Department of Electronics
and Telecommunications

TTT4110 Information and Signal Theory
Solution to the Exam August 2011

Problem 1

(a)

$$
\begin{aligned}
2y[n] - y[n-1] &= x[n] \\
\Rightarrow 2y[n] &= x[n] + y[n-1] \\
\Rightarrow y[n] &= \frac{1}{2}x[n] + \frac{1}{2}y[n-1]
\end{aligned}
$$

The block diagram that describe the system is shown in Figure 1.

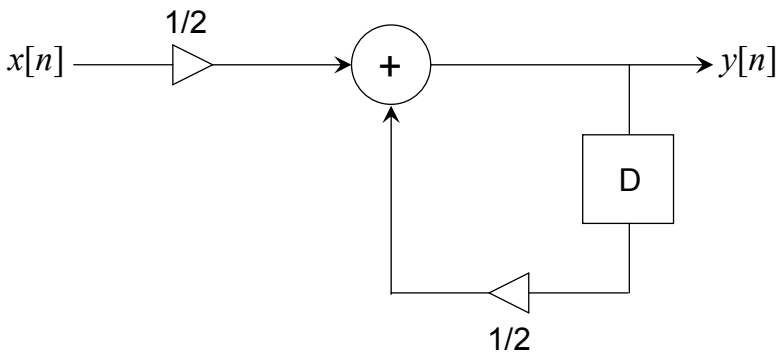

Figure 1: Block diagram of the system

(b) The frequency response is found as follows:

$$
\begin{aligned}
2y[n] - y[n-1] &= x[n] \\
\mathrm{DTFT}\{2y[n] - y[n-1]\} &= \mathrm{DTFT}\{x[n]\} \\
\Rightarrow 2Y(\hat{\omega}) - Y(\hat{\omega})e^{-j\hat{\omega}} &= X(\hat{\omega}) \\
\Rightarrow Y(\hat{\omega})(2 - e^{-j\hat{\omega}}) &= X(\hat{\omega}) \\
\Rightarrow H(\hat{\omega}) = \frac{Y(\hat{\omega})}{X(\hat{\omega})} &= \frac{1}{2 - e^{-j\hat{\omega}}} \qquad (1)
\end{aligned}
$$

To find the filter type, we must find $|H(\hat{\omega})|$.

$$
\begin{aligned}
|H(\hat{\omega})| &= \frac{1}{|2 - e^{-j\hat{\omega}}|} \\
&= \frac{1}{|2 - \cos\hat{\omega} + j\sin\hat{\omega}|} \\
&= \frac{1}{\sqrt{(2 - \cos\hat{\omega})^2 + \sin^2\hat{\omega}}} \\
&= \frac{1}{\sqrt{5 - 4\cos\hat{\omega}}}
\end{aligned}
$$

Now

$$\hat{\omega} = 0 \Rightarrow |H(\hat{\omega})| = 1$$

and

$$\hat{\omega} = \pi \Rightarrow |H(\hat{\omega})| = 1/3$$

$|H(\hat{\omega})|$ is monotonically decreasing function for $\hat{\omega} \in [0, \pi]$. Therefore, $H(\hat{\omega})$ is a low pass filter.

(c) We know that

$$H(\hat{\omega}) = \text{DTFT}\{h[n]\} = \sum_{n=-\infty}^{\infty} h[n]e^{-j\hat{\omega}n} \tag{2}$$

The $H(\hat{\omega})$ in (1) can be written as

$$H(\hat{\omega}) = \frac{1}{2 - e^{-j\hat{\omega}}} = \frac{1}{2}\frac{1}{1 - \frac{1}{2}e^{-j\hat{\omega}}} = \frac{1}{2}\sum_{n=0}^{\infty}(\frac{1}{2}e^{-j\hat{\omega}})^n = \sum_{n=0}^{\infty}\frac{1}{2^{n+1}}e^{-j\hat{\omega}n} \tag{3}$$

From (2) and (3),

$$h[n] = \begin{cases} 0, & n < 0 \\ \frac{1}{2^{n+1}}, & n \geq 0 \end{cases} = \frac{1}{2^{n+1}}u[n] \tag{4}$$

Alternative solution:

$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}y[n-1] \Rightarrow h[n] = \frac{1}{2}\delta[n] + \frac{1}{2}h[n-1]$$

Since it is a causal system, $h[n] = 0, n < 0$. We have

$$h[0] = \frac{1}{2}\delta[0] + \frac{1}{2}h[-1] = \frac{1}{2}$$

$$h[1] = \frac{1}{2}\delta[1] + \frac{1}{2}h[0] = \frac{1}{4} = \frac{1}{2^2}$$

$$h[2] = \frac{1}{2}\delta[2] + \frac{1}{2}h[1] = \frac{1}{8} = \frac{1}{2^3}$$

$$\Rightarrow h[n] = \frac{1}{2^{n+1}}, n \geq 0$$

This is an IIR filter because $h[n]$ is infinitely long.

(d) The input signal is

$$x[n] = 5 + 2\cos(\frac{\pi}{3}n).$$

Note that $x[n]$ consists of a DC-component ($\hat{\omega} = 0$) and a cosine-component with $\hat{\omega} = \frac{\pi}{3}$. The output signal will also contain these two frequency components only. The frequency response of the system determines the amplification/attenuation and phase shift of these two components.

We have found in 1b) that $|H(\hat{\omega})| = \frac{1}{\sqrt{5-4\cos\hat{\omega}}}$.

$\Rightarrow |H(0)| = 1$ and $|H(\frac{\pi}{3})| = \frac{1}{\sqrt{3}}$.

Phase shift of the system is given as $\measuredangle H(\hat{\omega}) = -\measuredangle(2 - e^{-j\hat{\omega}})$.

$2 - e^{-j\hat{\omega}} = 2 - (\cos\hat{\omega} - j\sin\hat{\omega}) = (2 - \cos\hat{\omega}) + j\sin\hat{\omega}$

$$\Rightarrow \measuredangle H(\hat{\omega}) = -\arctan\frac{\sin\hat{\omega}}{2 - \cos\hat{\omega}}$$

$\Rightarrow \measuredangle H(0) = -\arctan(0) = 0$ and $\measuredangle H(\frac{\pi}{3}) = -\arctan\frac{\frac{\sqrt{3}}{2}}{2-\frac{1}{2}} = -\arctan(\frac{1}{\sqrt{3}}) = \frac{-\pi}{6}$.

The output signal is therefore given as

$$y[n] = 5 + \frac{2}{\sqrt{3}}\cos(\frac{\pi}{3}n - \frac{\pi}{6}).$$

(e) The spectrum of the input signal is

$$X(\hat{\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\hat{\omega}}}.$$

The spectrum of the output signal is given as

$$\begin{aligned} Y(\hat{\omega}) &= X(\hat{\omega})\cdot H(\hat{\omega}) = \frac{1}{1+\frac{1}{2}e^{-j\hat{\omega}}}\cdot\frac{1}{2-e^{-j\hat{\omega}}} = \frac{1}{2-\frac{1}{2}e^{-j2\hat{\omega}}} = \frac{1}{2}\frac{1}{1-\frac{1}{4}e^{-j2\hat{\omega}}} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(\frac{1}{4}e^{-j2\hat{\omega}})^n = \sum_{k=0}^{\infty}\frac{1}{2^{2k+1}}e^{-j\hat{\omega}2k} \end{aligned} \tag{5}$$

The general form for $Y(\hat{\omega})$ is given as

$$Y(\hat{\omega}) = \text{DTFT}\{y[n]\} = \sum_{n=-\infty}^{\infty} y[n]e^{-j\hat{\omega}n} \tag{6}$$

(5) and (6) $\Rightarrow$

$$y[n] = \begin{cases} \frac{1}{2^{2k+1}}, & n = 2k, k \geq 0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{2^{n+1}}, & n \geq 0 \text{ and } n \text{ even} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Problem 2

(a) The signal $x(t)$ is a periodic with period $T_0 = 2$s, and $x(t) = t$ for $t \in [-1, 1]$. The coefficients c_k of the Fourier series are given as.

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\frac{2\pi k}{T_0}t}dt = \frac{1}{2}\int_{-1}^{1} x(t)e^{-j\pi kt}dt = \frac{1}{2}\int_{-1}^{1} te^{-j\pi kt}dt$$

For $k = 0$,

$$c_0 = \frac{1}{2}\int_{-1}^{1} tdt = \frac{t^2}{4}|_{-1}^{1} = 0$$

and for $k \neq 0$,

$$\begin{aligned} c_k &= \frac{1}{2}\int_{-1}^{1} te^{-j\pi kt}dt \\ &= \frac{1}{2}\left[-\frac{t}{j\pi k}e^{-j\pi kt}|_{-1}^{1} + \frac{1}{j\pi k}\int_{-1}^{1} e^{-j\pi kt}dt\right] \\ &= \frac{1}{2}\left[-\frac{1}{j\pi k}e^{-j\pi k} + \frac{-1}{j\pi k}e^{j\pi k} - \frac{1}{(j\pi k)^2}e^{-j\pi kt}|_{-1}^{1}\right] \\ &= \frac{1}{2}\left[-\frac{1}{j\pi k}(e^{j\pi k} + e^{-j\pi k}) + \frac{1}{(\pi k)^2}(e^{-j\pi k} - e^{j\pi k})\right] \\ &= \frac{1}{2}\left[-\frac{2}{j\pi k}\cos(\pi k) - \frac{2j}{(\pi k)^2}\sin(\pi k)\right] \\ &= \frac{j}{\pi k}(-1)^k \end{aligned}$$

(b) The sampling frequency is $f_s = 2.5$Hz.

$$\Rightarrow \omega_s = 2\pi f_s = 5\pi \ \frac{rad}{s}$$

To avoid aliasing we must limit the spectrum of $x(t)$ to $[-\frac{\omega_s}{2}, \frac{\omega_s}{2}] = [-2.5\pi, 2.5\pi]$. This can be achieved with an ideal low pass filter with cutoff frequency $\omega_c = 2.5\pi$. The magnitude response of the filter is shown in Figure 2.

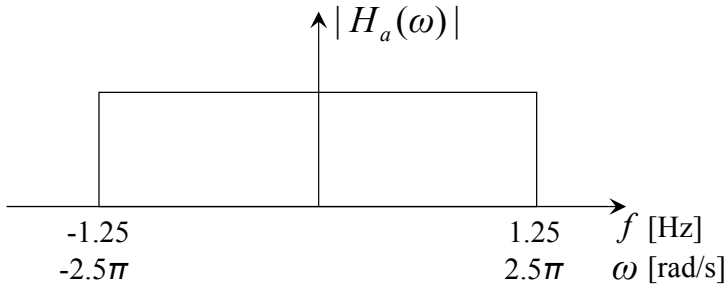

Figure 2: Magnitude response of the filter

(c) Signal $x(t)$ has frequency components in $w_k = \frac{2\pi k}{T_0} = \pi k$. After filtering, all components for $|k| > 2$ will be removed. Spectrum of $x'(t)$ is given by

$$c'_k = \begin{cases} c_k, & |k| \leq 2 \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

The magnitude spectrum of the signal $x'(t)$ is shown in the Figure 3.

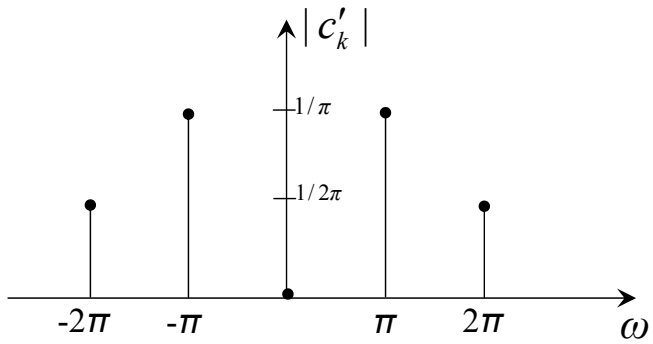

Figure 3: Magnitude spectrum of the signal $x'(t)$

(d) The spectrum of $x_s[n]$ is a periodic extension of the spectrum of $x'(t)$ with period $2\pi f_s = 5\pi$. The magnitude spectrum is shown in the Figure 4.

Problem 3

(a) Here $x_{min} = -1$, $x_{max} = 1$ and $L = 6$.

$$\Delta = \frac{x_{max} - x_{min}}{L} = \frac{1-(-1)}{6} = \frac{1}{3}$$

The decision levels $\{d_i\}$ and representation levels $\{r_i\}$ of the uniform quantiser are given as

$$d_i : -1, -\frac{2}{3}, -\frac{1}{3}, 0, \frac{1}{3}, \frac{2}{3}, 1$$

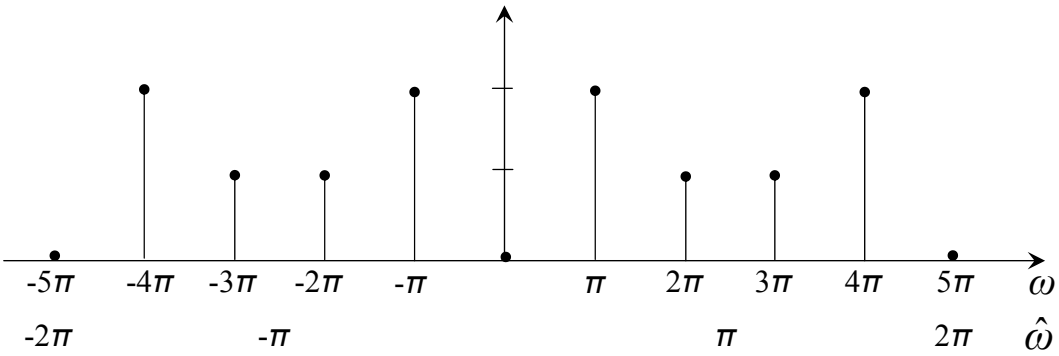

Figure 4: Magnitude spectrum of the sampled signal $x_s[n]$

$$r_i : -\frac{5}{6}, -\frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, \frac{1}{2}, \frac{5}{6}$$

(b) Quantization noise: $q = x - x_q$

Quantization noise Power:

$$\begin{aligned} P_q = E[q^2] = E[(x - x_q)^2] &= \int_{-\infty}^{\infty} (x - x_q)^2 f_X(x) dx \\ &= \sum_{i=1}^{6} \int_{d_{i-1}}^{d_i} (x - r_i)^2 f_X(x) dx \end{aligned}$$

Since $f_X(x)$ is constant at each quantization interval, the approximate formula will give the exact value of P_q.

$$P_q = \frac{\Delta^2}{12} = \frac{1/9}{12} = \frac{1}{108}$$

Alternative solution: Due to the symmetry of $f_X(x)$, we have

$$\begin{aligned} P_q &= 2\sum_{i=4}^{6} \int_{d_{i-1}}^{d_i} (x - r_i)^2 f_X(x) dx \\ &= 2\int_0^{1/3} (x - \frac{1}{6})^2 \cdot \frac{1}{4} dx + 2\int_{1/3}^{2/3} (x - \frac{1}{2})^2 \cdot \frac{1}{2} dx + 2\int_{2/3}^{1} (x - \frac{5}{6})^2 \cdot \frac{3}{4} dx \\ &= \frac{1}{2}\int_{-1/6}^{1/6} y^2 dy + \int_{-1/6}^{1/6} y^2 dy + \frac{3}{2}\int_{-1/6}^{1/6} y^2 dy \\ &= 3 \cdot 2\int_0^{1/6} y^2 dy \\ &= 6 \cdot \frac{y^3}{3}|_0^{1/6} = \frac{2}{6^3} = \frac{1}{108} \end{aligned}$$

To compute the signal-to-quantization noise ratio (SQNR), we first need to calculate

P_x.

$$\begin{aligned} P_x = E[x^2] &= \int_{-\infty}^{\infty} x^2 f_X(x)dx \\ &= 2\int_0^{\frac{1}{3}} \frac{3}{4}x^2dx + 2\int_{\frac{1}{3}}^{\frac{2}{3}} \frac{1}{2}x^2dx + 2\int_{\frac{2}{3}}^{1} \frac{1}{4}x^2dx \\ &= \frac{1}{2}x^3|_0^{\frac{1}{3}} + \frac{1}{3}x^3|_{\frac{1}{3}}^{\frac{2}{3}} + \frac{1}{6}x^3|_{\frac{2}{3}}^{1} \\ &= \frac{2}{9} \end{aligned}$$

The signal-to-quantization noise ratio (SQNR) is now given as

$$\text{SQNR} = \frac{Px}{P_q} = \frac{2/9}{1/108} = \frac{216}{9} = 24 = 13.8 \text{ dB}$$

(c) The entropy of the quantized signal is given by

$$H = E[I] = \sum_{i=1}^{6} p_i \log_2 \frac{1}{p_i}, \tag{9}$$

where p_i is the probability of ith representation value. We see from the graph of $f_X(x)$ that

$$p_1 = p_6 = \frac{1}{12}, p_2 = p_5 = \frac{1}{6}, p_3 = p_4 = \frac{1}{4}$$

(9) becomes

$$H = 2\left(\frac{1}{12}\log_2(12) + \frac{1}{6}\log_2(6) + \frac{1}{4}\log_2(4)\right) = 2.46 \text{ bits/symbol.}$$

(d) We have $L = 6$ representation levels. The minimum code word length that we have to use if all the code words should be of equal length is

$$b = \lceil \log_2 6 \rceil = 3 \text{ bits.}$$

A possible code is given in Table 1.

Table 1: An example of equal length code

Representation Level	Codeword
r_1	000
r_2	001
r_3	010
r_4	011
r_5	100
r_6	101

(e) We use entropy coding, i.e., we use different number of bits for the different symbols and allocate shortest codeword to the symbol that has the highest probability. (To get a uniquely decodable code, no codeword should be a prefix in another codeword.)

An example is given in Table 2: The average codeword length $\bar{L}$ in this case is given as

$$\bar{L} = \sum_{i=1}^{6} p_i l_i = 2 \cdot 4 \cdot \frac{1}{12} + 3 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 2 \cdot 2 \cdot \frac{1}{4} = 2.5 \text{ bits/symbol}$$

Table 2: An example of code with different codeword lengths

Representation Level	Codeword
r_1	1110
r_2	110
r_3	00
r_4	01
r_5	10
r_6	1111

Problem 4

(a) Transmission over this channel without ISI is possible if Nyquist criterion is fulfilled, i.e., we find some $T > 0$ and Δt such that.

$$g(lT + \Delta t) = \begin{cases} 1 & \text{for} \quad l = 0, \\ 0 & \text{for} \quad l \neq 0 \end{cases}$$

We see that $g(4) = 1 \Rightarrow \Delta t = 4\text{ms}$.

We see further that $g(lT + 4) = 0$ for $T \geq 2\text{ms}$, when $l \neq 0$. Therefore ISI-free transmission is possible if the distance between the symbols is $T \geq 2\text{ms}$.

The maximum signaling speed, i.e. the maximum number of channel symbols per second, for ISI-free transmission is $\frac{1}{T} = 500$ channel symbols/s.

(b) Here SNR [dB] $= 50 = 10 \log_{10} SNR \Rightarrow SNR = 10^5$.

To achieve error-free transmission of the signal, we must have

$$\frac{C}{T} \geq \frac{H}{T_s}$$

where $H = 2.46$ bits/sym (calculated in Problem 3c), $\frac{1}{T} = 500$ sym/s, $\frac{1}{T_s} = f_s = 2000$ sym/s and

$$C = \frac{1}{2} \log_2 (1 + SNR) = \frac{1}{2} \log_2 \left(1 + 10^5\right) = 8.3 \text{ bits/channel symbol},$$

$$\Rightarrow \frac{C}{T} = 8.3 \cdot 500 = 4152 \text{ bits/s}$$

$$\text{and } \frac{H}{T_s} = 2.46 \cdot 2000 = 4920 \text{ bits/s}$$

Since $\frac{C}{T} < \frac{H}{T_s}$, error-free transmission is not possible.