Norwegian University of Science and Technology
Department of Electronics and Telecommunications

TTT4120 Digital Signal Processing
Suggested Solution to Exam
Fall 2008

Problem 1

(a) The input and the input-output relation can be defined by

$$x(n) = \sum_{i=1}^{N} a_i x_i(n)$$
$$y_i(n) = H[x_i(n)]$$

- Linearity:

$$y(n) = H[x(n)] = H\left[\sum_{i=1}^{N} a_i x_i(n)\right] = \sum_{i=1}^{N} a_i H[x_i(n)] = \sum_{i=1}^{N} a_i y_i(n)$$

- Stability:

$$|x(n)| \leq \infty \quad n = -\infty, \infty \Longrightarrow$$
$$|y(n)| \leq \infty \quad n = -\infty, \infty$$

- Causality: $y(n)$ independent of $x(n+m)$, $m > 0$.
- Time invariance:

$$y(n) = H[x(n)] \Longrightarrow y(n-k) = H[x(n-k)], \quad n, k = -\infty, \infty$$

(b) A Linear time-invariant filter is called a Linar Time Invariant (LTI) filter. As consequence of the linearity and the time-invariance properties of the system, the response of the system to an arbitary input signal can be expressed in terms of the unit sample response.

The effect of the causality property on the unit sample response can be defined by

$$h(n) = 0 \qquad \text{for n<0}$$

Region of convergence:

$$\text{ROC:} \quad |H(z)| < \infty \Leftrightarrow |z| > |\alpha| \quad \text{where} \quad |\alpha| < 1 \Rightarrow z = e^{jw} \in \text{ROC}$$

(c) The unit sample response $h_1(n)$ and $h_2(n)$ can be found from the difference equations by setting $y(n) = h(n)$ and $x(n) = \delta(n)$

$$h_1(n) = b_0\delta(n) + b_1\delta(n-1) + b_2\delta(n-2)$$

Finding $h_1(n)$ for different values of n, gives us the unit sample response

$$\begin{aligned} h_1(0) &= b_0 \\ h_1(1) &= b_1 \\ h_1(2) &= b_2 \\ h_1(n) &= 0 \qquad \text{for n<0 and n>2} \end{aligned}$$

Similary we find $h_2(n)$

$$h_2(n) = ah_2(n-1) + b_0\delta(n)$$

Inserting for different values of n gives us the unit sample response

$$\begin{aligned} h_2(0) &= b_0 \\ h_2(1) &= ab_0 \\ h_2(2) &= a^2b_0 \\ &\vdots \\ h_2(n) &= a^nb_0 \qquad n \geq 0 \end{aligned}$$

The transfer functions $H_1(z)$ and $H_2(z)$ can be found by taking the Z transform of the unit sample responses.

$$\begin{aligned} H_1(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} \\ &= b_0 + b_1z^{-1} + b_2z^{-2} \end{aligned}$$

$$\begin{aligned} H_2(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} \\ &= \sum_{n=0}^{\infty} a^nb_0z^{-n} \\ &= \frac{b_0}{1-az^{-1}} \end{aligned}$$

ROC $H_1(z)$: Intire z-plane (except z=0)
ROC $H_2(z)$: $|z| > a \Rightarrow |a| < 1$

(d) The autocorrelation of a finite signal can be found by solving the equation

$$r_{hh}(m) = \sum_{n=0}^{N-|m|-1} h(n)h(n+m)$$

where N is the signal length.

$$\begin{aligned}
r_{h_1h_1}(0) &= \sum_{n=0}^{2} h_1^2(n) = b_0^2 + b_1^2 + b_2^2 \\
r_{h_1h_1}(1) &= \sum_{n=0}^{1} h_1(n)h_1(n+1) = b_0b_1 + b_1b_2 \\
r_{h_1h_1}(2) &= \sum_{n=0}^{0} h_1(n)h_1(n+2) = b_0b_2 \\
r_{h_1h_1}(m) &= 0 \qquad\quad m > 2
\end{aligned}$$

The autocorrelation of an infinite signal can be found by solving the equation

$$r_{hh}(m) = \sum_{n=0}^{\infty} h(n)h(n+m)$$

$$\begin{aligned}
r_{h_2h_2}(m) &= \sum_{n=0}^{\infty} h_2(n)h_2(n+m) \\
&= \sum_{n=0}^{\infty} b_0a^n b_0 a^{n+m} \\
&= \frac{b_0^2}{1-a^2}a^m \qquad\quad m \geq 0
\end{aligned}$$

Problem 2

(a) We find the transferfunction of the digital filter by inserting $s = \left(\frac{1-z^{-1}}{1+z^{-1}}\right)$ into the formula for the transfer function of the analog filter. We start with the first term:

$$\begin{aligned}
H_{a1}(s) &= \frac{s+1}{s+\frac{1}{3}} \\
&\Rightarrow \frac{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\frac{1}{3}} \\
&= \frac{1-z^{-1}+1+z^{-1}}{1-z^{-1}+\frac{1}{3}\left(1+z^{-1}\right)} \\
&= \frac{2}{\frac{4}{3}-\frac{2}{3}z^{-1}} \\
&= \frac{3}{2}\frac{1}{1-\frac{1}{2}z^{-1}}
\end{aligned}$$

Which is the same as $\frac{3}{2}H_1(z)$. Similarly for the second term:

$$\begin{aligned}
\frac{3}{4}H_{a2}(s) &= \frac{3}{4}\frac{s+1}{s+\frac{1}{2}} \\
&\Rightarrow \frac{3}{4}\frac{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1}{\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+\frac{1}{2}} \\
&= \frac{3}{4}\frac{1-z^{-1}+1+z^{-1}}{1-z^{-1}+\frac{1}{2}\left(1+z^{-1}\right)} \\
&= \frac{6}{6-2z^{-1}} \\
&= \frac{1}{1-\frac{1}{3}z^{-1}}
\end{aligned}$$

Since this is equal to $H_2(z)$, we have proven that the transfer function of the digital filter is $H(z) = \frac{3}{2}\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{3}z^{-1}}$. Further, we find the impulse response by finding the inverse Z-transform of the transfer function. By using the relation $\mathcal{Z}\{a^n u(n)\} = \frac{1}{1-az^{-1}} \quad |a| < 1$, we get:

$$h(n) = \frac{3}{2}\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n$$

(b) By combining the two terms into one term, we get:

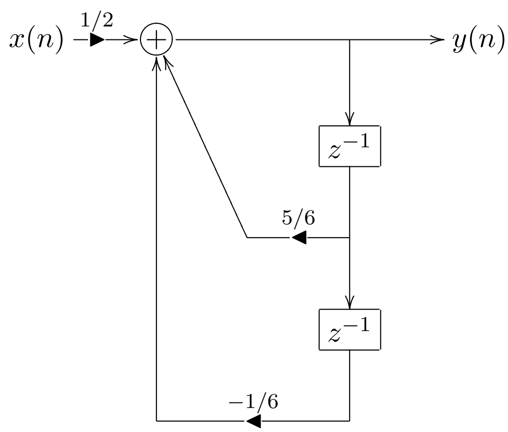
$$\begin{aligned}
H(z) &= \frac{3}{2}\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{3}z^{-1}} \\
&= \frac{\frac{3}{2}-\frac{1}{2}z^{-1}-\left(1-\frac{1}{2}z^{-1}\right)}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} \\
&= \frac{1}{2}\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}
\end{aligned}$$

Which is the wanted expression.

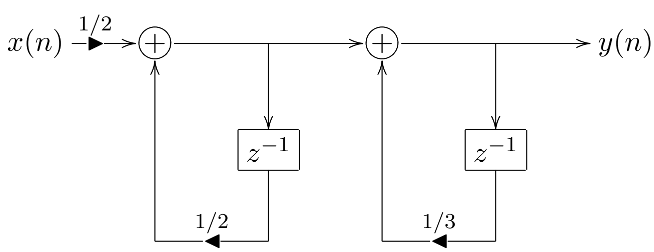
(c) Since we can rewrite H(z) as

$$H(z) = \frac{1/2}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

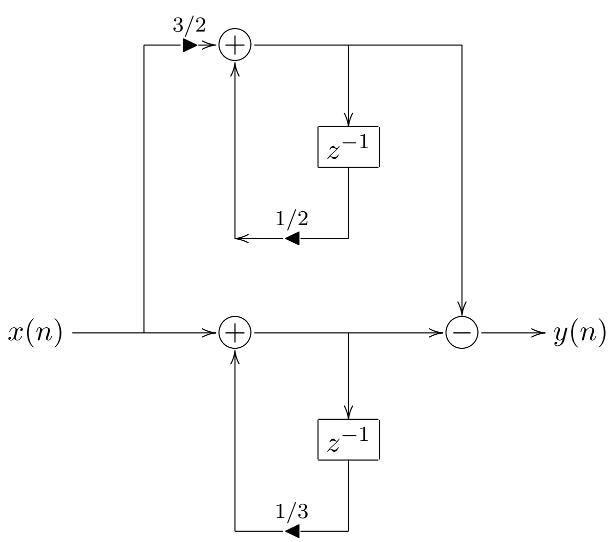
we get the DF2 structure:



By using $H(z) = \frac{1}{2}\frac{1}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$ the filter is represented by a cascade structure:



Finally, by using $H(z) = \frac{3}{2}\frac{1}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{3}z^{-1}}$ we get the filter represented by a parallell structure:



(d) The autocorrelation function of $h_1(n)$ may be found by convolution:

$$\begin{aligned}
h_1(n) &= \left(\frac{1}{2}\right)^n u(n) \\
r_{h1h1}(m) &= h_1(m) * h_1(-m) \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{n+m} \qquad m \geq 0 \\
&= \left(\frac{1}{2}\right)^m \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \\
&= \left(\frac{1}{2}\right)^m \frac{1}{1-\frac{1}{4}} \\
&= \frac{4}{3}\left(\frac{1}{2}\right)^m
\end{aligned}$$

The same procedure for $h_2(n)$ gives:

$$\begin{aligned}
h_2(n) &= \left(\frac{1}{3}\right)^n u(n) \\
r_{h2h2}(m) &= h_2(m) * h_2(-m) \\
&= \left(\frac{1}{3}\right)^m \frac{1}{1-\frac{1}{9}} \qquad m \geq 0 \\
&= \frac{9}{8}\left(\frac{1}{3}\right)^m
\end{aligned}$$

Finally, using convolution on h(n), gives:

$$\begin{aligned}
r_{hh}(m) &= \left(\frac{3}{2}h_1(m) - h_2(m)\right) * \left(\frac{3}{2}h_1(-m) - h_2(-m)\right) \\
&= \frac{9}{4}r_{h1h1}(m) + r_{h2h2}(m) - \frac{3}{2}h_1(m) * h_2(-m) - \frac{3}{2}h_2(m) * h_1(-m) \\
&= \frac{9}{4}r_{h1h1}(m) + r_{h2h2}(m) - \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n+m}\left(\frac{1}{3}\right)^n - \frac{3}{2}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^{n+m}\left(\frac{1}{2}\right)^n \qquad m \geq 0 \\
&= 3\left(\frac{1}{2}\right)^m + \frac{9}{8}\left(\frac{1}{3}\right)^m - \left(\frac{3}{2}\right)\left(\frac{6}{5}\right)\left(\frac{1}{2}\right)^m - \left(\frac{3}{2}\right)\left(\frac{6}{5}\right)\left(\frac{1}{3}\right)^m \\
&= \frac{6}{5}\left(\frac{1}{2}\right)^m - \frac{27}{40}\left(\frac{1}{3}\right)^m
\end{aligned}$$

Problem 3

(a) The round-off error e(n) is modeled as a uniform distributed random variable in the range $(-\frac{\Delta}{2}, \frac{\Delta}{2})$ where $\Delta = 2^{-B}$. The probability density function is therefore,

$$p_e(x) = \begin{cases} \frac{1}{\Delta} & |x| \leq \frac{\Delta}{2} \\ 0 & |x| > \frac{\Delta}{2} \end{cases}$$

The noise power σ_e^2 is given by the variance of $p_e(x)$.

$$\sigma_e^2 = \int_{-\infty}^{\infty} x^2 p_e(x)dx = \int_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} x^2 \frac{1}{\Delta} dx = \left[\frac{1}{3\Delta}x^3\right]_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}} = \frac{\Delta^2}{12} = \frac{2^{-2B}}{12}$$

(b) First we need to find an expression for the noise at the output q(n).

$$q(n) = e(n) * g_k(n) = \sum_{l=-\infty}^{\infty} g_k(l)e(n-l)$$

We then use this expression to find the noise power at the filter output.

$$\begin{aligned}
\sigma_{qk}^2 &= E[q^2(n)] = E\Big[\sum_{l=-\infty}^{\infty} g_k(l)e(n-l) \sum_{m=-\infty}^{\infty} g_k(m)e(n-m)\Big] \\
&= \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} g_k(l)g_k(m)E[e(n-l)e(n-m)] \\
&= \sum_{l=-\infty}^{\infty} g_k^2(l)E[e^2(n-k)] \\
&= \sigma_e^2 \sum_{l=-\infty}^{\infty} g_k^2(l) \\
&= \sigma_e^2 r_{g_k g_k}(0)
\end{aligned}$$

The third step is possible due to the fact that e(n) is white uncorrelated noise which makes $E[e(n-l)e(n-m)] = 0$ for $l \neq m$.

(c)
- By observing the DF2 filter structure from problem 2, we see that all three multiplications ”see” the same filter after the first multiplication to the output. Therefore the unit sample response is given by $g_k(n) = 2h(n)$ for k=1,2,3. First we find the rounding noise contribution q(n)

  $$q(n) = 3(2h(n) * e(n))$$

  To find the rounding error σ_q^2 we use the equation given in the previous problem.

  $$\sigma_q^2 = 3(4\sigma_e^2 r_{hh}(0)) = 12[\frac{6}{5} - \frac{27}{40}]\sigma_e^2 = \frac{63}{10}\sigma_e^2 \approx 6.3\sigma_e^2$$

- The cascaded filter structure has three multiplications and two of them "sees" the filter after the first multiplication to the output, $2h(n)$, while the third one "sees" $h_2(n)$. The unit sample respose is therefore $g_k(n) = 2h(n)$ for k=1,2 and $g_3(n) = h_2(n)$. The rounding error then becomes

$$q(n) = 2(2h(n) * e(n)) + h_2(n) * e(n)$$

$$\sigma_q^2 = 4(2\sigma_e^2 r_{hh}(0)) + \sigma_e^2 r_{h_2h_2}(0) = [8\frac{21}{40} + \frac{9}{8}]\sigma_e^2 = \frac{213}{40}\sigma_e^2 \approx 5,3\sigma_e^2$$

- The parallel filter structure also contains three multiplications where two of them "see" $h_1(n)$ and the last one "sees" $h_2(n)$. The unit sample respose is therefore $g_k(n) = h_1(n)$, for k=1,2 $g_3(n) = h_2(n)$.The rounding error is given by

$$q(n) = 2(h_1(n) * e(n)) + h_2(n) * e(n)$$

$$\sigma_q^2 = 2\sigma_e^2 r_{h_1h_1}(0) + \sigma_e^2 r_{h_2h_2}(0) = [\frac{18}{8} + \frac{4}{3}]\sigma_e^2 = \frac{430}{120}\sigma_e^2 \approx 3,6\sigma_e^2$$

As we can see from these calculations, the parallel filter structure introduces the least noise into the system, while the DF2 filter structure introduces almost twice as much noise into the system as the parallel structure.

(d) Overflow can happen after an addition if the sum is not in the dynamic range. To prevent overflow in the summation nodes we introduce an upper bound A_x on x(n).

$$|y_k(n)| = \Big| \sum_{m=-\infty}^{\infty} h_k(m)x(n-m) \Big| \leq A_x \sum_{m=-\infty}^{\infty} |h_k(m)|$$

Since the dynamic range is limited to (-1,1), A_x can be found by

$$A_x < \frac{1}{\sum_{m=-\infty}^{\infty} |h_k(m)|}$$

- In the DF2 structure we have 1 summation node which "sees" the entire filter h(n).

$$\sum_{m=0}^{\infty} |h(m)| = \sum_{m=0}^{\infty} \Big| \frac{3}{2}\left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n \Big| = 3 - \frac{3}{2} = \frac{3}{2}$$

$$A_x = \frac{2}{3}$$

- In the cascaded structure we have two summation nodes where the first node "sees" $\frac{1}{2}h_1(n)$ and the second node "sees" the entire filter h(n).

$$\sum_{m=0}^{\infty} \frac{1}{2}|h_1(m)| = \frac{1}{2}\sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^n = 1$$

$$A_{x_1} = 1$$
$$A_x = \frac{2}{3}$$

Since $A_x < A_{x1}$ we use A_x to scale the filter structure.

- The parallel structure has three summation nodes where the middle node "sees" $h(n)$, the top node "sees" $\frac{3}{2}h_1(n)$ and the button node "sees" $h_2(n)$.

$$\sum_{m=0}^{\infty} |h_2(m)| = \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{3}{2}$$
$$\frac{3}{2}\sum_{m=0}^{\infty} |h_1(m)| = 3$$

$$A_x = \frac{2}{3}$$
$$A_{x_1} = \frac{1}{3}$$
$$A_{x_2} = \frac{2}{3}$$

Since $A_{x1} < A_x \leq A_{x2}$ we use A_{x1} to scale the filter structure.

Problem 4

(a) We have the following situation:

$$\omega(n) \longrightarrow \boxed{H(z)} \longrightarrow y(n)$$

The output of H(z) is then a random process with autocorrelation function given by

$$\gamma_{yy}(m) = \sigma_\omega^2 r_{hh}(m)$$

where $r_{hh}(m)$ is the same as in problem 2.

The output y(n) of the filter is approximated with, $\hat{y}(n) = -ay(n-1)$. By applying the normal equations for this situation, we get:

$$a\gamma_{yy}(0) = -\gamma_{yy}(1) \Rightarrow$$
$$a = -\frac{\gamma_{yy}(1)}{\gamma_{yy}(0)}$$
$$= -\frac{\sigma_\omega^2 r_{hh}(1)}{\sigma_\omega^2 r_{hh}(0)}$$
$$= -\frac{\left(\frac{6}{5}\right)\left(\frac{1}{2}\right) - \left(\frac{27}{40}\right)\left(\frac{1}{3}\right)}{\frac{6}{5} - \frac{27}{40}}$$
$$= -\frac{15/40}{21/40} = -\frac{5}{7}$$

Further, the error power is given by:

$$
\begin{aligned}
\sigma_f^2 &= \gamma_{yy}(0) + a\gamma_{yy}(1) \\
&= \sigma_\omega^2 \frac{21}{40} - \sigma_\omega^2 \frac{5}{7}\frac{15}{40} \\
&= \sigma_\omega^2 \frac{9}{35}
\end{aligned}
$$

(b) We have the following situation:

$$
\omega(n) \xrightarrow{\ \sigma_\omega^2\ } \boxed{\frac{1/2}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})}} \longrightarrow y(n)
$$

Where y(n) is to be modelled by a linear prediction filter:

$$
\omega(n) \xrightarrow{\ \sigma_f^2\ } \boxed{\frac{1}{(1+a_1z^{-1}+a_2z^{-2}+\cdots-a_Pz^{-P})}} \longrightarrow y(n)
$$

Since H(z) is a second order allpole filter, y(n) is an AR[2] process. Thus we know that the best linear prediction filter is of order 2. This is because the parameters of the AR[2] process is related to the autocorrelation sequence by the Yule-Walker equations:

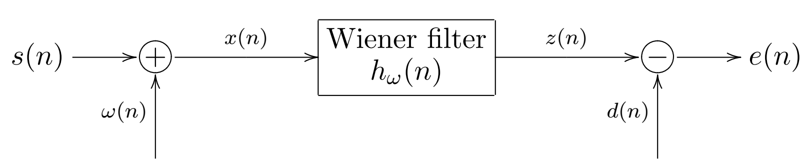
$$
\begin{bmatrix} \gamma_{yy}(0) & \gamma_{yy}(-1) & \gamma_{yy}(-2) \\ \gamma_{yy}(1) & \gamma_{yy}(0) & \gamma_{yy}(-1) \\ \gamma_{yy}(2) & \gamma_{yy}(1) & \gamma_{yy}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}\sigma_\omega^2 \\ 0 \\ 0 \end{bmatrix}
$$

where the noise power had to be scaled by $b_0^2 = \frac{1}{4}$ since the prediction filter has $b_0 = 1$. The corresponding relation for the linear predictor of order two is given by the normal equations:

$$
\begin{bmatrix} \gamma_{yy}(0) & \gamma_{yy}(-1) & \gamma_{yy}(-2) \\ \gamma_{yy}(1) & \gamma_{yy}(0) & \gamma_{yy}(-1) \\ \gamma_{yy}(2) & \gamma_{yy}(1) & \gamma_{yy}(0) \end{bmatrix} \begin{bmatrix} 1 \\ a_2(1) \\ a_2(2) \end{bmatrix} = \begin{bmatrix} \sigma_f^2 \\ 0 \\ 0 \end{bmatrix}
$$

Hence, we see that the coefficients, a_i, is the same as for the AR[2] process, and the error variance is equal to the noise variance divided by 4 ($\sigma_f^2 = \frac{\sigma_\omega^2}{4}$).

(c) Wiener filter is applied in cases where we are given a signal x(n) which consists of the sum of a signal s(n) and noise q(n). The objective of the filter is to suppress the undesired interference and recover as much of the signal as possible. The output of the filter is an approximation of the desired signal sequence $d(n) = s(n+D)$ where $D \in \mathbb{Z}$. The Wiener filter is designed to minimize the power of the error sequence $e(n)$.



(d) The Wiener filter is used as a noise reduction tool by setting the $d(n) = s(n)$.

There are three different SNR cases worth mentioning:

Low SNR ($\Gamma_{ss}(f) \ll \sigma_\omega^2$) The equation will be dominated by σ_ω^2 and therefore $H(f)$ will be close to zero.

Intermediate SNR Neither of the components will dominate, so both of them will contribute to $H(f)$.

High SNR ($\Gamma_{ss}(f) \gg \sigma_\omega^2$) The noise power may be neglected in the equation, hence $H(f)$ will be close to one.