SOLUTION PROBLEM 1

EXAM 12.8.2017

1

$$y[n] = x[n] + \alpha^2 y[n-2]$$

1a) $H(z) = \dfrac{Y(z)}{X(z)} \Rightarrow$

$$\mathcal{Z}\{y[n]\} = \mathcal{Z}\{x[n]\} + \mathcal{Z}\{\alpha^2 y[n-2]\}$$

$$\Leftrightarrow \quad Y(z) = X(z) + \alpha^2 z^{-2} Y(z)$$

$$\Rightarrow \quad H(z) = \frac{1}{1-\alpha^2 z^{-2}}$$

1b)

$$H(z) = \frac{1}{1-\alpha^2 z^{-2}} = \frac{1}{(1+\alpha z^{-1})} \cdot \frac{1}{(1-\alpha z^{-1})}$$

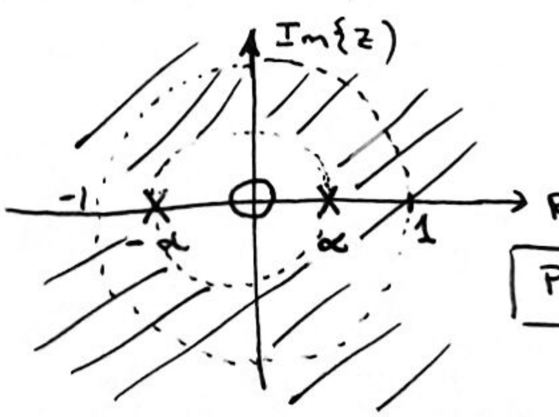
$$= \frac{z}{(z+\alpha)} \cdot \frac{z}{(z-\alpha)} \quad \left\{ \frac{(z-z_1)}{(z-p_1)} \cdot \frac{(z-z_2)}{z-p_2} \right.$$

$\therefore$ Two zeroes in $z_1 = z_2 = 0$

Poles in $p_1 = \alpha$ and $p_2 = -\alpha$

- Causal filter $\Rightarrow$ ROC exterior of the circle associated with poles

- Stable filter poles inside unit circle $\Rightarrow |\alpha| < 1$



$\sim$ ROC $\forall z : |z| > \alpha$

$\boxed{\text{PZ-PLOT}}$

1 c)

$$H(z) = \frac{1}{(1+\alpha z^{-1})} \cdot \frac{1}{(1-\alpha z^{-1})} = \frac{A}{1+\alpha z^{-1}} + \frac{B}{1-\alpha z^{-1}}$$

Residue calculus $\Rightarrow \quad A = B = \frac{1}{2}$

$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{1+\alpha z^{-1}}\right\} + \mathcal{Z}^{-1}\left\{\frac{1}{2}\cdot\frac{1}{1-\alpha z^{-1}}\right\}$$

$$= \frac{1}{2}(-\alpha)^n u[n] + \frac{1}{2}\alpha^n u[n] = \frac{1}{2}\left[(-\alpha)^n + \alpha^n\right] u[n]$$

$$= \begin{cases} \alpha^n & n \text{ even},\ n \geq 0 \\ 0 & n \text{ odd},\ n > 0 \end{cases}$$

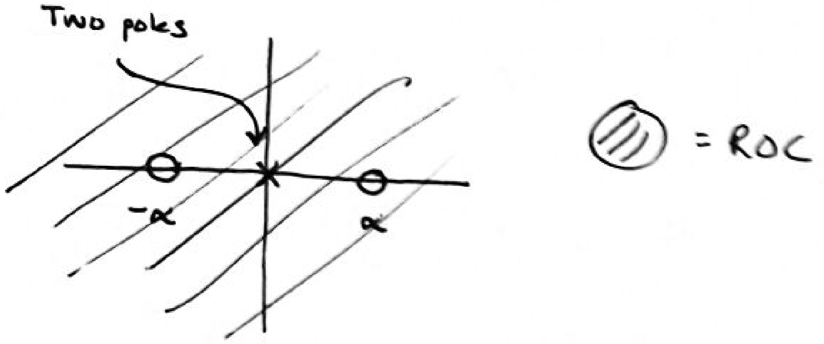
1 d)

i) $H_I(z) = \frac{1}{H(z)} = 1 - \alpha^2 z^{-2} = \frac{z^2 - \alpha^2}{z^2}$

ii) Zeroes in $z_{1,2} = \pm\alpha$

Two poles in $p_1 = p_2 = 0$

$\Rightarrow$ ROC $\forall z \setminus z = 0$



iii) Filter has zero gain at $\omega = 0$ & $\omega = \pi$

$\Rightarrow$ Bandpass

iv) For $H_I(z)$ to be a minimum-phase filter poles and zeros should be inside unit circle

$\Rightarrow \; |\alpha| < 1$

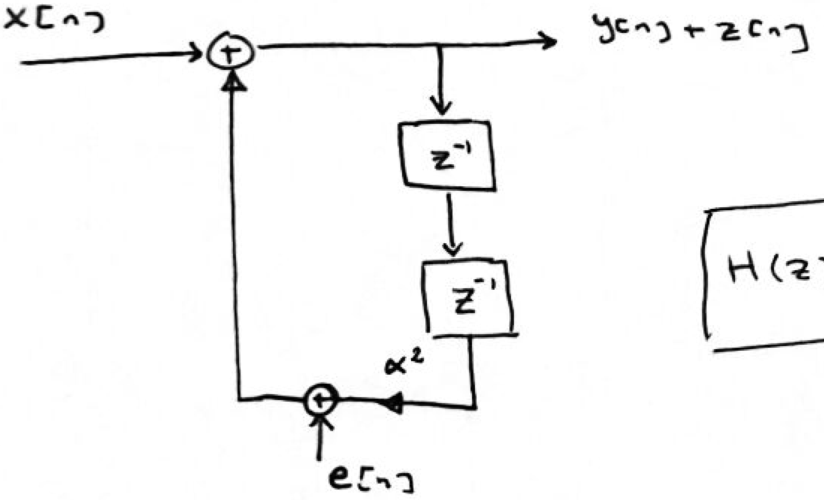
v) In general $H_I(z)$ is not linear phase since $h_I[n]$ needs to satisfy symmetry/antisymmetry conditions.

However, if $\alpha = \pm 1 \Rightarrow h_I[n] = 1 - z^{-2}$

or $H_I(z) = 1 - z^{-2} \Rightarrow H_I(\omega) = 1 - e^{-j2\omega} = j2\sin\omega \, e^{-j\omega}$

SOLUTION PROBLEM 2

2a)



$$\boxed{H(z) = \frac{1}{1-\alpha^2 z^{-2}}}$$

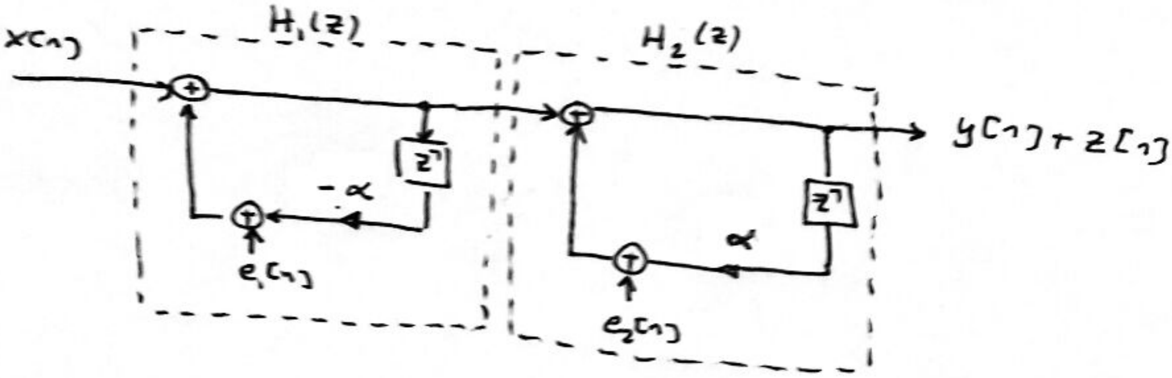
$$z[n] = e[n] * h[n]$$

$$\sigma_z^2 = \sigma_e^2\, r_{hh}[0] = \sigma_e^2 \sum_k h^2[n] = \sigma_e^2 \sum_{k \text{ even}} (\alpha^n)^2 =$$

$$= \sigma_e^2 \sum_{k=0}^{\infty} \alpha^{4n} = \frac{\sigma_e^2}{1-\alpha^4} = \{\alpha = 0{,}5\} = \underline{\underline{\frac{16}{15}\sigma_e^2}}$$

2b/

$$H(z) = \frac{1}{1+\alpha z^{-1}} \cdot \frac{1}{1-\alpha z^{-1}}, \quad \alpha = \frac{1}{2}$$

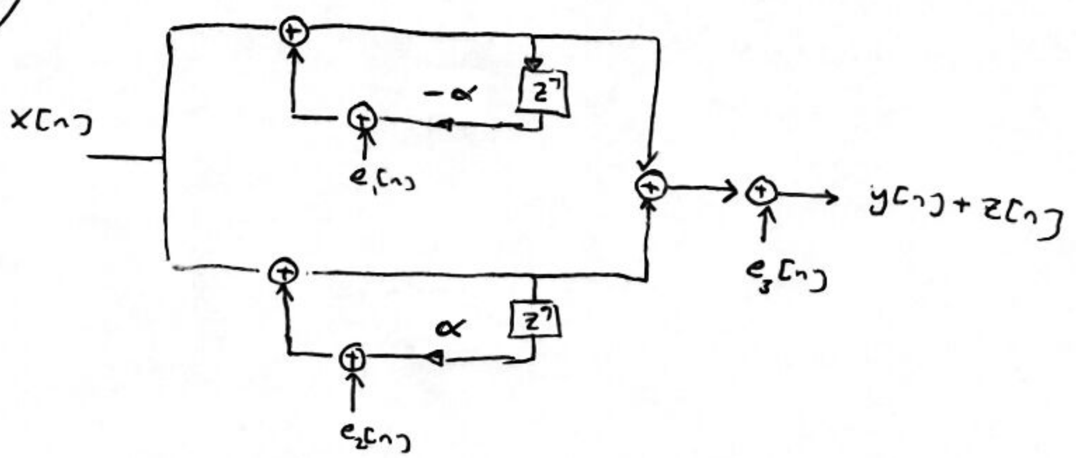


$$z[n] = e_1[n] * h[n] + e_2[n] * h_2[n]$$

$$\sigma_z^2 = \sigma_{e_1}^2 r_{hh}[0] + \sigma_{e_2}^2 r_{h_2h_2}[0] = \sigma_{e_1}^2 \sum_k h^2[n] + \sigma_{e_2}^2 \sum_k h_2^2[n]$$

$$= \frac{16}{15}\sigma_{e_1}^2 + \sigma_{e_2}^2 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2n} = \frac{16}{15}\sigma_{e_1}^2 + \frac{4}{3}\sigma_{e_2}^2 = \left\{\sigma_{e_1}^2 = \sigma_{e_2}^2\right\} = \underline{\underline{\frac{36}{15}\sigma_e^2}}$$

2c)



$$z[n] = e_1[n] * h_1[n] + e_2[n] * h_2[n] + e_3[n]$$

$$\sigma_z^2 = \sigma_{e_1}^2 r_{h_1h_1}[0] + \sigma_{e_2}^2 r_{h_2h_2}[0] + \sigma_{e_3}^2$$

$$= \sigma_{e_1}^2 \sum_n h_1^2[n] + \sigma_{e_2}^2 \sum_n h_2^2[n] + \sigma_{e_3}^2 = \left\{ \begin{matrix} \sigma_{e_1}^2 = \sigma_{e_2}^2 = \sigma_{e_3}^2 \\ h_1^2[n] = h_2^2[n] \end{matrix} \right\} =$$

$$= 2\sigma_e^2 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^{2n} + \sigma_e^2 = \underline{\underline{\frac{11}{3}\sigma_e^2}}$$

2d)

DF-II structure suffers the least.

Parallel structure the most.

SOLUTION PROBLEM 3

3a) Using the filter in 1a) we have

$$x[n] - \alpha^2 x[n-2] = w[n]$$

This is an AR(2)-model

Compare with the definition of an AR(p)-model

$$x[n] + \sum_{k=1}^{p} a_k x[n-k] = w[n]$$

Here $a_1 = 0$, $a_2 = \alpha^2$

3b)

$$\Gamma_{xx}(f) = |H(f)|^2 \Gamma_{ww}(f)$$

$$\Gamma_{ww}(f) = \mathcal{F}\{\gamma_{ww}[l]\} = \sigma_w^2 = 1$$

$$|H(f)|^2 = H(z)H(z^{-1})\Big|_{z=e^{j2\pi f}}$$

$$= \frac{1}{1-\alpha^2 z^{-2}} \cdot \frac{1}{1-\alpha z^2} = \frac{1}{1-\alpha^2(z^{-2}+z^2)+\alpha^4}\Bigg|_{z=e^{j2\pi f}}$$

$$= \frac{1}{1+\alpha^4 - 2\alpha^2\cos 4\pi f} = \{\alpha = \tfrac{1}{2}\} =$$

$$= \frac{1}{\frac{17}{16} - \frac{1}{2}\cos 4\pi f}$$

$$\Rightarrow \Gamma_{xx}(f) = \frac{\sigma_w^2}{\frac{17}{16} - \frac{1}{2}\cos 4\pi f} = \frac{1}{\frac{17}{16} - \frac{1}{2}\cos 4\pi f}$$

$$\underline{\qquad} \times \underline{\qquad}$$

#

$h[n] = \alpha^n$, n even

$l \geq 0$:

$$\gamma_{xx}[l] = \sum_n h[n]h[n+l] = \sum_{\substack{n \text{ even} \\ l \text{ even}}} \alpha^n \alpha^{n+l} = \alpha^l \sum_{n \text{ even}} \alpha^{2n}$$

$$= \alpha^l \sum_n \alpha^{4n} = \alpha^l \cdot \frac{1}{1-\alpha^4} = \frac{16}{15}\frac{1}{2^l}$$

$$\gamma[l] = \gamma[-l] \Rightarrow \gamma_{xx}[l] = \begin{cases} \frac{16}{15}\frac{1}{2^{|l|}}, & l \text{ even} \\ 0 & \text{otherwise} \end{cases}$$

3c) Optimal coefficients and prediction error power can be obtained from the normal equations

$$\begin{bmatrix} \gamma_{xx}[0] & \gamma_{xx}[-1] & \gamma_{xx}[-2] \\ \gamma_{xx}[1] & \gamma_{xx}[0] & \gamma_{xx}[-1] \\ \gamma_{xx}[2] & \gamma_{xx}[1] & \gamma_{xx}[0] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ 0 \end{bmatrix}$$

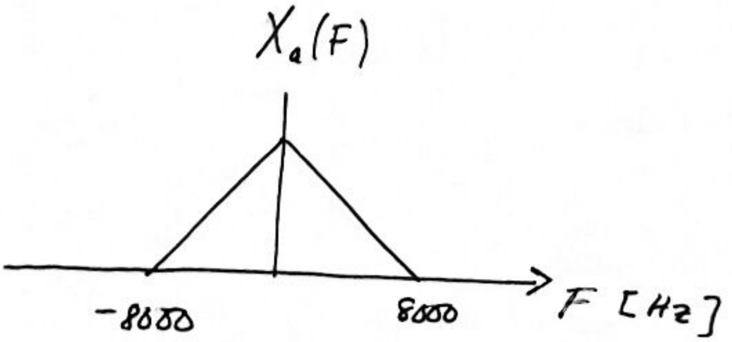
$$\Rightarrow \begin{bmatrix} \frac{16}{15}\sigma_w^2 & 0 & \frac{4}{15}\sigma_w^2 \\ 0 & \frac{16}{15}\sigma_w^2 & 0 \\ \frac{4}{15}\sigma_w^2 & 0 & \frac{16}{15}\sigma_w^2 \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad a_1 = 0$$

$$a_2 = -\frac{1}{4}$$

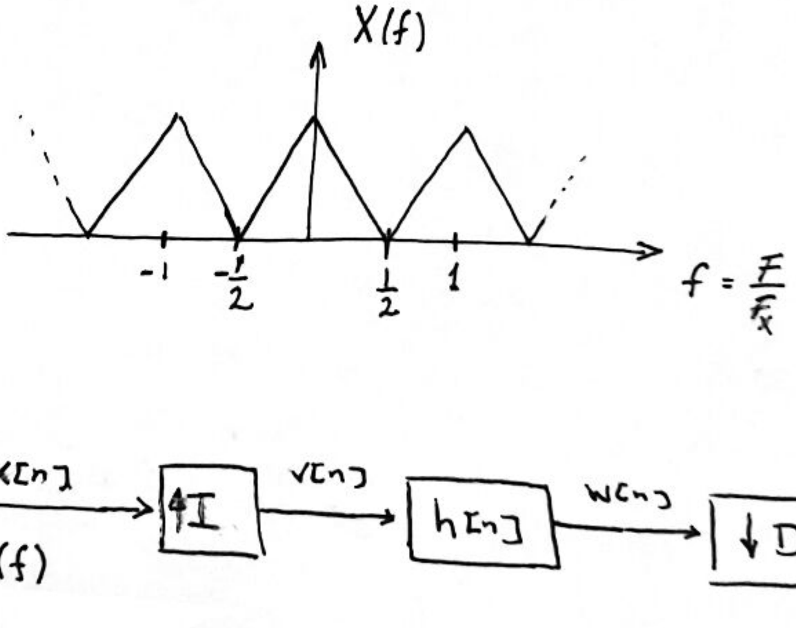
$$\sigma_e^2 = \sigma_w^2$$

SOLUTION PROBLEM 4

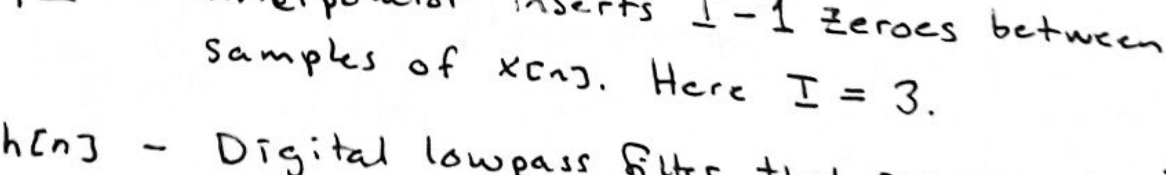


Sampling is to be reduced by a factor $\frac{12}{16} = \frac{3}{4}$
To achieve this we first increase the sampling frequency with 3 and thereafter reduce it with a factor of 4

$F_x = 16000$ Hz



a)

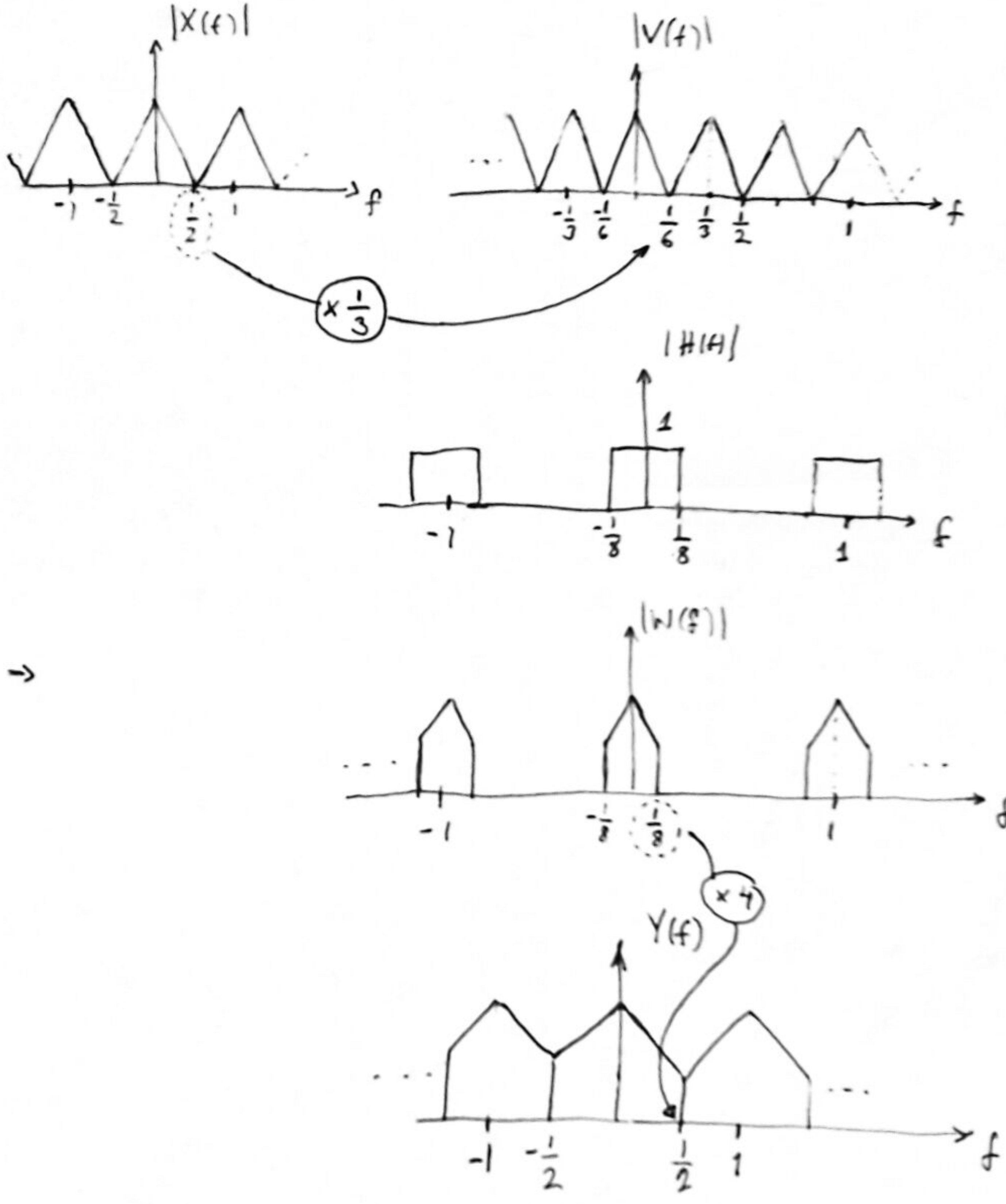


- ↑I – Interpolator inserts I − 1 zeroes between samples of x[n]. Here I = 3.
- h[n] – Digital lowpass filter that removes aliasing

$$H(f_v) = \begin{cases} 1, & |f_v| \leq \frac{1}{2\max\{I,D\}} = \frac{1}{8} \\ 0 & \text{otherwise} \end{cases}$$

- ↓D – Decimator retains every Dth sample of w[n]. Here D = 4

b) & c)



- Information is lost in rate conversion due to LP-filtering
- Rate $F_y = \frac{I}{D} F_x = \frac{3}{4} \cdot 16000 \text{ Hz} = 12000 \text{ Hz}$