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OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF ELECTRONICS AND TELECOMMUNICATIONS

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EXAMINATION IN COURSE TTT4110 SIGNAL PROCESSING AND COMMUNICATION SIGNAL PROCESSING

Date: Tuesday May 29, 2012
Time: 09.00 - 13.00

Permitted aids: D–No printed or handwritten material allowed.
Specified, simple calculator allowed..

INFORMATION

- The examination consists of 5 tasks.
- A list of formulas can be found in the appendix.
- Task weighting is given in parenthesis. Total amount of points is 62.
- The teacher will visit you twice, the first time around 10.00 and the second time around 12.00.

Problem 1 *(2+3+2+2+2=11)*

1a) Given the following analogue signal :

$$x_a(t) = cos(\frac{240}{3}\pi t + \frac{\pi}{6}) \tag{1}$$

Give respectively the period and the frequency of the signal.

A: $2\pi F_0 = 240\pi/3 \ \Rightarrow \ F_0 = 120/3 = 40 \Rightarrow \ T_0 = 1000/40 = 25$ (msec)

1b) The signal in eq. 1 is sampled with a rate of respectively $F_s = 100$ Hz and $F_s = 50$ Hz.

Give the normalized frequency of the resulting sequence for the two cases.

A: $f_0 = F_0/F_s$
$F_s = 100 \ \Rightarrow f_0 = 40/100 = 0.4$
$F_s = 50 \ \Rightarrow f_0 = 40/50 = 0.8 > 0.5 \ \Rightarrow \ f_0 = 1 - 0.8 = 0.2$ (due to aliasing)

Sketch the frequency content of the sequence for the two cases.

Is it possible to reconstruct the analogue signal from the two sequences? State the reason for your answer.

A: For the case $F_s = 100 \geq 2 * 40$ the signal can be reconstructed. In the other case $F_s = 50 < 2 * 40$, thus the reconstructed signal will have a frequency of $F_0 = 0.2 * F_s = 0.2 * 50 = 10$ Hz.

1c) Define respectively stability, linearity, time invariance (LTI) and causality for a discrete system.

A: Stability (BIBO) : $\max_n |x(n)| < \infty \ \Leftrightarrow \ \max_n |y(n)| < \infty$
Linearity : $x(n) = \sum_i a_i x_i(n) \ \Rightarrow y(n) = \sum_i a_i y_i(n)$ where $y_i(n) = H[x_i(n)]$
Time invariance : $y(n-k) = H[x(n-k)]$ for all k
Causality : $y(n)$ is independent of $x(n+k$ for all $k > 0$

1d) Given the following differential equation of an analogue system which fulfills all the criterias in subtask 1c :

$$y(t) + 2\frac{dy(t)}{dt} = x(t) \tag{2}$$

Assume $h(0) = 1/2$. Show that the system impulse response is given by :

$$h(t) = \frac{1}{2}e^{-t/2}u(t) \tag{3}$$

A : $x(t) = \delta(t) \;\Rightarrow\; y(t) = h(t)$
Assuming $t > 0 \Rightarrow\; h(t) + 2\frac{dh(t)}{dt} = 0 \;\Rightarrow\; h(t) = Ce^{at}u(t)$ and $\frac{dh(t)}{dt} = Cae^{at}u(t)$.
Inserting into the diff. eq. gives $a = -1/2$ and $h(0) = 1/2 \Rightarrow C = 1/2$. qed.

1e) Given the following difference equation of a discrete system which fulfills all the criterias in subtask 1c :

$$y(n) - \frac{1}{2}y(n-1) = x(n) + \frac{1}{2}x(n-1) \tag{4}$$

Sketch respectively the DF1 and DF2 structure of the system

A:

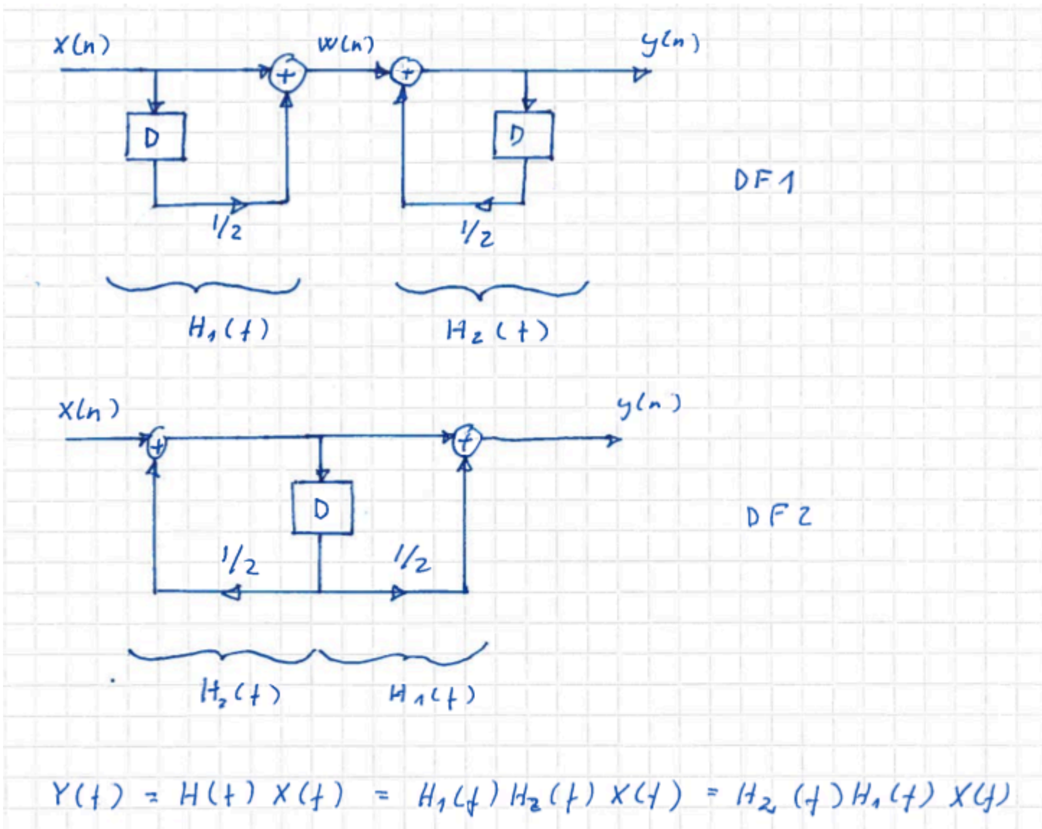

Figure 1: DF1 and DF2 structures

Problem 2 *(2+2+2+3=9)*

2a) Find the transfer function $H(F)$ of the analogue system in subtask 1d

A1 : easiest to use the diff .eq. :
$F[dy(t)/dt] = j2\pi FY(F) \Rightarrow Y(F)+2(j2\pi FY(F)) = X(F) \Rightarrow H(F) = Y(F)/X(F) = 1/(1+j4\pi F)$

A2 : somewhat more calculation by $FT[h(t)]$:
$H(F) = 1/2\int_0^\infty e^{-t/2}e^{-j2\pi Ft}dt = 1/2\int_0^\infty e^{-(1/2+j2\pi F)t}dt =$
$1/2(-1/(1/2+j2\pi F))e^{-(1/2+j2\pi F)t}|_0^\infty = 1/2(-1/(1/2+j2\pi F))[0-1] = 1/(1+j4\pi F)$

2b) Given a general sequence $x(n)$ with frequency response $X(f)$.

Show that the frequency response of the sequence $z(n) = x(n-k)$ is given by :

$$Z(f) = e^{-j2\pi kf}X(f) \tag{5}$$

A : $Z(f) = \sum_n x(n-k)e^{-j2\pi fn}$ inserting $m = n-k \Rightarrow$
$\sum_m x(m)e^{-j2\pi f(m+k)} = e^{-j2\pi k}\sum_m x(m)e^{-j2\pi fm} = e^{-j2\pi k}X(f)$

2c) Show that the transfer function $H(f)$ of the discrete system in subtask 1e is given by :

$$H(f) = \frac{1+\frac{1}{2}e^{-j2\pi f}}{1-\frac{1}{2}e^{-j2\pi f}} \tag{6}$$

A : Using the result from 2b $\Rightarrow Y(f)-(1/2)e^{-j2\pi f}Y(f) = X(f)+(1/2)e^{-j2\pi f}X(f) \Rightarrow$
$H(f) = Y(f)/X(f) = (1+(1/2)e^{-j2\pi f})/(1-(1/2)e^{-j2\pi f})$

2d) Show that the unit pulse response of the the discrete system is given by :

$$h(n) = \begin{cases} 1 & n = 0 \\ 2(\frac{1}{2})^n & n \geq 1 \end{cases} \tag{7}$$

A : Defining $G(f) = 1/(1-(1/2)e^{-j2\pi f}) \Rightarrow g(n) = (1/2)^n u(n)$
Further $H(f) = G(f)[1+(1/2)e^{-j2\pi f}] \Rightarrow h(n) = g(n)+(1/2)g(n-1) = (1/2)^n + (1/2)(1/2)^{n-1}u(n-1) = (1/2)^n[u(n)+u(n-1)]$
$n = 0$: $h(n) = (1/2)^n u(n) \Rightarrow h(0) = (1/2)^0 = 1$
$n > 0$: $h(n) = 2(1/2)^n$

Problem 3 *(2+2+2+2=8)*

The sequence $x(n) = (-\frac{1}{2})^n u(n)$ is input to the discrete system given by eq. 7.

3a) Show, by using linear convolution in the *frequency domain*, that the output sequence is given by

$$y(n) = (\frac{1}{2})^n u(n) \tag{8}$$

A : $X(f) = 1/(1 - (-1/2)e^{-j2\pi f}) = 1/(1 + (1/2)e^{-j2\pi f}) \Rightarrow Y(f) = H(f)X(f) = 1/(1 - (1/2)e^{-j2\pi f}) \Rightarrow y(n) = (1/2)^n u(n)$

3b) Show that the autocorrelation of the input sequence in subtask 3a is given by :

$$r_{xx}(m) = \begin{cases} \frac{4}{3}(-\frac{1}{2})^m & m \geq 0 \\ r_{xx}(-m) & m < 0 \end{cases} \tag{9}$$

A : Assuming positive indexes (symmetry) $m \geq 0$

$r_{xx}(m) = \sum_{n=0}^{\infty}(-\frac{1}{2})^n(-\frac{1}{2})n + m = (-\frac{1}{2})^m \sum_{n=0}^{\infty}(\frac{1}{4})^n = (-\frac{1}{2})^m \frac{1}{1-\frac{1}{4}} = \frac{4}{3}(-\frac{1}{2})^m$

3c) Why is the crosscorrelation between the output and input $r_{yx}(m) = 0$ for $m < 0$?

A : $r_{yx}(m) = \sum_n y(n+m)x(n)$
$r_{yx}(m) = 0$ for $m < 0$ for a causal system as $y(n+m) = y(n - |m|)$ can not be dependent on future input values $x(n)$; i.e $n > n - |m|$.
This can also be verified by that $r_{yx}(m) = h(m) * r_{xx}(m)$ in combination with $h(m) = 0$ for $m < 0$ for a causal system.

3d) Find the crosscorrelation $r_{yx}(m)$ for $m \geq 0$

A : For $m \geq 0$ we have : $r_{yx}(m) = \sum_{n=0}^{\infty}(\frac{1}{2})^{n+m}(-\frac{1}{2})^n = (\frac{1}{2})^m \sum_{n=0}^{\infty}(-\frac{1}{4})^n = (\frac{1}{2})^m \frac{1}{1+\frac{1}{4}} = \frac{4}{5}(\frac{1}{2})^m$

Problem 4 *(3+2+2+2+3+2=14)*

An analogue signal $x_a(t)$ has a dynamic range $(-1, 1)$ and amplitude distribution given by eq. 10 :

$$p_x(x) = \begin{cases} \frac{1}{4} & \frac{1}{2} \leq |x| < 1 \\ \frac{1}{2} & \frac{1}{4} \leq |x| < \frac{1}{2} \\ 1 & 0 \leq |x| < \frac{1}{4} \\ 0 & 1 \leq |x| \end{cases} \tag{10}$$

The signal is sampled by a rate $F_x = 1/T_x = 2000$ Hz. The sequence $x(n) = x_a(nT_x)$ is then digitized by a uniform quantizer with 8 levels x_k for $k = 1, 8$.

4a) Sketch the amplitude distribution and give the decision borders and representation values of the quantizer.

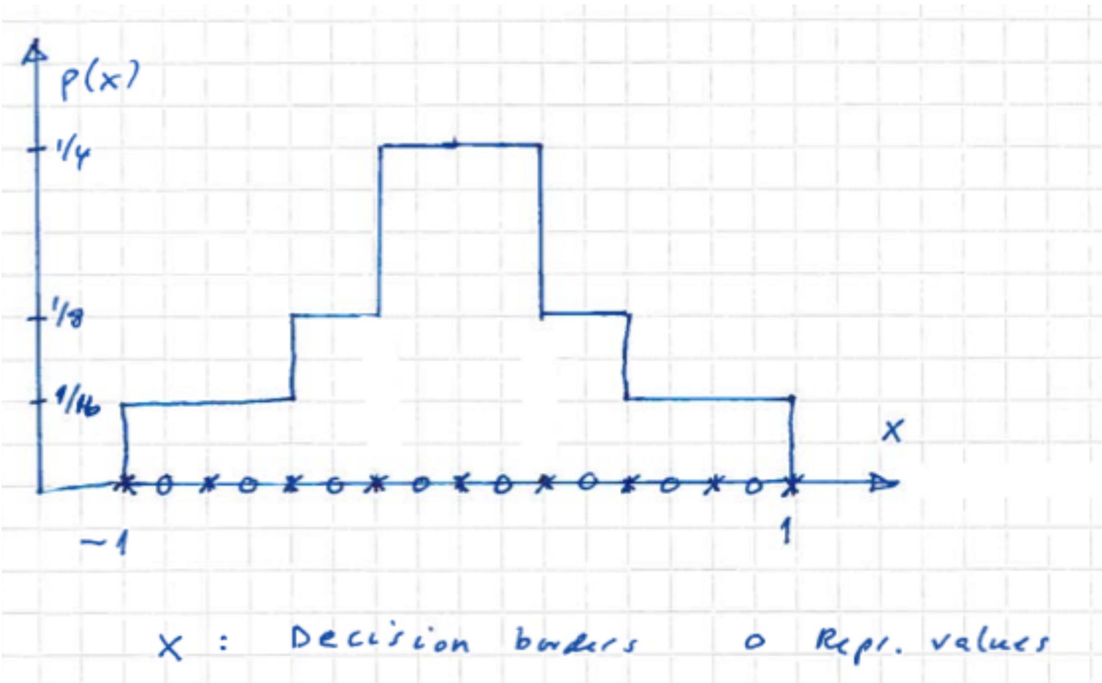

Figure 2: Amplitude distribution, decision borders and repr. values

A: Note that the correct values of the y-axis (p(x)) are the sketched ones multiplied by 4; i.e. $1/4, 1/2, 1$!!
Dec. borders : $[-1, -6/8, -4/8, -2/8, 0, 2/8, 4/8, 6/8, 1]$
Representation levels : $[x_1 = -7/8, x_2 = -5/8, x_3 = -3/8, x_4 = -1/8,$
$x_5 = 1/8, x_6 = 3/8, x_7 = 5/8, x_8 = 7/8]$

4b) Find the probability of each of the resulting 8 symbols.

A : The distance between two decision borders is 1/4.
Thus $p(x_1) = p(x_2) = p(x_7) = p(x_8) = (1/4) * (1/4) = 1/16$;
$p(x_3) = p(x_6) = (1/4) * (1/2) = 1/8, \quad p(x_4) = p(x_5) = (1/4) * 1 = 1/4$

4c) Find the entropy of the source (quantizer)

A : Using the notation $p_i = p(x_i)$ we have $H = \sum_{i=1}^{8} p_i log_2(1/p_i) = 4(1/16)log_2(16)+$
$2(1/8)log_2(8) + 2(1/4)log_2(4) = 1 + 3/4 + 1 = 2.75$ bits/sec

4d) How many bits are needed in order to transfer the symbols. ?

Give the bitrate (in bit/sec) for the source.

A : With $2 < H \leq 3$ we need 3 bits/symbol. A sampling rate of $F_x = 2000Hz$ gives a corresponding symbol rate after quantization. Thus the bitrate is $F_x H = 2000 * 3 = 6000$ bits/sec.

4e) Find the the decision borders and representation values if all the 8 symbols shall have equal probability.

A : We now want $p_i = p(x_i) = 1/8$; i.e. uniform distribution. The following decision borders of the amplitude distribution will achieve this. The representation values will be in the middle between the borders.

The corresponding borders and levels are :
Dec. borders : $[-1, -4/8, -2/8, -1/8, 0, 1/8, 2/8, 4/8, 1]$
Representation levels : $[x_1 = -6/8, x_2 = -3/8, x_3 = -3/16, x_4 = -1/16,$
$x_5 = 1/16, x_6 = 3/16, x_7 = 3/8, x_8 = 6/8]$

4f) Find the entropy for the case in subtask 4e.

A: $H = \sum_{i=1}^{8} p_i log_2(1/p_i) = 8(1/8)log_2(1/8) = 3$ bits/symbol.

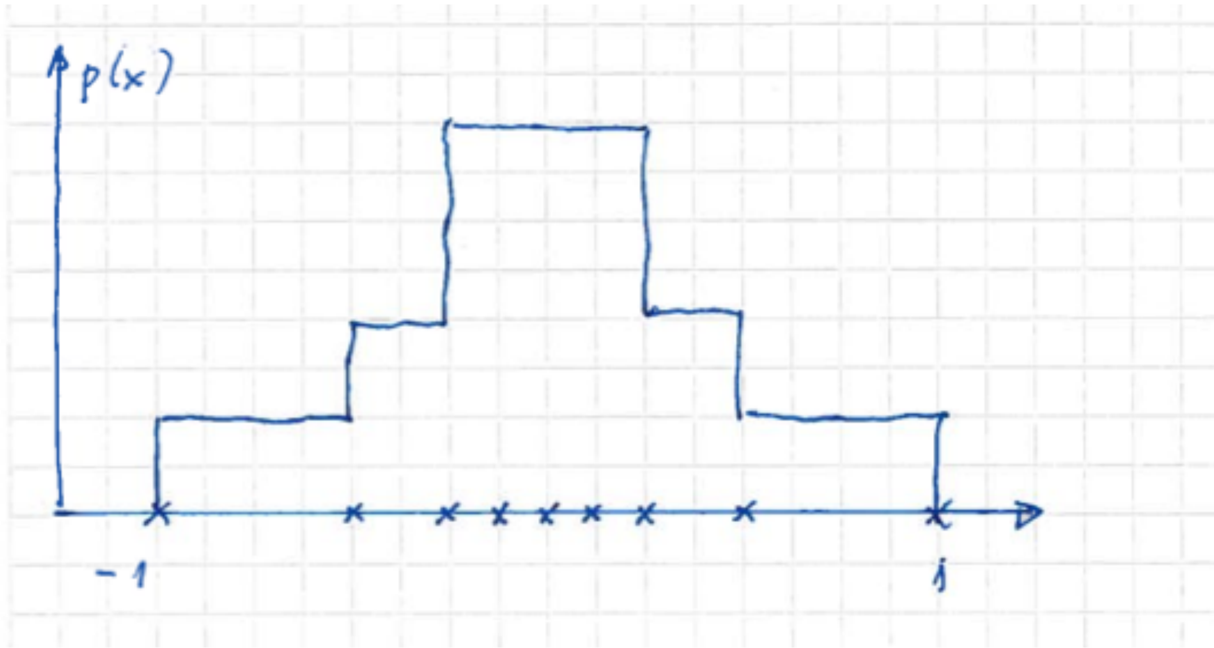

Figure 3: New decision borders - repr. values are not shown

Problem 5 *(2+2+2+2=8)*

A model of a baseband communication system is given in figure 1.

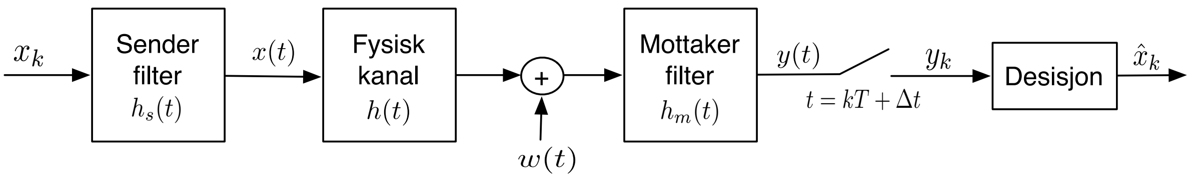

Figure 4: Modell for en digital overføringskanal

The received signal is given by :

$$y(t) = \sum_k x_k g(t - kT) \tag{11}$$

Here T is the distance between two adjacent transmitted channel symbols x_k. The complete channel is defined by $g(t) = h_s(t) * h(t) * h_m(t)$ while Δt is the delay due to the channel and $w(t)$ is white Gaussian channel noise .
In this task we assume for simplicity that $\Delta t = 0$.

5a) Show that $g(t)$ must fulfill the conditions below if it shall be possible to reconstruct the sequence $x(n)$ from the received signal $y(t)$

$$g(kT) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases} \tag{12}$$

A :
$y(nT) = y(n) = \sum_m x(m)g(nt - mT) = x(n)g(0) + \sum_{m \neq n} x(m)g((n-m)T)$. The last term is zero if $g(kT) = 0$ for any $k = n - m \neq 0$. The first term is $y(n) = x(n)$ if $g(0) = 1$.

5b) Show that the the corresponding conditions for $G(F)$, i.e. in the frequency domain is given by :

$$F_s \sum_k G(F - kF_c) = 1 \tag{13}$$

where $F_c = 1/T$.

A :
Given the above sequence $g(n)$ in eq. 12 the frequency response is given by $DTFT[g(n)] = \sum_n g(n)e^{-j2\pi fn} = 1$. Further $g(n)$ is a sampled version of $g(t)$. According to the Nyquist theorem in the frequency domain we then have $DTFT[g(n)] = 1 = F_s \sum_k G(F - kF_s)$.

5c) The channel capacity given below gives the upper limit for the channel symbol rate (in symbols/sec) for error free transmission :

$$C = \frac{1}{2}log_2(1 + \frac{P}{N}) \tag{14}$$

Here P/N is the signal-to-noise-ratio at the receiver.

Give the expression for the channel bitrate (in bits/sec) which corresponds to the channel capacity when $F_c = 1/T = 4000$

A : The max channel bit rate is given by $F_c C = 2000log_2(1 + \frac{P}{N})$

5d) Find the lowest value of P/N that makes it possible to achieve an error free transmission of the sequence $x(n)$ in task 4.

The bit rate in task 4 is 6000 bits/sec. Thus we must have :

$$2000log_2(1 + \frac{P}{N}) \geq 6000 \;\Rightarrow\; 1 + \frac{P}{N} \geq 2^3 \;\Rightarrow\; \frac{P}{N} \geq 7$$