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TTT4110 Information and Signal Theory
Solution to exam spring 2007

Problem 1

(a) The energy of $x(n)$ is given by

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \sum_{n=0}^{\infty} |a^n|^2 = \sum_{n=0}^{\infty} |a|^{2n}.$$

If $|a|$ is < 1, then the energy is given by:

$$E = \frac{1}{1-|a|^2}.$$

If $|a|$ is ≥ 1 the sum diverges if the signal $x(n)$ is infinitely long.

(b) The signal energy up to n=N-1 (N terms) is given by

$$E = \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} |a^n|^2 = \sum_{n=0}^{N-1} |a|^{2n} = \frac{1-|a|^{2N}}{1-|a|^2}.$$

where $N \in \mathbb{N}$ and $a \in \mathbb{C}$.

(c) The impulse response of a linear, time-invariant systems satisfies

$$h(n) = 0 \qquad \text{for } n < 0$$

if and only if the system is also causal.
In this case the system is causal since the impulse response h(n) is zero valued for all $n < 0$.

BIBO-stability (Bounded-input-bounded output), means that a bounded input results in a bounded output.

BIBO-stability is guaranteed if the sum of the magnitude of the impulse response samples is bounded: $\sum_{n=-\infty}^{\infty} |h(n)| < \infty$.
In our case $h(n)$ fulfills the BIBO-stability criterion:

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=-\infty}^{\infty} |b^n| u(n) = \frac{1}{1-|b|} < \infty \qquad \text{if } |b| < 1.$$

(d) The frequency response of the filter with impulse response

$$h(n) = b^n \cdot u(n), \qquad |b| < 1,$$

is given by

$$H(e^{j\omega}) = \mathcal{F}\{h(n)\} = \sum_{n=-\infty}^{\infty} b^n u(n) e^{-j\omega n} = \sum_{n=0}^{\infty} (be^{-j\omega})^n = \frac{1}{1-be^{-j\omega}}.$$

(e) The output signal can be found by the convolution

$$\begin{aligned} y(n) &= x(n) * h(n) = \sum_{k=\infty}^{\infty} a^k u(k) b^{n-k} u(n-k) \\ &= b^n \sum_{k=0}^{n} (ab^{-1})^k = b^n \frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}}. \end{aligned}$$

(f) From the previous expression we seem to run into trouble when a=b. But if we take a look at the summation in the case of $a = b$,

$$\begin{aligned} y(n) &= b^n \sum_{k=0}^{n} (ab^{-1})^k \\ &= b^n \sum_{k=0}^{n} (1)^k = b^n (n+1), \end{aligned}$$

it turns out that in this case there are n+1 terms of value 1 which sums to n+1.

(g) When performing convolution one of the signals is reversed in time and shifted relative to the other signal. For each shift an output signal is calculated by multiplying the two sequences and summing the result. In this case one of the signals is a reversed version of the other. When reversing the signal for convolution both will have the same orientation. The maximum value of the output signal is when the filter and the input signal overlap completely:

$$y_M = \sum_{k=-\infty}^{\infty} \tilde{x}(N-k)\tilde{x}(N-k) = \sum_{k=0}^{N-1} a^{2k} = \frac{1-|a|^{2N}}{1-|a|^2}.$$

As we can see from the equation, this is equal to the signal energy.

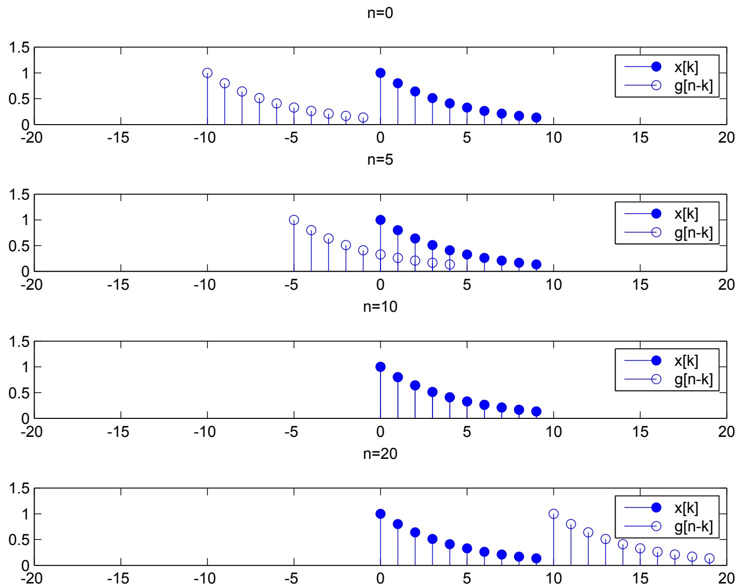

Figure 1: Convolution with matched filter

(h) The output power spectrum of an LSI filter can be found by

$$S_{YY}(\omega) = |G(e^{j\omega})|^2 S_{XX}(\omega).$$

In this case the input is white noise

$$S_{XX}(\omega) = S_{NN}(\omega) = \frac{N_0}{2}.$$

We have to find frequency response and the magnitude of the matched filter.

$$\begin{aligned} G(e^{j\omega}) = \mathcal{F}\{g(n)\} &= \sum_{n=-\infty}^{\infty} a^n (u(n) - u(n-N)) e^{-j\omega n} \\ &= \sum_{n=0}^{N-1} (ae^{-j\omega})^n = \frac{1-(ae^{-j\omega})^N}{1-ae^{-j\omega}}. \end{aligned}$$

$$\begin{aligned} |G(e^{j\omega})|^2 &= G(e^{j\omega}) \cdot G(e^{j\omega})^* \\ &= \frac{1-(ae^{-j\omega})^N}{1-ae^{-j\omega}} \cdot \frac{1-(ae^{j\omega})^N}{1-ae^{j\omega}} \\ &= \frac{1+a^{2N}-2a^N\cos(\omega N)}{1+a^2-2a\cos(\omega)}. \end{aligned}$$

This gives the power spectral density of the output noise component,

$$\begin{aligned} S_{DD}(\omega) &= |G(e^{j\omega})|^2 S_{XX}(\omega) \\ &= \frac{N_0}{2} \frac{1+a^{2N}-2a^N\cos(\omega N)}{1+a^2-2a\cos(\omega)}. \end{aligned}$$

The variance can be found by solving the integral,

$$\begin{aligned}\sigma_D^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{DD}(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 \frac{N_0}{2} d\omega.\end{aligned}$$

(i) Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

helps us solve the integral:

$$\sigma_D^2 = \frac{N_0}{2}\frac{1}{2\pi}\int_{-\pi}^{\pi} |G(e^{j\omega})|^2 d\omega = \frac{N_0}{2}\sum_{n=-\infty}^{\infty} |g(n)|^2 = \frac{N_0}{2}\cdot\frac{1-|a|^{2N}}{1-|a|^2}.$$

($\sum_{n=-\infty}^{\infty} |g(n)|^2$ is equal to y_M found in problem 1g.)

(j) The input signal $s(n) = \tilde{x}(n) + n(n)$ is transmitted through the matched filter. The output from the filter is given by

$$y_s(n) = s(n) * g(n) = \tilde{x}(n) * g(n) + n(n) * g(n).$$

Since the signal and the noise are uncorrelated the variance of the output signal is given by,

$$\sigma_{y_s}^2 = E[y_s(n)] = E[(\tilde{x}(n) * g(n))^2] + E[(n(n) * g(n))^2],$$

where

$$E[(\tilde{x}(n) * g(n))^2] = \left(\frac{1-|a|^{2N}}{1-|a|^2}\right)^2 = E_g^2,$$

as found in 1g, and

$$E[(n(n) * g(n))^2] = \sigma_D^2 = \frac{N_0}{2}E_g.$$

The signal-to-noise ratio is therefore,

$$SNR = \frac{E_g^2}{E_g\frac{N_0}{2}} = \frac{E_g}{N_0/2}.$$

As we can see from the expression the SNR does not depend and the shape of the recieved pulse when a matched filter is used.

Problem 2

(a) The values for α and β can be found by solving the integral,

$$\begin{aligned}\sigma_X^2 &= \int_{-\alpha}^{\alpha} \beta x^2 dx = 2\beta \int_0^{\alpha} x^2 dx \\ &= 2\beta \left[\frac{1}{3}x^3\right]_0^{\alpha} = \frac{2}{3}\beta\alpha^3,\end{aligned}$$

and the setting area of the pdf equal to one:

$$2\alpha\beta = 1 \rightarrow \beta = \frac{1}{2\alpha}.$$

Solving the equation set gives the following values for α and β:

$$\begin{aligned}\alpha &= \sqrt{3\sigma_X^2}, \\ \beta &= \frac{1}{\sqrt{12\sigma_X^2}}.\end{aligned}$$

(b) The quantization noise can be approximated by the equation,

$$\sigma_Q^2 = \frac{\Delta^2}{12},$$

where Δ in this case is,

$$\Delta = \frac{2\alpha}{L}.$$

The quantization noise is therefore,

$$\sigma_Q^2 = \frac{\left(\frac{2\alpha}{L}\right)^2}{12} = \frac{\alpha^2}{3L^2} = \frac{\sigma_X^2}{L^2}.$$

The approximate formula is exact whenever the pdf is linear in each quantization interval. In this case the pdf is a constant in all quantization intervals and therefore the approximate formula is exact.

(c) The signal-to-noise ratio is given by,

$$SNR = \frac{\sigma_x^2}{\sigma_Q^2} = \frac{\sigma_X^2}{\frac{\sigma_X^2}{L^2}} = L^2.$$

(d) Since the pdf is flat and the signal is quantized by a uniform quantizer, the probabilities, p_n, will all be equal to $\frac{1}{L}$. The entropy is therefore,

$$H = \sum_{n=1}^{L} p_n \log_2 \left\{\frac{1}{p_n}\right\} = \sum_{n=1}^{L} \frac{1}{L} \log_2 L = \log_2 L.$$

(e) First we calculate the differential entropy

$$\begin{aligned} h(X) &= -\int_{\infty}^{\infty} f_X(x)\log_2(f_X(x))dx = -\int_{-\alpha}^{\alpha} \beta\log_2(\beta)dx \\ &= -2\alpha\beta\log_2\beta = \log_2\left(\sqrt{12\sigma_X^2}\right) = \log_2(2\sqrt{3}\sigma_X). \end{aligned}$$

Using the approximation formula and the differential entropy, the entropy of the quantized signal is

$$\begin{aligned} H(X) &= h(X) - \log_2(\Delta) = \log_2(2\sqrt{3}\sigma_X) - \log_2(\Delta) \\ &= \log_2\left(\frac{2\sqrt{3}\sigma_X}{\Delta}\right) = \log_2\left(\frac{2\sqrt{3}\sigma_X L}{2\alpha}\right) \\ &= \log_2\left(\frac{2\sqrt{3}\sigma_X L}{2\sqrt{3}\sigma_X}\right) = \log_2 L. \end{aligned}$$

For this approximation to be valid, in general, the value of Δ must be very small, in other words the pdf must be constant over the subinterval. In this case this is true regardless of the value of L, which is why the approximate formula gives the same result as the exact formula.

(f) According to the Nyquist-Shannon sampling theorem perfect reconstruction requires that the signal is bandlimited and that the sampling frequency is greater than twice the bandwidth. With ideal bandlimiting the lowest possible rate is therefore

$$F_s = 2B.$$

To avoid Inter Symbol Interference (ISI) the channel must be an ideal Nyquist channel. The Nyquist criterion in the frequency domain can be written

$$\frac{1}{T}\sum_{m=-\infty}^{\infty} G(j(\Omega + m\frac{2\pi}{T}) = 1.$$

The smallest possible bandwidth is obtained when the channel is an ideal lowpass channel with constant amplification and abrupt transition from the passband to the stopband. In this case we can transmit 2 symbols per second per Hz. This means that for a noiseless channel, a signal of bandwidth B can be transmitted over this channel error free when ideally sampled. When the channel bandwidth is smaller, as given here, some distortion will be present as revealed in the following.

(g) The entropy of a source is the minimum average number of bits per symbol the source can be represented by. We use the entropy of the quantized source found earlier in the exercise.

$$H(X) = \log_2 L.$$

We use the Nyquist theorem which states that the maximum B symbols can be transmitted on a channel with bandwidth $\frac{B}{2}$ per second. This means that we can transmit 2 symbols with entropy $H(X)$ on the channel per second. The minimum number of levels, M, is therefore,

$$M = 2^{2H(X)} = 2^{2\log_2 L} = L^2.$$

(h) The channel signal-to-noise-ratio is found by setting the channel capacity equal to the source entropy:

$$\begin{aligned} C = \frac{B}{2}\log_2(1+CSNR) &= 2BH = 2B\log_2 L. \\ \log_2(1+CSNR) &= \log_2(L^4). \\ CSNR &= L^4 - 1. \end{aligned}$$

(i) We found in c) that $SNR = L^2$. Inserting for L^4 in the result in h), we find

$$CSNR = SNR^2 - 1,$$

or solved with respect to SNR:

$$SNR = \sqrt{CSNR + 1}.$$

(j) In a correlated signal there are statistical dependencies between the samples. We try to remove this by decomposing the signal into components that are uncorrelated. This can be done in filter-banks/transforms, or by prediction as in DPCM system. In the first case the samples in different bands are uncorrelated provided the filters separate the frequency bands entirely (ideal filters), and the signals in each band become uncorrelated if the bands are narrow enough (implying that the power spectral density inside each band is constant). In the latter case the predictor tries to whiten the signal. If this is successful, the prediction error is white, which is equivalent to an uncorrelated signal. Thus, transmitting the prediction error instead of the actual signal will improve the system's total gain. Actually, this is obtained only if we use closed loop DPCM. For open loop DPCM the signal and noise are amplified in the same way in the receiver, for which case full whitening will have no effect on the obtained SNR as compared to no prediction at all. Half-whitening will, however, give a prediction gain of half the prediction gain, measured in dB, obtainable in closed-loop DPCM.