SOLUTION PROBLEM 1

$$H(z) = \frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}$$

1a) $H(z) = \dfrac{Y(z)}{X(z)} \Rightarrow \left(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right)Y(z) = X(z)$

$$\Rightarrow \quad Y(z) = X(z) + \frac{3}{4}z^{-1}Y(z) - \frac{1}{8}z^{-2}Y(z)$$

$$\mathcal{Z}^{-1}\{Y(z)\} = \mathcal{Z}^{-1}\left\{X(z)+\frac{3}{4}z^{-1}Y(z)-\frac{1}{8}z^{-2}Y(z)\right\}$$

$$\Leftrightarrow \quad y[n] = x[n] + \frac{3}{4}y[n-1] - \frac{1}{8}y[n-2]$$

1b)

$$H(z) = \frac{1}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$= H_1(z)\,H_2(z) \quad \text{with} \quad H_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$$

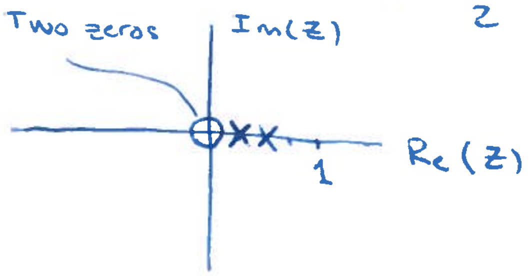
$$H_2(z) = \frac{1}{1-\frac{1}{4}z^{-1}}$$

Pole-zero plot:

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$\Rightarrow$ Two zeros at $z = 0$

One pole at $z = \frac{1}{2}$ and one pole at $z = \frac{1}{4}$



$\Rightarrow$ Lowpass filter

1c) i) System is causal and poles cannot be part of ROC $\Rightarrow$ $|z| > \frac{1}{2}$

ii) Unit circle is included in ROC $\Rightarrow$ System is stable.

iii) The filter has IIR so it cannot have exactly linear phase. (No point of symmetry)

iv) The filter is minimum-phase because zeros and poles are inside the unit circle.

1d)

$$H(z) = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-1}} = \frac{1}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{4}z^{-1})}$$

$$= \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{4}z^{-1}}$$

Residue calculus $\Rightarrow$ $A = 2$, $B = -1$

$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{2}{1-\frac{1}{2}z^{-1}} - \frac{1}{1-\frac{1}{4}z^{-1}}\right\}$$

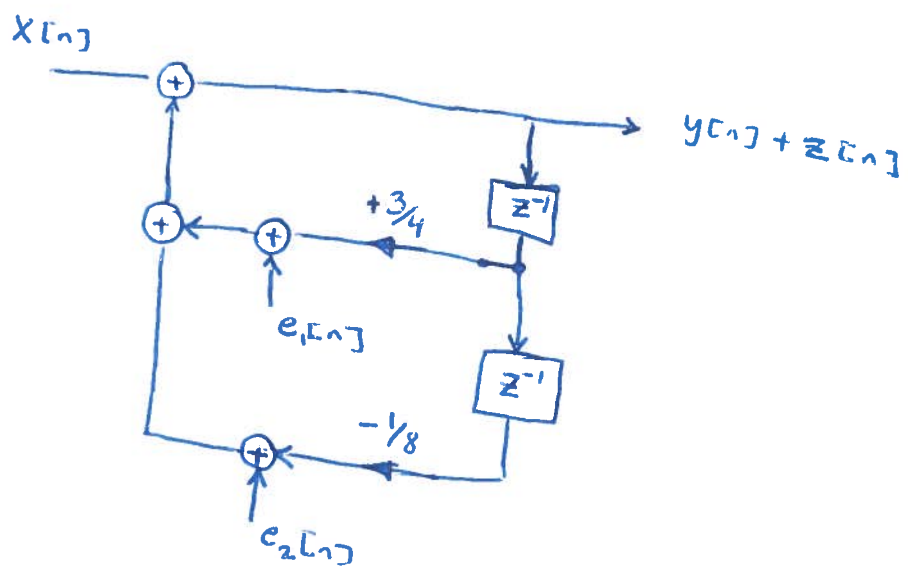
$$= 2\,\mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{1}{2}z^{-1}}\right\} - \mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{1}{4}z^{-1}}\right\}$$

$$= 2\left(\frac{1}{2}\right)^n u[n] - \left(\frac{1}{4}\right)^n u[n]$$

$$= \left[2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right] u[n]$$

ANSWER PROBLEM 2

2a)



$$z[n] = (e_1[n] + e_2[n]) * h[n]$$

$$\sigma_z^2 = \sigma_{z_1}^2 + \sigma_{z_2}^2 = \sigma_{e_1}^2 r_{hh}[0] + \sigma_{e_2}^2 r_{hh}[0] = \{\sigma_{e_1}^2 = \sigma_{e_2}^2\} =$$

$$= 2\sigma_{e_1}^2 r_{hh}[0] = 2\sigma_e^2 \sum_{n=0}^{\infty} h^2[n]$$

$$r_{hh}[0] = \sum_{k=0}^{\infty} h^2[n] = \sum_{k=0}^{\infty} \left(2\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n\right)^2$$

$$= \sum_{k=0}^{\infty} \left(4\left(\tfrac{1}{2}\right)^{2n} - 4\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{4}\right)^n + \left(\tfrac{1}{4}\right)^{2n}\right)$$

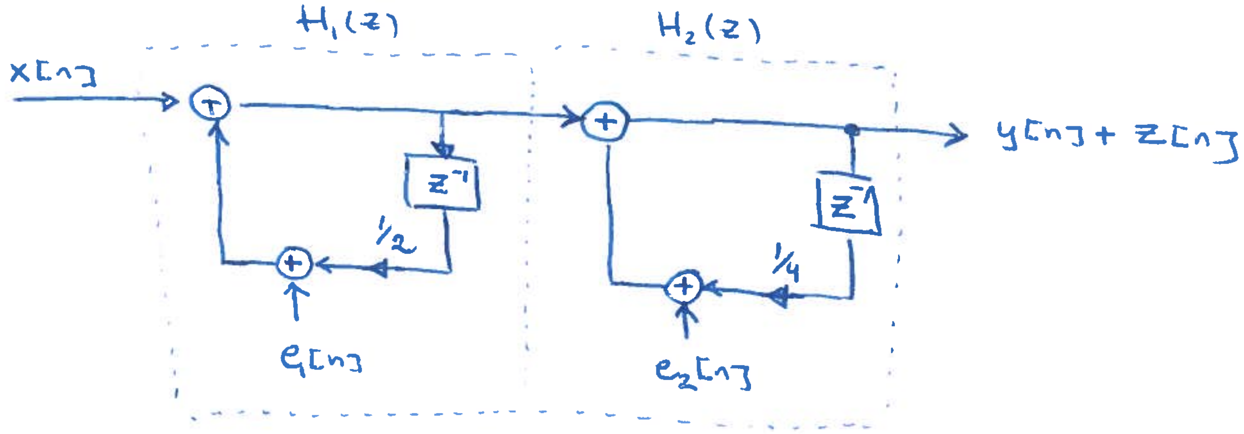
$$= 4 \cdot \frac{1}{1-\frac{1}{4}} - 4 \cdot \frac{1}{1-\frac{1}{8}} + \frac{1}{1-\frac{1}{16}} = \frac{16}{3} - \frac{32}{7} + \frac{16}{15} = \frac{64}{35}$$

$$\Rightarrow \sigma_z^2 = 2 \cdot \frac{64}{35} \sigma_{e_1}^2 = \frac{128}{35} \cdot \frac{2^{-2B}}{12} = \frac{128}{35} \cdot \frac{2^{-8}}{12} = \frac{1}{840}$$

$$\approx 1{,}24 \cdot 10^{-3}$$

2b) $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1}{1-\frac{1}{4}z^{-1}} = \frac{1}{1-\frac{1}{4}z^{-1}} \cdot \frac{1}{1-\frac{1}{2}z^{-1}}$

i)



$$z[n] = e_1[n] * h[n] + e_2[n] * h_2[n]$$

$$\sigma_z^2 = \sigma_{e_1}^2 \, r_{hh}[0] + \sigma_{e_2}^2 \, r_{h_2h_2}[0]$$

$$= \sigma_{e_1}^2 \cdot \frac{64}{35} + \sigma_{e_2}^2 \sum_{n=0}^{\infty} h_2^2[n] = \sigma_{e_1}^2 \frac{64}{35} + \sigma_{e_2}^2 \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{2n}$$

$$= \left\{ \sigma_{e_1}^2 = \sigma_{e_2}^2 \right\} = \sigma_{e_1}^2 \left(\frac{64}{35} + \frac{1}{1-\frac{1}{16}} \right) = \frac{304}{105} \cdot \sigma_{e_1}^2$$

$$= \frac{304}{105} \cdot \frac{2^{-2B}}{12} = \left\{ 304 = 2^4 \cdot 19 \right\} = \frac{19}{105} \cdot \frac{2^4 \cdot 2^{-8}}{12} = \frac{19}{20160}$$

$$\approx 9{,}42 \cdot 10^{-4}$$

ii) Change order of blocks in above figure

$$z[n] = e_2[n] * h[n] + e_1[n] * h_1[n]$$

$$\sigma_z^2 = \sigma_{e_1}^2 \left(r_{hh}[0] + r_{h_1h_1}[0] \right) = \sigma_{e_1}^2 \left(\frac{64}{35} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \right)$$

$$= \sigma_{e_1}^2 \left(\frac{64}{35} + \frac{1}{1-\frac{1}{4}} \right) = \frac{332}{105} \cdot \frac{2^{-8}}{12} = \frac{83}{80640} \approx 1{,}03 \cdot 10^{-3}$$

2c) i) DF II suffers most from rounding errors

ii) Cascade implementation $\frac{1}{1-\frac{1}{2}z^{-1}} \cdot \frac{1}{1-\frac{1}{4}z^{-1}}$ suffers the least

iii) As the number of bits increases, the rounding noise at the output will decrease. For a large number of bits, all realizations will perform the same.

If the number of bits decreases, the noise power at the output will increases for all realizations

- Note that the solutions to σ_z^2 for the different implementations and $B=4$ were given in 2a) and 2b) You may have stated them here instead, which is also fine. As long as the numerical values are stated somewhere in your solution.

3a) i) $X[n]$ is a moving average process (MA) of order $q = 2$, i.e.,

$$X[n] = W[n] - \frac{3}{4} W[n-1] + \frac{1}{8} W[n-2]$$

ii) - Non-parametric methods require a lot of samples for good frequency resolution

- Non-parametric models suffer from spectral leakage due to windowing $\Rightarrow$ can mask weak signals

- Basic assumption of non-parametric methods is that $\gamma_{xx}[l] = 0$ for $l \geq N$, N being the data length.

- Parametric methods eliminates need for windowing and assumption that $\hat{\gamma}_{xx}[l] = 0$ for some $l \geq N$

- Approximation characterized by a few parameters

- Parametric models allow us to extrapolate missing values.

The answer shall cover something relevant to the above points

3 b) $\Gamma_{xx}(f) = |H(f)|^2 \Gamma_{ww}(f)$, where

$$\Gamma_{ww}(f) = \sigma_w^2$$

$$H(f) = H(z)\Big|_{z=e^{jw}} = \left(1-\tfrac{1}{2}z^{-1}\right)\left(1-\tfrac{1}{4}z^{-1}\right)\Big|_{z=e^{jw}}$$

$$= \left(1-\tfrac{1}{2}e^{-jw}\right)\left(1-\tfrac{1}{4}e^{-jw}\right)$$

$$|H(f)|^2 = H(f)H^*(f) = \left(1+\tfrac{1}{4}-\cos w\right)\left(1+\tfrac{1}{16}-\tfrac{1}{2}\cos w\right)$$

$$\gamma_{xx}[l] = \begin{cases} \sigma_N^2 \cdot \left(1+\left(\tfrac{3}{4}\right)^2+\left(\tfrac{1}{8}\right)^2\right) = \sigma_w^2 \cdot \tfrac{101}{64}, & l=0 \\ \sigma_w^2\left(-\tfrac{3}{4}-\tfrac{3}{4}\cdot\tfrac{1}{8}\right) = -\sigma_w^2\,\tfrac{27}{32}, & l=\pm 1 \\ \sigma_w^2\left(1\cdot\tfrac{1}{8}\right) = \sigma_w^2\cdot\tfrac{1}{8} & l=\pm 2 \\ 0 & |l|>2 \end{cases}$$

3c) $\sigma_f^2 = a_0 \gamma_{xx}[0] + a_1 \gamma_{xx}[-1], \quad n=0$

$0 = a_0 \gamma_{xx}[-1] + a_1 \gamma_{xx}[0], \quad n=1$

$$a_0 = 1 \Rightarrow \quad a_1 = \frac{-\gamma_{xx}[-1]}{\gamma_{xx}[0]} = \frac{27}{32} \cdot \frac{64}{101} = \frac{54}{101} \approx \underline{\underline{0{,}53}}$$

$$\sigma_f^2 = \sigma_w^2 \frac{101}{64} + \sigma_w^2 \frac{54}{101}\left(-\frac{27}{32}\right) = \sigma_w^2 \frac{7285}{6464}$$

$$\approx \underline{\underline{1{,}13\,\sigma_w^2}}$$

$$\Gamma_{xx}(f) = |H_1(f)|^2 \Gamma_{ww}(f) = \frac{\sigma_f^2}{|1 - a_1 e^{-j2\pi f}|^2} =$$

$$= \frac{1}{1 + a_1^2 - 2a_1 \cos 2\pi f}, \quad \text{where } a_1 = 0{,}53$$

3d) $$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}}$$

$$\Gamma_{zz}(z) = H_2(z) H_2(z^{-1}) \Gamma_{xx}(z)$$

$$= H_2(z) H_2(z^{-1}) H_1(z) H_1(z^{-1}) \sigma_w^2$$

$$= \sigma_w^2 \quad \text{when } H_2(z) = \frac{1}{H_1(z)}.$$

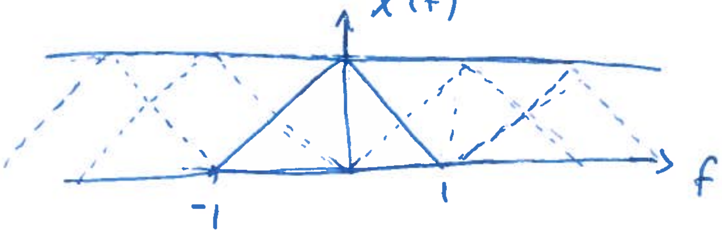
ANSWER PROBLEM 4

4a) $F_x \geq 2B$ where B is the single-sided bandwidth of $x_a(t)$. From figure we get $B = 4000\,\text{Hz} \Rightarrow F_x \geq 8000\,\text{Hz}$

If $F_x < 8000\,\text{Hz}$ we get aliasing in $X(f)$ and $x_a(t)$ cannot be perfectly reconstructed from $x[n]$.

4b) $X(f) = X(F/F_x) = F_x \sum_k X([f-k]F_s)$

$F_x = 4000\,\text{Hz} \Rightarrow f = 1$



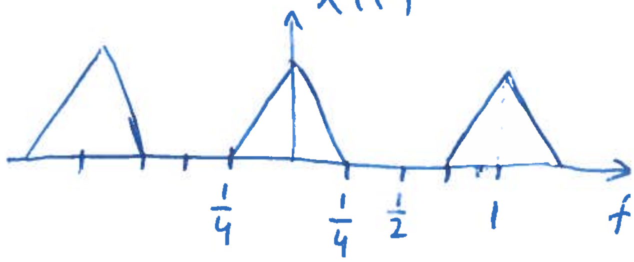
$F_x = B$,

$X(f) = \text{const} \quad \forall |f| < 1$

$X(f+k) = X(f), \quad k \in \mathbb{Z}$

Aliasing!

$F_x = 4000\,\text{Hz} \Rightarrow f = \frac{1}{4}$



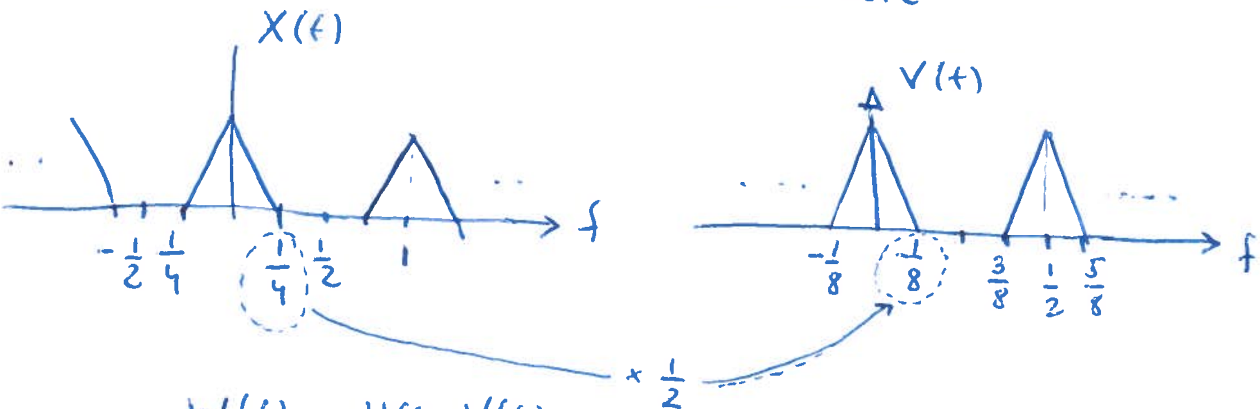
$F_x = 4B$

$X(f+k) = X(f)$

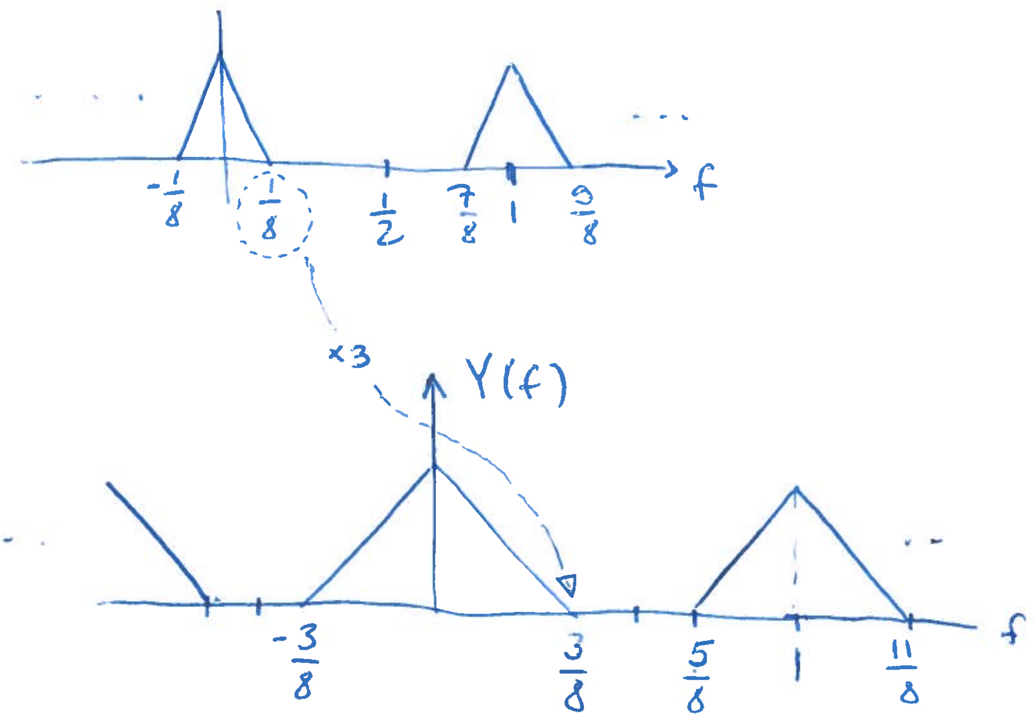
No aliasing!

4c) $F_x = 16000\,Hz \quad I = 2\,,\ D = 3$

$$H(f_v) = \begin{cases} 1\,, & |f_v| < \frac{1}{6} \\ 0 & \text{elsewhere} \end{cases}$$



$W(f) = H(f)\cdot V(f)$



- No information lost in the rate conversion, since $H(f)$ did not filter away information in $X(f)$

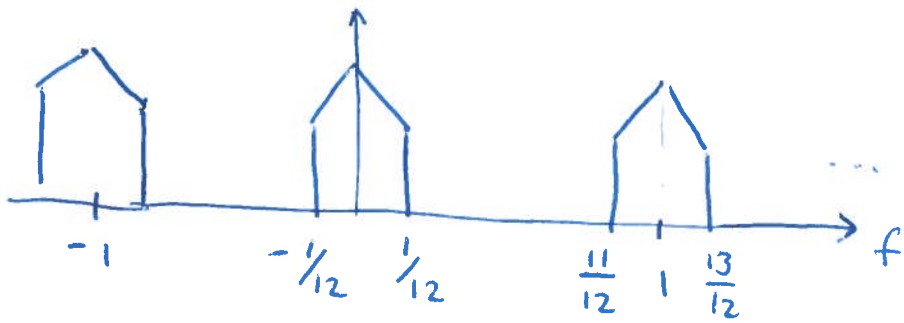
- Rate $F_y = \frac{I}{D} F_x = \frac{2}{3}\,16000\,Hz = \frac{32000}{3}\,Hz$

4d) $F_x = 4B$, $I = 2$, $D = 6$

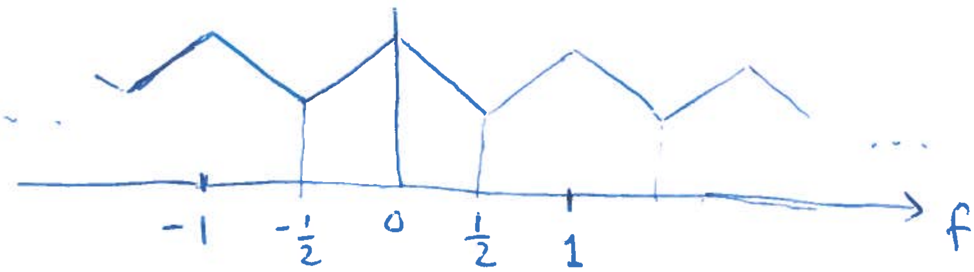
$V(f)$ same as in 4c

$$H(f) = \begin{cases} 1 & |f| \leq \frac{1}{12} \\ 0 & \text{elsewhere} \end{cases}$$

$W(f) = H(f)V(f)$



$Y(f)$



- Information was lost in the rate conversion
- Rate $F_y = \frac{I}{D} \cdot F_x = \frac{2 \cdot 16000\,Hz}{6} = \frac{16000}{3}\,Hz$