpha + b)x(n-1) + a\beta e(n)\Big)\Bigg\}$$
$$= \Big(a\alpha + b\Big)^2 \sigma_X^2 + \alpha^2\beta^2\sigma_E^2$$

In order for the filter to be a whitening filter, y(n) should only consist of a scaled version of the white noise term $e(n)$. From $y(n) = \Big(a\alpha + b\Big)x(n-1) + a\beta e(n)$ we get the equation:

$$a\alpha + b = 0$$

Which means that $b = -a\alpha$. We have the additional condition that $y(n)$ should have the same variance as $x(n)$, $\sigma_Y^2 = \sigma_X^2$. From (d) we have that $\sigma_X^2 = \sigma_E^2\beta^2/(1-\alpha^2)$. This gives us the equation:

$$\frac{\sigma_E^2\beta^2}{1-\alpha^2} = a^2\beta^2\sigma_E^2$$

Solving for the filter parameters we get $a = \frac{1}{\sqrt{1-\alpha^2}}$ and by using $b = -a\alpha$ we also get $b = -\frac{\alpha}{\sqrt{1-\alpha^2}}$ We now want to compare the quantization noise when quantizing $x(n)$ directly to the case when we quantize $y(n)$. The variance of the signals on the input to the quantizers are the same, and the signals have the same pdf. The only difference is that $x(n)$ is a correlated signal, while $y(n)$ is uncorrelated/white. The quantization noise variance on the output of the de-quantizer can be denoted by σ_q^2 and it is same for both cases. But when we are using a whitening filter we have to use a reconstruction filter at the receiver. Which means that the quantization noise passes through this filter. We have to find this reconstruction filter first. Using the filter parameters we just found we express $y(n)$ as:

$$y(n) = \frac{x(n)}{\sqrt{1-\alpha^2}} - \frac{\alpha}{\sqrt{1-\alpha^2}}x(n-1)$$

Which suggests that we reconstruct the signal by:

$$\hat{x}(n) = \sqrt{1-\alpha^2}y_q(n) + \alpha\hat{x}(n-1)$$

To find the quantization noise on the output of this filter which we denote by $z(n)$ we assume that only the quantization noise is applied at the input of the filter. We then get this difference equation for the noise:

$$z(n) = \sqrt{1-\alpha^2}q(n) + \alpha\hat{z}(n-1)$$

We now find the variance of this noise:

$$\sigma_Z^2 = \mathcal{E}\{z(n)z(n)\} = \mathcal{E}\Big\{\Big(\sqrt{1-\alpha^2}q(n) + \alpha\hat{z}(n-1)\Big)\cdot\Big(\sqrt{1-\alpha^2}q(n) + \alpha\hat{z}(n-1)\Big)\Big\}$$

$$= \Big(1-\alpha^2\Big)\sigma_q^2 + \alpha^2\sigma_Z^2$$

With the result that $\sigma_Z^2 = \sigma_q^2$. This means that the noise variance when quantizing $x(n)$ directly gives the same same quantization noise variance as when quantizing the white signal $y(n)$.

(h) Even though the quantization noise variance is the same for both cases, this does not mean that the noise spectra are the same. When quantizing $x(n)$ directly the quantization noise is white and the spectrum is given by:

$$S_{qq}(\omega) = \sigma_q^2$$

To find the noise spectrum for the other case we first take the Fourier transform of $z(n)$:

$$F\{z(n)\} = F\Big\{\sqrt{1-\alpha^2}q(n) + \alpha\hat{z}(n-1)\Big\} = \sqrt{1-\alpha^2}Q(e^{j\omega}) + \alpha Z(e^{j\omega})e^{-j\omega}$$

$$Z(e^{j\omega}) = \frac{\sqrt{1-\alpha^2}}{1-\alpha e^{-j\omega}}Q(e^{j\omega})$$

We have used the difference equation for $z(n)$ which we found in (g). From this we find the spectral density:

$$S_{ZZ}(\omega) = E\Big\{Z(e^{j\omega})Z^*(e^{j\omega})\Big\} = \frac{(1-\alpha^2)\sigma_q^2}{\Big(1-\alpha e^{-j\omega}\Big)\Big(1-\alpha e^{j\omega}\Big)} = \frac{(1-\alpha^2)\sigma_q^2}{1-2\alpha\cos(\omega)+\alpha^2}$$

We see that $S_{ZZ}(\omega)$ has the exact same spectral shape as $S_{XX}(\omega)$. While the quantization noise when quantizing $x(n)$ directly is white.

Problem III

(a) A probability density function (pdf) $f_X(x)$ has to satisfy $\int_{-\infty}^{\infty} f_X(x)dx = 1$. If the pdf is symmetric around $x=0$ then we need only to integrate half of the interval and multiply by 2, i.e $2\int_0^{\infty} f_X(x)dx = 1$. This is the case for our pdf :

$$2\int_0^{\infty} f_X(x)dx = 2\int_0^{\infty} Ae^{-\alpha x}dx = 2A\Big[-\frac{1}{\alpha}e^{-\alpha x}\Big]_0^{\infty} = 2A/\alpha$$

Since the integral of the pdf from $x = -\infty$ to ∞ should be equal to 1, we get the equation:

$$2A/\alpha = 1$$

Which leads to $A = \alpha/2$. We also know that the variance of x should be equal to 1. Since the pdf is symmetric around $x = 0$ we know that the mean is zero, i.e $\mu_X = 0$. This simplifies the calculation of the variance:

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 f_X(x)dx = 2\int_0^{\infty} x^2 f_X(x)dx = 2\alpha/2 \cdot \int_0^{\infty} x^2 e^{-\alpha x}dx = \alpha\frac{2!}{\alpha^3} = \frac{2}{\alpha^2}$$

Because of the condition $\sigma_X^2 = 1$, we get $\alpha = \sqrt{2}$.

(b) The quantization noise for an uniform quantizer with stepsize $\Delta << 1$ is given by:

$$\sigma_q^2 = \Delta^2/12$$

This formula applies for uniform quantization independent of which pdf the signal follows.

(c) The probability of the signal lying inside the interval $[(k-1)\Delta, k\Delta]$ is given by:

$$\begin{aligned} P_k &= \int_{(k-1)\Delta}^{k\Delta} f_X(x)dx = \int_{(k-1)\Delta}^{k\Delta} \frac{1}{\sqrt{2}}e^{-\sqrt{2}x}dx = \frac{1}{\sqrt{2}}\Big[-\frac{1}{\sqrt{2}}e^{-\sqrt{2}x}\Big]_{(k-1)\Delta}^{k\Delta} \\ &= \frac{1}{2}e^{-\sqrt{2}k\Delta}\left(e^{\sqrt{2}\Delta}-1\right) \end{aligned}$$

Note that we have found the probability for signals lying in positive intervals, but $P_k = P_{-k}$. This is because the pdf is symmetric around $x = 0$. Now we are going to use these probabilities to find the smallest rate R for the quantized source. The smallest rate is given by the entropy. The entropy is found according to this formula:

$$\begin{aligned} H &= -\sum_{k=-\infty}^{\infty} P_k \log_2(P_k) = -2\cdot\sum_{k=1}^{\infty} P_k\log_2(P_k) \\ &= -2\cdot\sum_{k=1}^{\infty} 1/2e^{-\sqrt{2}k\Delta}(e^{\sqrt{2}\Delta}-1)\log_2\left(1/2e^{-\sqrt{2}k\Delta}(e^{\sqrt{2}\Delta}-1)\right) \\ &= -2\cdot\sum_{k=1}^{\infty} 1/2e^{-\sqrt{2}k\Delta}(e^{\sqrt{2}\Delta}-1)\cdot\left(\log_2(1/2)+\log_2(e^{\sqrt{2}\Delta}-1)+\log_2(e^{-\sqrt{2}k\Delta})\right) \end{aligned}$$

Now we realize that $\sum_{k=-\infty}^{\infty} P_k \cdot C = C$, where C is assumed to be a constant. And therefore:

$$-2\cdot\sum_{k=1}^{\infty} 1/2e^{-\sqrt{2}k\Delta}(e^{\sqrt{2}\Delta}-1)\cdot\left(\log_2(1/2)+\log_2(e^{\sqrt{2}\Delta}-1)\right) = -\log_2(1/2)-\log_2(e^{\sqrt{2}\Delta}-1)$$

This is used to simplify some of the terms of the entropy expression:

$$
\begin{aligned}
H &= -\log_2(1/2) - \log_2(e^{\sqrt{2}\Delta} - 1) - 2 \cdot \sum_{k=1}^{\infty} 1/2 e^{-\sqrt{2}k\Delta}(e^{\sqrt{2}\Delta} - 1) \cdot \log_2(e^{-\sqrt{2}k\Delta}) \\
&= 1 - \log_2(e^{\sqrt{2}\Delta} - 1) - \sum_{k=1}^{\infty} e^{-\sqrt{2}k\Delta}(e^{\sqrt{2}\Delta} - 1) \cdot -\sqrt{2}k\Delta \log_2(e) \\
&= 1 - \log_2(e^{\sqrt{2}\Delta} - 1) + \sqrt{2}\Delta \log_2(e)(e^{\sqrt{2}\Delta} - 1)\frac{e^{-\sqrt{2}\Delta}}{(1 - e^{-\sqrt{2}\Delta})^2} \\
&= 1 - \log_2(e^{\sqrt{2}\Delta} - 1) + \sqrt{2}\Delta \log_2(e) e^{\sqrt{2}\Delta}(1 - e^{-\sqrt{2}\Delta})\frac{e^{-\sqrt{2}\Delta}}{(1 - e^{-\sqrt{2}\Delta})^2} \\
&= 1 - \log_2(e^{\sqrt{2}\Delta} - 1) + \frac{\sqrt{2}\Delta \log_2(e)}{1 - e^{-\sqrt{2}\Delta}}
\end{aligned}
$$

(d) Now we consider that we represent the source samples by a rate $R = H$. The source has a bandwidth W which means that we sample the source $2W$ times per second. The source therefore produces $2WR$ bits per second. The channel capacity when we occupy a frequency bandwidth of B Hz is given by $C = B \log_2(1 + SNR)$. By setting the expression for the number of bits the source produces, equal to the channel capacity, we are able to find the minimum SNR to achieve error free transmission:

$$
\begin{aligned}
2WR &= B \log_2(1 + SNR) \\
\frac{2WR}{B} &= \log_2(1 + SNR) \\
SNR &= 2^{2WR/B} - 1 \\
SNR &= 2^{4R} - 1
\end{aligned}
$$

In the last step we have inserted $W = 4$ khz and $B = 2$ khz.