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TTT4110 Information and Signal Theory
Solution to exam August 9, 2007

Problem 1

(a) The relation between the output signal $y(n)$ and the input signal $x(n)$ expressed by the impulse response $h(n)$ is given by the convolution:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) \\ &= \sum_{k=-\infty}^{\infty} h(n-k)x(k) \\ &= h(n) * x(n) = x(n) * h(n). \end{aligned}$$

(b) The output signal from a LSI filter with unit sample response $h(n)$ where the input signal is $x(n) = e^{jwn}$:

$$\begin{aligned} y(n) &= \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{jw(n-k)} \\ &= e^{jwn} \sum_{k=-\infty}^{\infty} h(k)e^{-jwk} = x(n) \sum_{k=-\infty}^{\infty} h(k)e^{-jwk} \\ &= x(n)H(e^{jw}). \end{aligned}$$

The last equality, however, is only valid when $x(n) = e^{jwn}$.

The relation between the unit sample response and the frequency response is thus given by

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} h(k)e^{-jwk}.$$

(c) The input-output relation of the system is given by:

$$\begin{aligned} y(n) &= \alpha(x(n) * h(n)) + (v(n) * g(n))\beta \\ &= \alpha(x(n) * h(n)) + (h(n) * x(n) * g(n))\beta \\ &= (\alpha + g(n)\beta) * h(n) * x(n). \end{aligned}$$

As we can see from the result in a), the unit sample response of the total system is

$$h_{tot}(n) = (\alpha + g(n)\beta) * h(n).$$

(d) The frequency response of the total system with impulse response $h_{tot}(n)$ is given by

$$\begin{aligned} H_{tot}(e^{j\omega}) = \mathcal{F}\left\{h_{tot}(n)\right\} &= \mathcal{F}\left\{(\alpha + g(n)\beta)\right\} \cdot \mathcal{F}\left\{h(n)\right\} \\ &= (\alpha + \beta G(e^{jw}))H(e^{jw}). \end{aligned}$$

(e) The frequency response of the two LSI filters can be found by taking the Fourier transforms of the two impulse responses respectively.

$$H(e^{jw}) = \mathcal{F}\left\{a_0\delta(n) + a_1\delta(n-1)\right\} = a_0 + a_1 e^{-jw},$$

$$G(e^{jw}) = \mathcal{F}\left\{\gamma^n u(n)\right\} = \sum_{n=0}^{\infty} \gamma^n e^{-jwn} = \frac{1}{1-\gamma e^{-jw}}.$$

The total frequency response is now given by:

$$H_{tot}(e^{j\omega}) = (\alpha + \beta G(e^{jw}))H(e^{jw}) = (\alpha + \frac{\beta}{1-\gamma e^{-jw}})(a_0 + a_1 e^{-jw}).$$

BIBO-stability (Bounded-Input-Bounded Output) means that a bounded input results in a bounded output. BIBO-stability is guaranteed if the sum of the magnitude of the impulse response is bounded: $\sum_{n=0}^{\infty} |h_{tot}(n)| < \infty$.

$h(n)$ fulfills the BIBO-stability criterion if $\gamma < 1$. If this is the case, the series converge for all w.

Problem 2

(a) The Discrete Time Fourier Transform (DTFT):

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jwn}.$$

The DTFT applies for all w and can transform infinite discrete-time signals.

The Discrete Fourier Transform (DFT):

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn}.$$

The DFT can thus transform a finite length discrete time signal. The result is a sequence of the same length as the input signal. If an infinite length signal has non-zero values in $n \in [0, N-1]$, then

$$X(e^{jw}) = X(k).$$

In other words, DFT gives exact values of the DTFT in the points $w = \frac{2\pi}{N}k$.

(b) The inverse DFT is expressed by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j\frac{2\pi}{N}kn}.$$

To prove this we insert the expression for $X(k)$:

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(l)e^{-j\frac{2\pi}{N}kl}e^{j\frac{2\pi}{N}kn} \quad = \quad \frac{1}{N}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} x(l)e^{-j\frac{2\pi}{N}k(n-l)}$$

We observe that the sum

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(l-n)} = \delta(l-n) = \begin{cases} 1 & \text{for } l = n \\ 0 & \text{otherwise,} \end{cases}$$

that is, it only has a non-zero value when $l = n$, because $e^{-j2\pi n} = 1$ when n is an integer.
The expression for the IDFT now becomes

$$\sum_{l=0}^{N-1} x(l)\delta(l-n) = x(n). \qquad\qquad \text{q.e.d}$$

(c) The DTF of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} e^{j\omega_0 n}e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} e^{j(\omega_0 - \frac{2\pi}{N}k)n}.$$

If $\omega_0 = \frac{2\pi}{N}l$ then

$$X(k) = \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}(l-k)n} = \frac{1 - e^{j2\pi(l-k)}}{1 - e^{j\frac{2\pi}{N}(l-k)}}.$$

In the case of $l = n$ we can see that both the numerator and the denominator are equal to zero. However, if we inspect the sum directly, we observe the fact that

$$X(k) = \sum_{n=0}^{N-1} 1 = N.$$

Other values of k will only make the numerator equal to zero, not the denominator. The DFT of $x(n)$ is therefore,

$$X(k) \quad = \quad \begin{cases} N & \text{for k=l} \\ 0 & \text{otherwise.} \end{cases}$$

We can easily see that other values of ω_0 will not satisfy the condition of the numerator being equal to zero.

(d) To prove the relation

$$X(k) = X^*(N-k),$$

we use the DFT transform

$$\begin{aligned} X(N-k) &= \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(N-k)n} \\ &= \sum_{n=0}^{N-1} x(n) e^{j\frac{2\pi}{N}kn}, \end{aligned}$$

since $e^{-j2\pi n} = 1$ for all n (integer). Since $x(n)$ is real, the complex conjugate of the DFT becomes

$$X^*(N-k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = X(k) \qquad \text{q.e.d}$$

Solving the above equation for $k = 0$ and $k = \frac{N}{2}$, we see that both $X(0)$ and $X(N/2)$ are real.

(e) To be able to exactly represent a discrete signal with finite length N by series expansion, the N basis functions have to be linearly independent of each other. This means that any one basis function cannot be expressed by a linear combination of the other basis functions.

Signals of infinite length have to be represented by a infinite number of basis functions. However, it is important to notice that an infinite number of terms does not necessarily imply uniform convergence.

Problem 3

(a) The quantization noise can be approximated by

$$\sigma_Q^2 = \frac{\Delta^2}{12},$$

where Δ is the quantization interval. This approximation is exact if the pdf is constant or any linear function over each quantization interval. In this case it is constant over each interval, which implies that the exact value of the quantization noise is

$$\sigma_Q^2 = \frac{(\frac{1}{8})^2}{12} = \frac{1}{768}.$$

(b) The power of the reconstructed signal is given by

$$P = \sum_{i=1}^{8} x_i^2 p_i.$$

The probabilities are

$$\begin{aligned} p_1 = p_8 &= \frac{1}{4}, \\ p_2 = p_7 &= \frac{1}{8}, \\ p_3 = p_4 = p_5 = p_6 &= \frac{1}{16}. \end{aligned}$$

Then the power can be found as

$$P = 2\left[\left(\frac{7}{16}\right)^2 \frac{2}{8} + \left(\frac{5}{16}\right)^2 \frac{1}{8} + \left(\frac{3}{16}\right)^2 \frac{1}{16} + \left(\frac{1}{16}\right)^2 \frac{1}{16}\right] = \frac{1}{8}.$$

(c) The lowest possible average bit rate is given by the entropy

$$\begin{aligned} H &= -\sum_{i=1}^{8} p_i \log_2 p_i = -2\left(\frac{1}{4}\log_2\frac{1}{4} + \frac{1}{8}\log_2\frac{1}{8} + 2\frac{1}{16}\log_2\frac{1}{16}\right) \\ &= 2\left(\frac{2}{4} + \frac{3}{8} + \frac{4}{8}\right) = \frac{22}{8} \text{ bits/symbol.} \end{aligned}$$

(d) The necessary channel signal-to-noise ratio to assure error-free transmission can be found by solving for the CSNR from the channel capacity equation and setting the channel capacity equal to the entropy:

$$C = H = \frac{1}{2}\log_2(1 + CSNR) \quad \rightarrow \quad CSNR = 2^{2H} - 1 = 2^{\frac{22}{4}} - 1 = 44.25.$$

(e) The lowest possible symbol energy occurs when we transmit the most probable symbols on the channel symbols with the smallest amplitude. Average symbol energy per sample then becomes:

$$P = 2\left[\left(\frac{1}{16}\right)^2 \frac{1}{4} + \left(\frac{3}{16}\right)^2 \frac{1}{8} + \left(\frac{5}{16}\right)^2 \frac{1}{16} + \left(\frac{7}{16}\right)^2 \frac{1}{16}\right] = \frac{3}{64} = 0,0469.$$