



TTT4110 Information and Signal Theory

Solution to exam spring 2008

Problem 1

- (a) FIR filter: FIR (Finite Impulse Response) filters have, as the name implies, a finite impulse response because they have no feedback. This means that the output signal is a weighted sum of the input signal only. Therefore, if an impulse is sent through the system, the output will become zero as soon as the impulse runs through the filter. Because of this property, a FIR filter is always stable.

IIR filter: IIR (Infinite Impulse Response) filters have an infinite impulse response. These filters have feedback, which means that the filter output is a linear combination of previous output signals and the input signal. These feedbacks can produce an unstable filter.

- (b) A The feedback tells us that this is an IIR filter. The difference equation can be found by defining a function $u(n)$ just after the addition. This gives us the two equations

$$\begin{aligned}y(n) &= bu(n - 2) \\ u(n) &= x(n) + \frac{a}{b}y(n + 1)\end{aligned}$$

Inserting the second equation into the first one we get this difference equation,

$$\begin{aligned}y(n) &= bu(n - 2) \\ &= ay(n - 1) + bx(n - 2)\end{aligned}$$

As we can see this equation corresponds to differential equation 2.

- B This is also an IIR filter so we follow the same strategy as for filter A.

$$\begin{aligned}y(n) &= bu(n - 1) \\ u(n) &= x(n) + \frac{a}{b}y(n - 1)\end{aligned}$$

Inserting the second equation into the first one we get this difference equation,

$$\begin{aligned}y(n) &= bu(n - 1) \\ &= ay(n - 2) + bx(n - 1)\end{aligned}$$

This corresponds to differential equation 3.

- C This is a FIR filter so we can easily find the difference equation by following an impulse through the filter.

$$y(n) = ax(n) + bx(n - 2)$$

which is the same as equation number 1.

- (c) The unit sample response for the filter structure is the response $y(n) = h(n)$ when the input $x(n) = \delta(n)$. The differential equation can now be written as,

$$h(n) = ah(n-2) + b\delta(n-1)$$

To derive the unit sample response we find $h(n)$ for $n = [0, 1, 2, 3, \dots]$.

$$\begin{aligned} h(0) &= ah(-2) + b\delta(-1) = 0 \\ h(1) &= ah(-1) + b\delta(0) = b \\ h(2) &= ah(0) + b\delta(1) = 0 \\ h(3) &= ah(1) + b\delta(2) = ab \\ h(4) &= ah(2) + b\delta(3) = 0 \\ h(5) &= ah(3) + b\delta(4) = a^2b \\ h(6) &= ah(4) + b\delta(5) = 0 \\ h(7) &= ah(5) + b\delta(6) = a^3b \\ &\vdots \end{aligned}$$

Hence,

$$h(n) = \begin{cases} 0 & n \leq 0 \\ a^{\frac{n-1}{2}}b & n = \text{Odd samples} \\ 0 & n = \text{Even samples} \end{cases}$$

- (d) The frequency response can be found in three different ways.

- Directly from the difference equation $y(n) - ay(n-2) = bx(n-1)$,

$$\begin{aligned} H(e^{jw}) &= \frac{\sum_{k=-\infty}^{\infty} b_k e^{-jwk}}{\sum_{k=-\infty}^{\infty} a_k e^{-jwk}} \\ &= \frac{be^{-jw}}{1 - ae^{-2jw}} \end{aligned}$$

- From the unit sample response,

$$H(e^{jw}) = \sum_{k=-\infty}^{\infty} h(k) e^{-jwk}$$

Since we only are interested in odd samples we substitute k with $2l + 1$

$$\begin{aligned}
H(e^{jw}) &= \sum_{k=-\infty}^{\infty} h(2l + 1)e^{-jw(2l+1)} \\
&= \sum_{k=-\infty}^{\infty} a^{\frac{(2l+1)-1}{2}} b e^{-jw(2l+1)} \\
&= \sum_{k=0}^{\infty} (a^l b) e^{-2jwl} e^{-jw} \\
&= b e^{-jw} \sum_{k=0}^{\infty} (a e^{-2jw})^l \\
&= \frac{b e^{-jw}}{1 - a e^{-2jw}}
\end{aligned}$$

- By observing the difference equation $y(n) = ay(n-2) + bx(n-1)$ in the frequency plane we get the following relation,

$$Y(e^{jw}) = aY(e^{jw})e^{-2jw} + bX(e^{jw})e^{-jw}$$

By remembering that the relationship between the input signal and the output signal in the frequency domain we can find the frequency response,

$$H(e^{jw}) = \frac{Y(e^{jw})}{X(e^{jw})} = \frac{b e^{-jw}}{1 - a e^{-2jw}}$$

The magnitude response then becomes,

$$\begin{aligned}
|H(e^{jw})|^2 &= H(e^{jw})H(e^{jw})^* \\
&= \frac{b e^{-jw}}{1 - a e^{-2jw}} \cdot \frac{b e^{jw}}{1 - a e^{2jw}} \\
&= \frac{b^2}{1 - a e^{2jw} - a e^{-2jw} + a^2} \\
&= \frac{b^2}{1 - 2a \cos(2w) + a^2}
\end{aligned}$$

The magnitude response is plotted in figure 1.

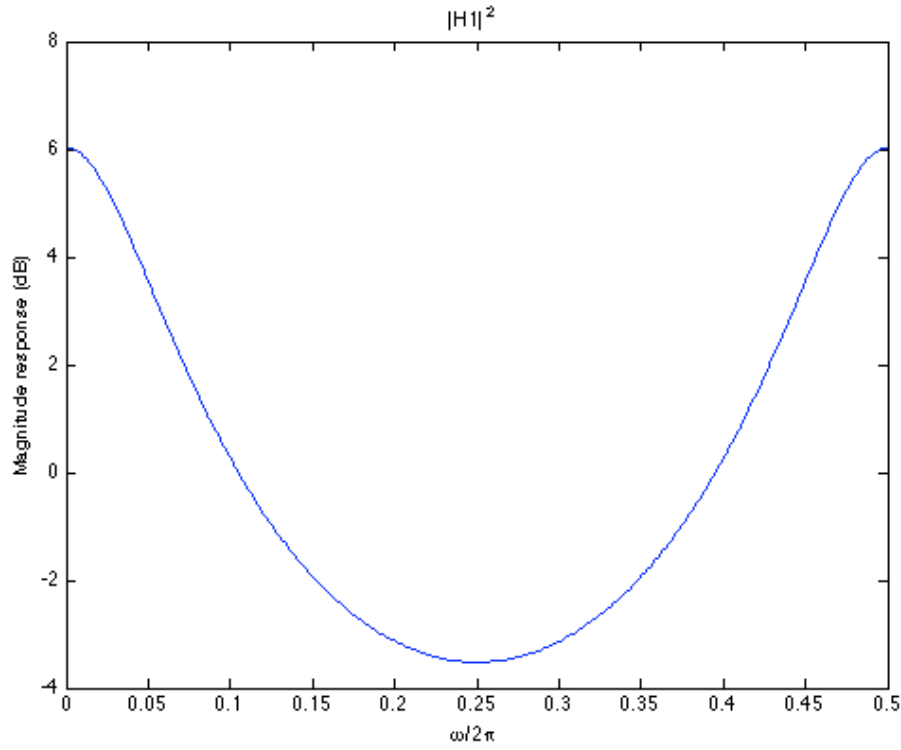


Figure 1: Magnitude response(dB) for a=0.5 and b=1

- (e) Given that the input power spectral density is $S_{XX}(\omega)$ the power spectral density can be expressed by the equation,

$$\begin{aligned} S_{YY} &= |H(e^{jw})|^2 S_{XX} \\ &= \frac{\sigma_x^2 b^2}{(1 - 2a \cos(2w) + a^2)} \end{aligned}$$

- (f) The variance can be found by exploiting the fact that R_{YY} and S_{YY} are fourier pairs which means that we have to solve the integral

$$\sigma_Y^2 = R_{YY}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{YY}(\omega) d\omega$$

Parseval's theorem,

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{-j\omega})|^2 d\omega$$

helps us solve the integral.

$$\begin{aligned}
\sigma_Y^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{YY}(\omega) d\omega \\
&= \frac{\sigma_x^2}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 d\omega \\
&= \sigma_x^2 \sum_{n=-\infty}^{\infty} |h(n)|^2 \\
&= \sigma_x^2 \sum_{l=0}^{\infty} |h(2l+1)|^2 \\
&= \sigma_x^2 |b|^2 \sum_{l=0}^{\infty} |a|^{2l} \\
&= \frac{\sigma_x^2 |b|^2}{(1 - |a|^2)}
\end{aligned}$$

Problem 2

(a) The variance of $f_X(x)$ can be found by solving the integral,

$$\begin{aligned}
\sigma_X^2 &= 2 \int_0^1 x^2 f_X(x) dx \\
&= 2 \int_0^{\frac{1}{2}} \frac{3}{4} x^2 dx + 2 \int_{\frac{1}{2}}^1 \frac{1}{4} x^2 dx \\
&= \frac{10}{48}
\end{aligned}$$

(b) For a b bit uniform quantization with A=1, the quantization interval becomes $\Delta = \frac{2A}{2^b} = 2^{1-b}$. The noise variance can now easily be found

$$\sigma_Q^2 = \frac{\Delta^2}{12} = \frac{2^{2(1-b)}}{12} = \frac{1}{3} \cdot 2^{-2b}$$

The signal to noise ratio is therefore,

$$SNR = \frac{\sigma_X^2}{\sigma_Q^2} = \frac{4.8}{\frac{2^{-2b}}{3}} = \frac{10}{16} \cdot 2^{2b}$$

(c) To find the entropy we need to find the probability of the signal being in different intervals. In this case we can see that the pdf consists of eight equal sized squares. The probabilities for the 4 level case can therefore easily be found,

$$\begin{aligned}
p_1 &= p_4 = \frac{1}{8} \\
p_2 &= p_3 = \frac{3}{8}
\end{aligned}$$

The entropy is now given by,

$$\begin{aligned} H &= -\sum_{n=1}^4 p_n \log_2 p_n \\ &= -\frac{2}{8} \log_2\left(\frac{1}{8}\right) - \frac{6}{8} \log_2\left(\frac{3}{8}\right) = 1.81 \text{ bits/symbol} \end{aligned}$$

- (d) The code is a prefix-free code and it is therefore uniquely decodable. Prefix-free means that no codeword is the beginning of another codeword.
- (e) The minimum average number of bits per symbol is obtained by letting the most probable symbols be represented by the shortest code words. The minimum average number of bits per symbol is therefore,

$$L = \sum_{n=1}^N p_n l_n = 3 \cdot \frac{1}{8} + \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = 1.875 \text{ bits/symbol}$$

This is better than the fixed rate coding, but still a little bit higher than the lower bound which is the entropy $H = 1.81$ bits/symbol

- (f) A pdf-optimised quantiser takes the pdf of the signal levels into account. This gives the possibility to find the decision intervals and the representation values that minimizes the average distortion when the number of intervals are fixed.

Problem 3

- (a) The matched filter is a reversed and shifted version of the received signal. When the received pulse is given by $ag(t)$ and the impulse response of the matched filter is given by,

$$h(t) = ag(T_0 - t),$$

The received pulse shape and the matched filter is sketched in figure 2.

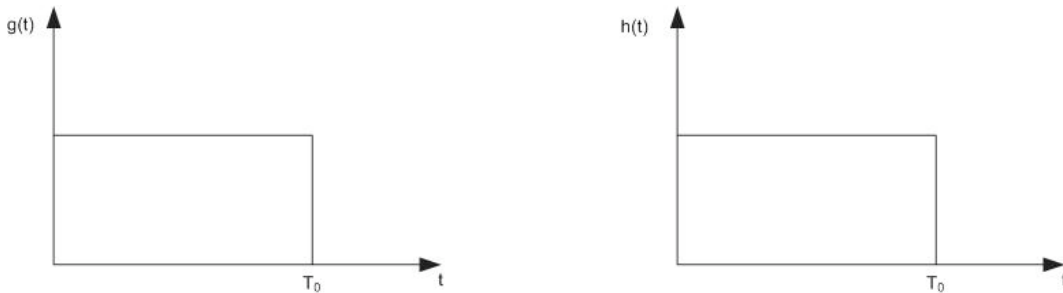


Figure 2: $g(t)$ and $h(t)$

- (b) The signal bandwidth is can be found by taking the Fourier transform of $g(t)$

$$\begin{aligned} G(j\Omega) &= \int_{-\infty}^{\infty} g(t) e^{-j\Omega t} dt = \int_0^{T_0} e^{-j\Omega t} dt = -\frac{1}{j\Omega} [e^{-j\Omega T_0} - 1] \\ &= -\frac{1}{j\Omega} e^{-\frac{j\Omega T_0}{2}} \left(e^{\frac{-j\Omega T_0}{2}} - e^{\frac{j\Omega T_0}{2}} \right) = e^{-\frac{j\Omega T_0}{2}} T_0 \frac{\sin\left(\frac{\Omega T_0}{2}\right)}{\frac{\Omega T_0}{2}} \end{aligned}$$

As we can see from this expression, the signal bandwidth is infinite. As a consequence of this, the channel bandwidth must also be infinite.

- (c) The output $y(n)$ can be found by the convolution: $y(t) = h(t) * g(t)$
Figure 3 illustrates the convolution. We see that if $t < T_0$ the convolution can be found

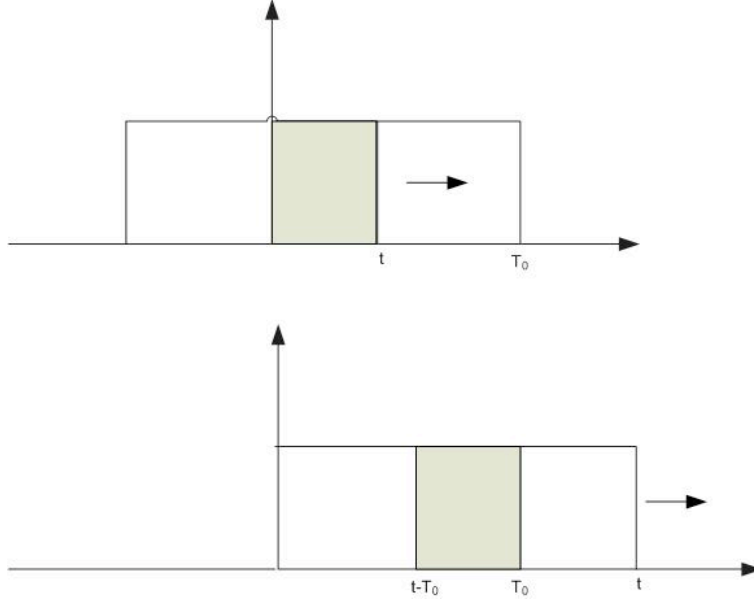


Figure 3: Convolution

by

$$y(t) = \int_0^t a d\tau = at,$$

and if $t > T_0$,

$$y(t) = \int_{t-T_0}^{T_0} a d\tau = a(2T_0 - t).$$

To summarize, we have:

$$y(t) = \begin{cases} 0 & \text{if } t < 0 \\ at & \text{if } 0 \leq t < T_0 \\ a(2T_0 - t) & \text{if } T_0 < t < 2T_0. \end{cases}$$

A sketch of $y(t)$ shows that the convolution of two equal boxes is a triangle.

- (d) The received signal $x(t)$, for $T_0 = 1$ and $a = [1, 2, 1, 1]$,

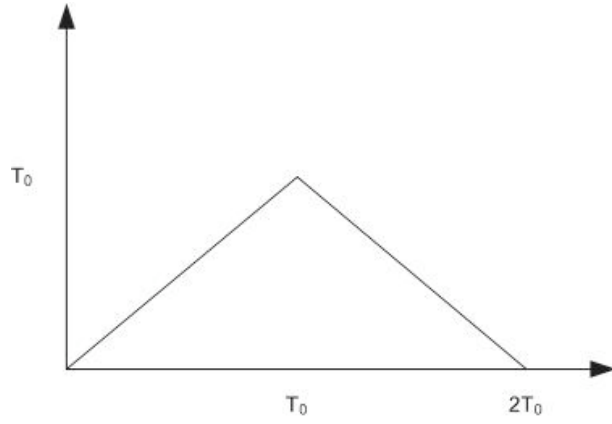


Figure 4: $y(t)$

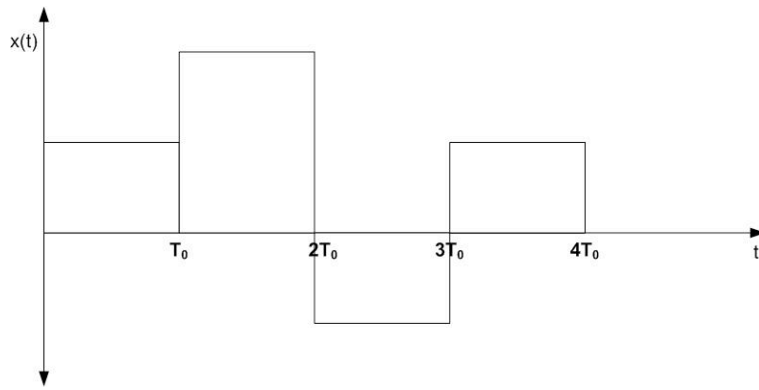


Figure 5: Received signal $x(t)$

- (e) Since the filter is linear and time invariant, the output signal can be found by adding the scaled and shifted versions of the output shown in figure 3. The output signal from the matched filter is shown in figure 5.

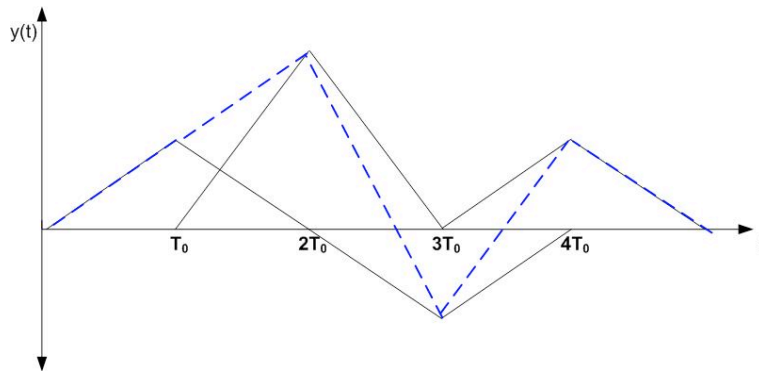


Figure 6: Output signal from the matched filter

In order for a communication channel to be called a *Nyquist* channel, its impulse response $g(t)$ must satisfy

$$g(lT + \Delta t) = \begin{cases} 1 & \text{for } l = 0 \\ 0 & \text{for } l \neq 0, \end{cases}$$

for some $\Delta t > 0$ and $T > 0$.

From the figure we see that the pulses effect each other for all t except for $t = iT_0$, where i is an integer. Here the amplitude is equal to the input signal which means that it is possible to recover the sent amplitudes at times iT_0 . The channel is therefore a Nyquist channel.

- (f) The two recieved samples have the same distribution as the noise around the detection points. An error free transmission implies that there is no overlap between the two distributions. Therefore, if $A \geq 1$ both symbols can be transmitted without error.

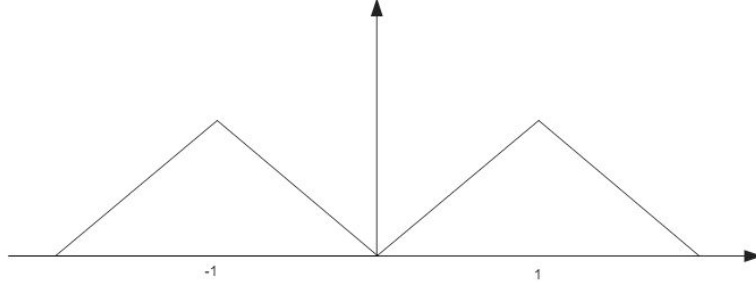


Figure 7: Received amplitudes in the case of error free transmission and $A = \pm 1$

If there is an overlap between the distributions, the error rate is given by the shaded area shown in figure 8. This area is the probability of detecting $A = -1$ when $A = 1$ is the sent symbol. Both transmitted symbols will experience the same error rate due to symmetry.

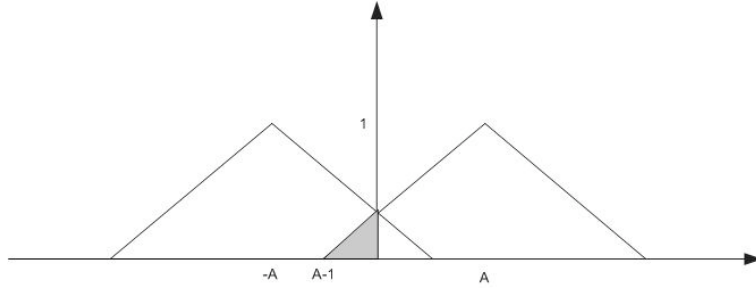


Figure 8: Recieved amplitudes in the case of erroneous transmission

The shaded part of the distribution is a triangle. The error rate is therefore the area of the triangle,

$$P(A = 1 \rightarrow A = -1) = \frac{1}{2}(A - 1)^2$$

