NORWEGIAN UNIVERSITY
OF SCIENCE AND TECHNOLOGY
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EXAMINATION IN COURSE TTT4120 DIGITAL SIGNAL PROCESSING

Date: Friday Desember 3rd, 2010
Time: 09.00 - 13.00

Permitted aids: D–No printed or handwritten material allowed.
Specified, simple calculator allowed..

INFORMATION

- The examination consists of 4 problems.
  - Problem 1 concerns analysis of digital filters.
  - Problem 2 concerns rational process models.
  - Problem 3 concerns fixed point implementation.
  - Problem 4 concerns Wiener filters.
  - All tasks are weighted equally.
  - A list of formulas can be found in the appendix.
- All answers should be justified!
- The teacher will visit you twice, the first time around 10.00 and the second time around 11.45.

Problem 1

1a) Given a causal, linear and time invariant (LTI) system with unit pulse response $h(n)$.

Give the region of convergence (ROC) in the z-plane for the system.

Where must the zeros and poles be placed in the z-plane?

Answer : ROC is $|z| > |a|$ and $|a| < 1$
No restriction on zeros, however poles must be inside unit circle, i.e. $\max_k |p_k| < |a|$

1b) A causal LTI filter is described by the following difference equation :

$$y(n) + \frac{1}{6}y(n-1) - \frac{1}{6}y(n-2) = 2x(n) + \frac{1}{6}x(n-1), \ n = -\infty, \infty \qquad (1)$$

Show that the transfer function $H(z)$ is given by :

$$H(z) = \frac{2 + \frac{1}{6}z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})} \qquad (2)$$

Answer :

$$Y(z) + \frac{1}{6}Y(z)z^{-1} - \frac{1}{6}Y(z)z^{-2} = 2X(z) + \frac{1}{6}X(z)z^{-1} \quad \Rightarrow$$

$$Y(z)(1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}) = X(z)(2 + \frac{1}{6}z^{-1}) \quad \Rightarrow$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 + \frac{1}{6}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}}$$

It is easy to see that the roots of the denominator is respectively $z = -\frac{1}{2}$ and $z = \frac{1}{3}$

1c) Give the ROC area in the z-plane for the filter in task 1b.

Is the filter stable?

Does the filter have minimum phase?

Answer : $\max[|p_1|, |p_2|] = \frac{1}{2} \quad \Rightarrow \text{ ROC} : |z| > \frac{1}{2}$
The causality and the ROC together guarantees stability
Minimum phase requires that both poles and zeros are inside the unit circle. We have a zeros at $n_1 = -\frac{1}{12}$. Thus the filter has minimum phase.

1d) Show that the unit impulse response of the filter is given by :

$$h(n) = \begin{cases} (-\frac{1}{2})^n + (\frac{1}{3})^n & n \geq 0 \\ 0 & n < 0 \end{cases} \tag{3}$$

Answer :

$$\frac{2 + \frac{1}{6}z^{-1}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} = \frac{C}{1 + \frac{1}{2}z^{-1}} + \frac{D}{1 - \frac{1}{3}z^{-1}} \quad \Rightarrow$$

$$C + D \;=\; 2$$

$$(-\frac{C}{3} + \frac{D}{2})z^{-1} \;=\; \frac{1}{6}z^{-1} \quad \Rightarrow \quad -2C + 3D = 1$$

which give $C = D = 1$!

Thus :

$$H(z) \;=\; \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{1}{1 - \frac{1}{3}z^{-1}} \quad \Rightarrow$$

$$h(n) \;=\; \begin{cases} (-\frac{1}{2})^n + (\frac{1}{3})^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Problem 2

2a) Sketch the DF2 and the parallel form structures for the filter in task 1.

2b) The autocorrelation sequence of a general filter is defined as :

$$r_{hh}(m) = \sum_{n=-\infty}^{\infty} h(n)h(n+m) \quad m = -\infty, \infty \tag{4}$$

Show that the filter in task 1 has the following autocorrelation sequence :

$$r_{hh}(m) = \begin{cases} \frac{46}{21}(-\frac{1}{2})^m + \frac{111}{56}(\frac{1}{3})^m & m \geq 0 \\ r_{hh}(-m) & m < 0 \end{cases} \tag{5}$$

Answer : we assume $m \geq 0$

$$r_{hh}(m) = \sum_n [(-\frac{1}{2})^n + (\frac{1}{3})^n][(-\frac{1}{2})^{n+m} + (\frac{1}{3})^{n+m}] = A(-\frac{1}{2})^m + B(\frac{1}{3})^m$$

where

$$\begin{aligned} A &= \sum_n [(-\frac{1}{2})^n + (\frac{1}{3})^n](-\frac{1}{2})^n = \sum_n (\frac{1}{4})^n + (-\frac{1}{6})^n \\ B &= \sum_n [(-\frac{1}{2})^n + (\frac{1}{3})^n](\frac{1}{3})^n = \sum_n (-\frac{1}{6})^n + (\frac{1}{9})^n \end{aligned}$$

Utilizing that the sums go from $0 \to \infty$ we get :

$$\begin{aligned} A &= \frac{1}{1-\frac{1}{4}} + \frac{1}{1+\frac{1}{6}} = \frac{4}{3} + \frac{6}{7} = \frac{4*7+3*6}{3*7} = \frac{46}{21} \approx 2.2 \\ B &= \frac{1}{1+\frac{1}{6}} + \frac{1}{1-\frac{1}{9}} = \frac{6}{7} + \frac{9}{8} = \frac{6*8+7*9}{7*8} = \frac{111}{56} \approx 2.0 \end{aligned}$$

2c) White noise $w(n)$ with power $\sigma_w^2 = 1$ is input to the filter in task 2a.

Which kind of parametric processes are respectively the internal signals of the DF2 structure and the output signal?

Answer : The difference equation can be rewritten to include the internal sequence g(n) in the DF2-structure :

$$
\begin{aligned}
g(n) &= -\frac{1}{6}g(n-1) + \frac{1}{6}g(n-2) + w(n) \qquad (6)\\
y(n) &= 2g(n) + \frac{1}{6}g(n-1) \qquad (7)
\end{aligned}
$$

The internal sequence $g(n)$ is thus an AR[2]-process, while the output sequence $y(n)$ is an ARMA[1,2]-prosess.

2d) Find, by using linear prediction (i.e. the Yule-Walker equations), the process parameters for the best AR[1]-model of the filter output signal $y(n)$.

Answer : The Yule-Walker equations for the best AR[1] -model are given by

$$
\begin{aligned}
\gamma_{yy}(1) &= -a\gamma_{yy}(0) \quad \Rightarrow \quad a = -\frac{\gamma_{yy}(1)}{\gamma_{yy}(0)}\\
\gamma_{ff}(0) &= \gamma_{yy}(0) + a\gamma_{yy}(1)
\end{aligned}
$$

where $f(n)$ is the prediction error.

Using equation 5 and $\sigma_w^2 = 1$ we have

$$
\begin{aligned}
\gamma_{yy}(0) &= \sigma_w^2 r_{hh}(0) = A + B \approx 4.2\\
\gamma_{yy}(1) &= \sigma_w^2 r_{hh}(1) = -\frac{A}{2} + \frac{B}{3} \approx -1.1 + 0.67 = -0.43
\end{aligned}
$$

Thus we find

$$
\begin{aligned}
a &= -\frac{-0.43}{4.2} \approx 0.10\\
\sigma_{ff}^2 &= 4.2 - 0.43 * 0.10 \approx 4.16
\end{aligned}
$$

Problem 3

The discrete filter with structures as in task 2a is to be implemented in fixed point representation using $B+1$ bits and dynamic range $[-1,1)$.
Rounding (quantization) is performed after each multiplication and the rounding error can be regarded as white noise with power $\sigma_e^2 = \frac{2^{-2B}}{12}$. All the rounding noise sources lead to a resulting noise signal $z(n)$ with power σ_z^2 at the output.

3a) Find the resulting noise power σ_z^2 at the filter output as a function of σ_e^2 for the DF2 structure.

Answer : We have four multiplication sources. Two of them are in the feedback part , (i.e. equation 6) and two in the forward part (i.e. equation 7). The corresponding rounding noise sources in the feedback part must go through the total filter $h(n)$ to reach the output. The noise sources in the forward part are in fact already at the output.

Thus $\sigma_z^2 = 2\sigma_e^2 r_{hh}(0) + 2\sigma_e^2 = (2 * 4.2 + 2)\sigma_e^2 = 10.4\sigma_e^2$

3b) Also find the resulting noise power σ_z^2 at the filter output as a function of σ_e^2 for the parallel structure.

Answer : The parallel structure consists of two first order filters with unit pulse responses (see equation 3) given by $h_1(n) = (-\frac{1}{2})^n u(n)$ and $h_2(n) = (\frac{1}{3})^n u(n)$. We need the energy (zero'th autocorrelation value) for them; i.e.

$$\begin{aligned}
\sum_{n=0}^{\infty} h_1^2(n) &= \sum_{n=0}^{\infty}(-\frac{1}{2})^{2n} = \sum_{n=0}^{\infty}(\frac{1}{4})^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3} \\
\sum_{n=0}^{\infty} h_2^2(n) &= \sum_{n=0}^{\infty}(\frac{1}{3})^{2n} = \sum_{n=0}^{\infty}(\frac{1}{9})^n = \frac{1}{1-\frac{1}{9}}) = \frac{9}{8}
\end{aligned}$$

Thus the resulting noise power at the output is $\sigma_z^2 = \frac{4}{3}\sigma_e^2 + \frac{9}{8}\sigma_e^2 = \frac{59}{24}\sigma_e^2 \approx 2.5\sigma_e^2$

Note that the parallel structure is much better than the DF2-structure with respect to noise power due to rounding!

3c) The filter input $x(n)$ has full dynamic range,i.e $x_{max} = \max_n |x(n)| = 1$.

Show that one has to scale the input by 2/7 (downscaling by 7/2) in order to avoid overflow in the parallel structure.

Answer : We have one internal summary node in each of the first order branches. In addition we of course have the summary at output $y(n)$. From the input to the internal nodes we have unit pulse responses corresponding to the two terms in $h(n)$ in equation 3.

Thus we have :

$$\begin{aligned} \sum_{n=0}^{\infty} |h_1(n)| &= \sum_{n=0}^{\infty} |(-\frac{1}{2})^n| = \sum_{n=0}^{\infty} (\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2 \\ \sum_{n=0}^{\infty} |h_2(n)| &= \sum_{n=0}^{\infty} |(\frac{1}{3})^n| = \sum_{n=0}^{\infty} (\frac{1}{3})^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2} \end{aligned} \tag{8}$$

Further, it is obvious from equation 3 that the sum of magnitudes of the total unit pulse response $h(n)$ is the sum of the above two values, i.e $2 + \frac{3}{2} = \frac{7}{2}$. Thus it is this last value $\frac{7}{2}$ which we have to use as a scale factor.

3d) Find the reduction in signal to noise ratio $SNR = \sigma_y^2/\sigma_z^2$ due to scaling at the output of the parallel structure.

Answer : Scaling down the input from $x(n)$ to $x(n)/S$ leads to that the output also is scaled by the same amount, i.e $y(n)/S$. Thus the output power σ_y^2 is reduced by a factor S^2. The noise power σ_z^2 is independent of the scaling factor as scaling is done before internal multiplications (and thus roundings). Thus the SNR is reduced by a factor $S^2 = (\frac{7}{2})^2 = \frac{49}{4} = 12.25$

Problem 4

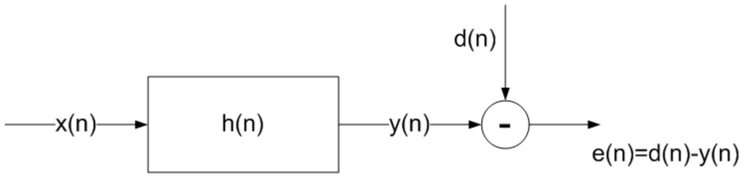

Figure 1: The general Wiener filter

A stationary signal $s(n)$ is contaminated by white additive noise $w(n)$, i.e. the observed signal is $x(n) = s(n) + w(n)$.
A Wiener filter $h(n)$ will minimize the mean square error $\sigma_e^2 = E[e^2(n)] = E[(d(n)-y(n))^2]$.
The statistical properties of $s(n)$, $d(n)$ and the noise power σ_w^2 are supposed to be known.

4a) Show that the mean square error can be written as

$$\sigma_e^2 = \gamma_{dd}(0) - 2\sum_k h(k)\gamma_{dx}(k) + \sum_k\sum_l h(k)h(l)\gamma_{xx}(l-k) \tag{9}$$

Answer : When we replace $y(n)$ by $h(n) * x(n) = \sum_k h(k)x(n-k)$ we get :

$$\sigma_e^2 = E[d^2(n)] - 2E[d(n)\sum_k h(k)x(n-k)] + E[(\sum_k h(k)x(n-k))(\sum_l h(l)x(n-l))] \Rightarrow$$

$$\sigma_e^2 = = \gamma_{dd}(0) - 2\sum_k h(k)E[(d(n)x(n-k)] + \sum_k\sum_l h(k)h(l)E[x(n-k)x(n-l)] \Rightarrow$$

$$\sigma_e^2 = = \gamma_{dd}(0) - 2\sum_k h(k)\gamma_{dx}(k) + \sum_k\sum_l h(k)h(l)\gamma_{xx}(l-k)$$

4b) Derive the formula for the FIR Wiener filter of length M

Answer : The sums in the MSE-expression is now finite $(0 \rightarrow M-1)$. Thus we can define two M-dimensional column vectors $\vec{h}$ and $\vec{\gamma}_{dx}$ and a MxM matrix $\underline{\gamma}_{xx}(k,l) = \gamma_{xx}(k-l)$.
The MSE can now be written $MSE = \gamma_{dd}(0) - 2\vec{h}^T\vec{\gamma}_{dx} + \vec{h}^T\underline{\gamma}_{xx}\vec{h}$

We now find the optimal filter which gives MMSE by setting the gradient equal to zero :

$$\nabla_{\vec{h}} MSE = -2\vec{\gamma}_{dx} + 2\underline{\gamma}_{xx}\vec{h} = \vec{0}$$

Thus we have :

$$\underline{\gamma}_{xx}\vec{h} = \vec{\gamma}_{dx} \quad \Rightarrow \quad \vec{h} = \underline{\gamma}_{xx}^{-1}\vec{\gamma}_{dx} \tag{10}$$

since the matrix is Toeplitz and thus always invertable.

4c) Assume a filter length of $M = 3$. Describe the differences in the formula when the FIR filter is to be used for respectively :

- straight noise reduction, i.e. $d(n) = s(n)$
- smoothing, i.e. $d(n) = s(n-1)$
- prediction, i.e. $d(n) = s(n+1)$

Answer : since $w(n)$ is white noise it is uncorrelated both with $s(n)$ and with $d(n)$. Thus $\gamma_{dx}(l) = \gamma_{ds}(l)$ and $\gamma_{xx}(l) = \gamma_{ss}(l) + \sigma_w^2\delta(l)$

- straight noise reduction, i.e. $d(n) = s(n) \;\rightarrow \gamma_{ds}(l) = \gamma_{ss}(l)$. The left formula in equation 9 can be written elementwise as :

$$\gamma_{ss}(l) = \sum_{k=0}^{M-1} h(k)\gamma_{xx}(l-k) \quad l = 0, 1, 2$$

- smoothing, i.e. $d(n) = s(n-1) \;\rightarrow \gamma_{ds}(l) = \gamma_{ss}(l-1)$. The left formula in equation 9 can be written elementwise as :

$$\gamma_{ss}(l-1) = \sum_{k=0}^{M-1} h(k)\gamma_{xx}(l-k) \quad l = 0, 1, 2$$

- prediction, i.e. $d(n) = s(n+1) \;\rightarrow \gamma_{ds}(l) = \gamma_{ss}(l+1)$. The left formula in equation 9 can be written elementwise as :

$$\gamma_{ss}(l+1) = \sum_{k=0}^{M-1} h(k)\gamma_{xx}(l-k) \quad l = 0, 1, 2$$

Thus the difference lies in the indexes of $\gamma_{ss}(l)$ on the left hand side :

- for $d(n) = s(n)$ we use $\vec{\gamma}_{dx} = [\gamma_{ss}(0), \gamma_{ss}(1), \gamma_{ss}(2)]^T$
- for $d(n) = s(n-1)$ we use $\vec{\gamma}_{dx} = [\gamma_{ss}(1), \gamma_{ss}(0), \gamma_{ss}(1)]^T$
- for $d(n) = s(n+1)$ we use $\vec{\gamma}_{dx} = [\gamma_{ss}(1), \gamma_{ss}(2), \gamma_{ss}(3)]^T$

Some basic equations and formulaes.

A. Sequences :

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \Longleftrightarrow |\alpha| < 1$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

B. Linear convolution :

$$y(n) = h(n) * x(n) = \sum_k h(k)x(n-k) = \sum_k x(k)h(n-k)$$

$$Y(z) = H(z)X(z) \Rightarrow Y(f) = H(f)X(f) \Rightarrow Y(k) = H(k)X(k) \quad k = 0, ..., N-1$$

C. Transforms :

$$H(z) = \sum_n h(n)z^{-n} \Rightarrow H(f) = \sum_n h(n)\ e^{-j2\pi nf}$$

$$H(k) = \sum_n h(n)\ e^{-j2\pi nk/N} \qquad k = 0, ..., N-1$$

$$h(n) = \sum_k H(k)\ e^{j2\pi nk/N} \qquad n = 0, ..., N-1$$

D. The sampling theorem :

$$x(n) = x(nT_s) = x_a(t)|_{t=nT_s} \qquad n = -\infty, ..., \infty$$

$$X(f) = X(F/F_s) = F_s \sum_k X_a[(f-k)F_s]$$

E. Autocorrelation, energy spectrum and Parsevals theorem :

Given a sequence $x(n)$ with finite energy E_x :

Autocorrelation : $r_{xx}(m) = \sum_n x(n)x(n+m) \quad m = -\infty, ..., \infty$

Energy spectrum : $S_{xx}(z) = X(z)X(z^{-1}) \Rightarrow S_{xx}(f) = |X(f)|^2$

Parsevals theorem: $E_x = \sum_n x^2(n) = \int_0^{2\pi} |X(f)|^2 df = \int_0^{2\pi} S_{xx}(f) df$

F. Multirate formulaes :

Decimation where $T_{sy} = DT_{sx}$:

$$v(mT_{sy}) = \sum_k h[(mD-k)T_{sx}]\ x(kT_{sx}) \quad m = -\infty, ..., \infty$$

Upsampling where $T_{sx} = UT_{sy}$:

$$y(lT_{sy}) = \sum_n h[(l-nU)T_{sy}]\ x(nT_{sx}) \quad l = -\infty, ..., \infty$$

Interpolation where $T_{sy} = DT_{sv} = \frac{D}{U}T_{sx}$:

$$y(lT_{sy}) = \sum_m h[(lD-mU)T_{sv}]\ x(mT_{sx}) \quad l = -\infty, ..., \infty$$

G. Autocorrelation, power spectrum and Wiener-Khintchin theorem :

Given a stationary, ergodic sequence $x(n)$ with infinite energy :

Autocorrelation : $\gamma_{xx}(m) = E[x(n)x(n+m)]\ m = -\infty, ..., \infty$

Power spectrum: $\Gamma_{xx}(z) = Z[\gamma_{xx}(m)] \quad \Rightarrow$

Wiener-Khintchin : $\Gamma_{xx}(f) = DTFT[\gamma_{xx}(m) = \sum_m \gamma_{xx}(m)\ e^{-j2\pi mf}$

H. The Yule-Walker and Normal equations where $a_0 = 1$:

Yule-Walker equations : $$\sum_{k=0}^{P} a_k \gamma_{xx}(m-k) = \sigma_f^2 \; \delta(m) \quad m = 0, ..., P$$

Normal equations: $$\sum_{k=1}^{P} a_k \gamma_{xx}(m-k) = -\gamma_{xx}(m) \quad m = 1, ..., P$$