NTNU
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Department of Electronics and Telecommunications

Examination paper for TTT4120 Digital Signal Processing

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Examination date: Monday, December 12, 2016

Examination time (from-to): 09:00-13:00

Permitted examination support material:

D - Basic calculator allowed

No printed or handwritten materials allowed

Other information:

- Exam consists of four (4) problems.
- A few basic formulas are provided in the Appendix
- Lecturer will visit you twice during exam: first time around 10 o'clock, and; second time around 12 o'clock.

Language: English.

Number of pages (front page excluded): 6

Number of pages enclosed: 2

Informasjon om trykking av eksamensoppgave

Originalen er:

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Date Signature

Problem 1 (2+6+4+6=18): Basics of filter theory and design

The system function of a causal filter is given by

$$H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

1a) Provide the difference equation corresponding to $H(z)$ in the form

$$y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M] - a_1 y[n-1] - \cdots - a_N y[n-N]$$

1b) Express the filter as a cascade of two filters, i.e.

$$H(z) = H_1(z) \cdot H_2(z) = \frac{1}{(1 - p_1 z^{-1})} \cdot \frac{1}{(1 - p_2 z^{-1})}$$

Additionally, draw the pole-zero plot based on your findings, and discuss what type of filter $H(z)$ is (lowpass, highpass, bandpass, allpass).

1c) Provide answers and motivations for the following:

- What is the region of convergence (ROC) for filter $H(z)$?
- Is the filter stable?
- Does the filter have linear phase?
- Is the filter minimum-phase?

1d) Express the filter in its parallel form

$$H(z) = H_3(z) + H_4(z) = \frac{A}{(1 - p_1 z^{-1})} + \frac{B}{(1 - p_2 z^{-1})}$$

Show that the unit impulse response is given by

$$h[n] = 2h_3[n] - h_4[n]$$

where

$$h_3[n] = \left(\frac{1}{2}\right)^n u[n]$$

$$h_4[n] = \left(\frac{1}{4}\right)^n u[n]$$

Problem 2 (6+9+3 = 18): Filter structures and implementations

The filter in **Problem 1** is implemented using fixed-point representation with $B+1$ bits and dynamic range $[-1,1)$. Rounding is performed after each multiplication and the rounding error $e[n]$ can be modeled as white noise with variance $\sigma_e^2 \; = 2^{-2B}/12$. Rounding noise sources combine into a noise signal $z[n]$ at the filter output with variance σ_z^2.

In other words, each multiplier in the fixed-point implementation is modeled as

$$Q(ay[n-k]) = ay[n-k] + e[n]$$

which is equivalent to adding noise sources after multipliers in the infinite-precision realization.

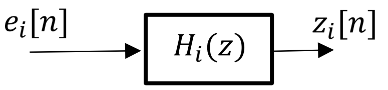
2a) Draw the direct-form structure II (DF-II) of $H(z)$ with noise sources due to rounding included. Determine the variance of the round-off noise at the filter output.

2b) Draw the two possible cascade-structures, $H_1(z)H_2(z)$ and $H_2(z)H_1(z)$, with noise sources due to rounding included. For each combination, determine the variance of the round-off noise at the filter output.

2c) Considering $B = 4$ bits are used in the implementations above. Which of the three implementations above suffers the most from rounding noise? Which implementation suffers the least? What can you say about the performances of the different realizations as the number of bits, B, increases or decreases?

[Hint:] Assuming noise source $e_i[n]$ with variance σ_{ei}^2 acts as input to (sub-)filter $h_i[n]$ that terminates at the output. The variance of the noise signal $z_i[n]$, due to $e_i[n]$, is given by

$$\sigma_{zi}^2 = \sigma_{ei}^2 r_{h_i h_i}[0] = \sigma_{ei}^2 \sum_k h_i^2[k]$$



$\sigma_{ei}^2 = E\{e_i^2[n]\}$ $\qquad$ $\sigma_{zi}^2 = E\{z_i^2[n]\}$

Problem 3 (4+6+6+2=18): Parametric modeling

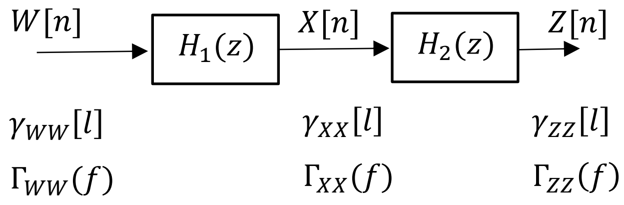


Fig. 1: Filtering of stochastic processes

The stochastic process $X[n]$ is modeled by filtering a white noise process $W[n]$, with autocorrelation sequence $\gamma_{WW}[l] = \sigma_W^2 \delta[l]$, through a filter $H_1(z)$. When $H_1(z)$ is causal and stable, the autocorrelation sequence and spectrum of $X[n]$ can be obtained from

$$\gamma_{XX}[l] = \begin{cases} \sigma_W^2 \sum_{n=0}^{\infty} h_1[n] h_1[n+l], & l \geq 0 \\ \gamma_{XX}[-l], & l < 0 \end{cases}$$

$$\Gamma_{XX}(f) = |H_1(f)|^2 \Gamma_{WW}(f)$$

3a) Provide answers (with motivations) to the following two questions:

- What type of process, AR(p), MA(q), or ARMA(p, q), is $X[n]$ when the noise is filtered through $H_1(z) = 1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}$? Provide the model order.
- What are the advantages of using a model-based (parametric) approach to spectrum estimation when compared to a non-parametric approach?

3b) Compute $\Gamma_{XX}(f)$ and $\gamma_{XX}[l]$ when the filter is known to be $H_1(z) = 1 - \frac{3}{4} z^{-1} + \frac{1}{8} z^{-2}$.

3c) Assume that you are only given the values $\gamma_{XX}[0]$ and $\gamma_{XX}[1]$ from **3b)**. You decide to model $X[n]$ using an AR(1) model.

- Compute the best AR(1) estimate for process $X[n]$, i.e., find $\hat{a}_1$ and σ_f^2.
- Form the spectrum estimate $\hat{\Gamma}_{XX}(f)$ that results from the AR(1) model.

3d) Given $H_1(z)$ in **3a)**, provide the whitening filter $H_2(z)$ such that $\Gamma_{ZZ}(f) = \sigma_W^2$.

Problem 4 (2+4+6+6): Sampling and rate-conversion

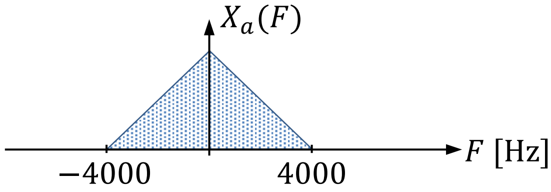


Fig. 2: Spectrum $X_a(F)$ of continuous-time signal $x_a(t)$

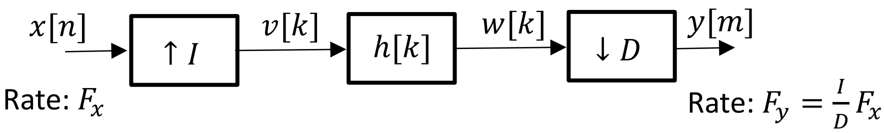


Fig. 3: Rate-conversion by a fractional factor $\frac{I}{D}$

Fig. 2 shows the spectrum $X_a(F)$ of the continuous-time signal $x_a(t)$. Signal $x_a(t)$ is sampled at rate $F_x = 1/T_x$ to generate sequence $x[n] = x_a(t)|_{t\ =nT_x}$. The sampling rate is thereafter changed in digital domain from F_x to F_y by passing $x[n]$ through the system in Fig. 3. Filter $h[k]$ is a lowpass filter with frequency response

$$H(f_v) = \begin{cases} 1, & |f_v| \leq \frac{1}{2\max(I,D)} \\ 0, & \text{otherwise} \end{cases}$$

4a) What is the minimum sampling rate F_x that can be used without losing information in the sampling process $x[n] = x_a(t)|_{t\ =nT_x}$? Describe what happens if the sampling rate is chosen smaller than this minimum rate.

4b) Sketch the spectrum $X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi f n}$ for the cases when $F_x = 4000$ Hz and $F_x = 16000$ Hz. Make appropriate comments relevant to aliasing and periodicity.

4c) For the case when $F_x = 16000$ Hz, $I = 2$ and $D = 3$

- Sketch the spectra of signals $v[k]$, $w[k]$ and $y[m]$ (i.e., $V(f)$, $W(f)$ and $Y(f)$)
- What is the rate of sequence $y[m]$?
- Comment on whether any information is lost in the rate conversion

4d) Repeat question **4c)** for the case when $F_x = 16000$ Hz, $I = 2$ and $D = 6$

Appendix: TTT4120 Table of formulas, 2016

A. Sequences:

$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ and $-\sum_{n=-1}^{-\infty} \alpha^n = \frac{1}{1-\alpha}$

$\sum_{n=0}^{N-1}(n+1)\alpha^n = \frac{1-\alpha^N}{(1-\alpha)^2} - \frac{N\alpha^N}{1-\alpha}$; $\alpha \neq 1$

$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty}(n+1)\alpha^n = \frac{1}{(1-\alpha)^2}$

B. Linear convolution:

$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$Y(z) = H(z)X(z)$

$Y(f) = H(f)X(f)$

$Y(k) = H(k)X(k)$, $k = 0,1,\dots,N-1$ where $Y(k) = Y(f_k)$ with $f_k = k/N$

C. Transforms:

Z-transform: $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$

DTFT: $H(f) = \sum_{n=-\infty}^{\infty} h[n]e^{-j2\pi fn}$

DFT: $H(k) = \sum_{n=0}^{N-1} h[n]e^{-j2\pi fnk/N}$ $k = 0,1,\dots,N-1$

IDFT: $h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi fnk/N}$ $n = 0,1,\dots,N-1$

D. Sampling theorem:

Given an analog signal $x_a(t)$ sampled at $F_s = 1/T$. The DTFT of the resulting discrete-time sequence $x[n] = x_a(t)|_{t=nT}$ is given by

$$X(f) = X(F/Fs) = F_s \sum_{k=-\infty}^{\infty} X([f-k]F_s)$$

E. Autocorrelation, energy spectrum and Parseval:

Given a sequence $h[n]$ with finite energy E_h

Autocorrelation: $r_{hh}[l] = \sum_{n=-\infty}^{\infty} h[n]h[n+l]$ $l \in \mathbb{Z}$

Energy spectrum: $S_{hh}(z) = H(z)H(z^{-1}) \Rightarrow S_{hh}(f) = |H(f)|^2$

Parseval's theorem: $E_h = r_{hh}[0] = \sum_{n=-\infty}^{\infty} h^2[n] = \int_0^{2\pi} |H(f)|^2 df$

F. Multirate:

Decimation (downsampling) where $T_y = DT_x$

$v(mT_y) = \sum_{k=-\infty}^{\infty} h[(mD-k)T_x]x(kT_x) \quad m \in \mathbb{Z}$

Interpolation (upsampling) where $T_y = T_x/I$

$y(lT_y) = \sum_{n=-\infty}^{\infty} h\big[(l-nI)T_y\big]x(nT_x) \qquad l \in \mathbb{Z}$

Rate coversion where $T_y = DT_v = \frac{D}{I}T_x$

$y(lT_y) = \sum_{m=-\infty}^{\infty} h[(lD-mI)T_v]x(mT_x) \quad l \in \mathbb{Z}$

G. Autocorrelation, power density spectrum and Wiener-Khintchin:

Given a wide-sense stationary and ergodic sequence $X[n]$ with infinite energy

Autocorrelation: $\gamma_{XX}[l] = E\{X[n]X[n+l]\} \qquad l \in \mathbb{Z}$

Power spectrum: $\Gamma_{XX}(z) = \mathcal{Z}\{\gamma_{XX}[l]\} \Rightarrow$

Wiener-Khintchin: $\Gamma_{XX}(f) = \mathrm{DTFT}\{\gamma_{XX}[l]\} = \sum_{l=-\infty}^{\infty} \gamma_{XX}[l]e^{-j2\pi fl}$

H. Yule-Walker and Normal equations where $a_0 = 1$:

Autocorrelation: $\sum_{k=0}^{P} a_k\gamma_{XX}[n-k] = \sigma_f^2\delta[n] \qquad n = 0, \dots, p$

Normal equations: $\sum_{k=1}^{P} a_k\gamma_{XX}[n-k] = -\gamma_{XX}[n] \qquad n = 1, \dots, p$

I. Some common z-transform pairs:

Sequence	Transform	ROC
$\delta[n]$	1	$\forall z$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-b^n u[-n-1]$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$
$(a^n \sin\omega_0 n)u[n]$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$(a^n \cos\omega_0 n)u[n]$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-nb^n u[-n-1]$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert < \lvert b\rvert$