NORWEGIAN UNIVERSITY
OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF TELECOMMUNICATIONS

Contact during examination:
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EXAMINATION IN COURSE TTT4120 DIGITAL SIGNAL PROCESSING

Date: Thursday, August 18th, 2011
Time: 09.00 - 13.00

Permitted aids: D–No calculators allowed.
No printed or handwritten materials allowed.

INFORMATION

- The examination includes 4 problems, each of which has 4 subsections.
- Problem 1 deals with basic properties of systems/filters.
- Problem 2 deals with filter structures.
- Problem 3 deals with stationary processes and parametric estimation.
- Problem 4 deals with filtering in the frequency domain
- The weight of each subproblem is given in parenthesis at problem start
- The course responsible will visit you twice, the first time around 10.00 o'clock and the second time between 12.00 - 12.30.

Problem 1 : (3+5+4+4)

1a) Which properties have to be fulfilled in order to describe a system by its unit pulse response $h(n)$?

Given that the above properties are fulfilled, define the two properties stability and causality in terms of $h(n)$.

1b) Define the z-transform $H(z)$ in terms of $h(n)$, $n = -\infty, \infty$.

What is meant by the term "region of convergence" (ROC) of the transfer function $H(z)$?

Sketch ROC in the z-plane for respectively a causal and anti-causal system.

What area in the z-plane must be included in ROC if the system is to be stable? State the reason for your answer.

1c) Given the following stable filter $H_1(z)$.

$$H_1(z) = \frac{z^{-1} - a}{1 - az^{-1}} \tag{1}$$

Show that the filter is allpass.

For which values of the filter coeffisient a is the filter causal?

1d) Define the autocorrelation sequence $r_{hh}(m), \quad m = -\infty, \infty$ of a general, stable filter $h(n)$.

Explain why the autocorrelation sequence of the allpass filter in subtask 1c has the form

$$r_{h_1h_1}(m) = \delta(m), \quad m = -\infty, \infty \tag{2}$$

Problem 2 : (4+3+6+3)

Given a stable, causal filter $H(z)$ on the form

$$H(z) = H_1(z)H_2(z) = \frac{z^{-1} - \frac{2}{3}}{1 - \frac{2}{3}z^{-1}} \frac{1}{1 - \frac{1}{2}z^{-1}} = \frac{z^{-1} - \frac{2}{3}}{1 - \frac{5}{6}z^{-1} + \frac{1}{3}z^{-2}} \tag{3}$$

i.e. $H_1(z)$ is given by the allpass filter in subtask 1c (using $a = \frac{2}{3}$) and $H_2(z)$ is given by

$$H_2(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} \tag{4}$$

2a) Show that $H(z)$ can be written on the following parallel form

$$H(z) = H_3(z) + H_4(z) = \frac{\frac{10}{3}}{1 - \frac{2}{3}z^{-1}} + \frac{-4}{1 - \frac{1}{2}z^{-1}} \tag{5}$$

2b) Derive the unit impulse response $h(n)$ of the filter $H(z)$

2c) Sketch the following structures for $H(z)$:

- Direct form 2 (DF2)
- Parallel
- Cascade

2d) Explain why the autocorrelation sequences of $H(z)$ and $H_2(z)$ are identical.

Problem 3 : (6+4+4+4)

Given a causal, stable filter with unit pulse response $g(n), \quad n = 0, \infty$. White noise $w(n)$ with power σ_w^2 is input to the filter.
The autocorrelation function $\gamma_{yy}(m), \quad m = -\infty, \infty$, and the power spectrum $\Gamma_{yy}(z)$ of the resulting output signal $y(n)$ are given by

$$\gamma_{yy}(m) = \begin{cases} \sigma_w^2 \sum_{n=0}^{\infty} g(n)g(n+m) & m \geq 0 \\ \gamma_{yy}(-m) & m < 0 \end{cases} \tag{6}$$

$$\Gamma_{yy}(z) = \sigma_w^2 \; G(z)G(z^{-1}) \tag{7}$$

3a) Define respectively an ARMA, AR and MA process.

What is the principial difference between a physical process and a process model?

3b) Explain which type of parametric process we will find at the filter output $y(n)$ when white noise with power σ_w^2 is input to respectively :

- $H_1(z)$
- $H(z)$

where the filters are defined in task 2.

3c) Find the autocorrelation sequence of the output $y(n)$ when white noise with power σ_w^2 is input to $H(z)$.

3d) Give the process parameters of the best AR[1] model for each of the two output signals $y(n)$ in subtask 3b.

State your reason for the answers.

Problem 4 : (3+6+5+3)

4a) Set up the formulas for a N-point Diskret Fourier Transform (DFT) and its inverse (IDFT) for a sequence $x(n)$ of finite length M

How must N be chosen if one wishes reproduce $x(n)$ from the DFT values?

4b) One wants to filter an infinitely long sequence $x(n), \ n = -\infty, \infty$ by a FIR-filter $h(n)$ of length L.

Explain how the filtering can be performed in the frequency domain by using the so called "overlap-add" method.

Compare the"overlap-add" method to standard time domain filtrering with respect to the number of multiplications and addition per output sample.

4c) One wants to use DFT to perform a frequency analysis of an infinitely long sequence $x(n), \ n = -\infty, \infty$. In real one has to base the analysis of a finite segment of length K of the sequence.

Discuss the problems regarding frequency resolution and frequency "leackage" (side-lobes) as a function of the segment length K.

How can one manage to achieve a compromise with respect to the two nonidealities in the frequency domain?

4d) The radix-2 Fast Fourier Transform (FFT) is a fast algorithm for calculating the DFT of a sequence when the length N is a power of 2, i.e. $N = 2^R$

Explain shortly the *principle* of the radix-2 FFT algorithm.

Formulas given at exam August 2011 TTT4120 DSP

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Some basic equations and formulas.

A. Sequences :

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \Longleftrightarrow |\alpha| < 1$$

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

B. Linear convolution :

$$y(n) = h(n) * x(n) = \sum_k h(k)x(n-k) = \sum_k x(k)h(n-k)$$

$$Y(z) = H(z)X(z) \Rightarrow Y(f) = H(f)X(f) \Rightarrow$$

$$Y(f_k) = H(f_k)X(f_k) \quad f_k = k/N \text{ for } k = 0, \ldots, N-1 \text{ where we write } Y(k) = Y(f_k)$$

C. Transforms :

$$H(z) = \sum_n h(n)z^{-n} \Rightarrow H(f) = \sum_n h(n)\ e^{-j2\pi nf}$$

$$\text{DFT} : H(k) = \sum_{n=0}^{L-1} h(n)\ e^{-j2\pi nk/N} \quad k = 0, ..., N-1$$

$$\text{IDFT} : h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)\ e^{j2\pi nk/N} \quad n = 0, ..., L-1$$

D. The sampling (Nyquist) theorem :

Given an analog signal $x_a(t)$ with bandwidth $\pm B$ which is sampled by $F_s = 1/T_s$:

$$x(n) = x(nT_s) = x_a(t)|_{t=nT_s} \quad n = -\infty, ..., \infty$$

$$X(f) = X(F/F_s) = F_s \sum_k X_a[(f-k)F_s]$$

$$x_a(t) \text{ can be recovered from } x(n) \quad \Leftrightarrow \quad F_s \geq 2B \tag{1}$$

E. Autocorrelation, energy spectrum and Parsevals theorem :

Given a sequence $h(n)$ with finite energy E_h :

Autocorrelation : $r_{hh}(m) = \sum_n h(n)h(n+m) \quad m = -\infty, ..., \infty$

Energy spectrum : $S_{hh}(z) = H(z)H(z^{-1}) \Rightarrow S_{hh}(f) = |H(f)|^2$

Parsevals theorem: $E_h = r_{hh}(0) = \sum_n h^2(n) = \int_0^{2\pi} |H(f)|^2 df$

F. Multirate formulaes :

Decimation where $T_{sy} = DT_{sx}$:

$$v(mT_{sy}) = \sum_k h[(mD-k)T_{sx}]\ x(kT_{sx}) \quad m = -\infty, ..., \infty$$

Upsampling where $T_{sx} = UT_{sy}$:

$$y(lT_{sy}) = \sum_n h[(l-nU)T_{sy}]\ x(nT_{sx}) \quad l = -\infty, ..., \infty$$

Interpolation where $T_{sy} = DT_{sv} = \frac{D}{U}T_{sx}$:

$$y(lT_{sy}) = \sum_m h[(lD-mU)T_{sv}]\ x(mT_{sx}) \quad l = -\infty, ..., \infty$$

G. Autocorrelation, power spectrum and Wiener-Khintchin theorem :

Given a stationary, ergodic sequence $x(n)$ with infinite energy :

Autocorrelation : $\gamma_{xx}(m) = E[x(n)x(n+m)] \; m = -\infty, ..., \infty$

Power spectrum: $\Gamma_{xx}(z) = Z[\gamma_{xx}(m)] \quad \Rightarrow$

Wiener-Khintchin : $\Gamma_{xx}(f) = DTFT[\gamma_{xx}(m)] = \sum_{m} \gamma_{xx}(m) \; e^{-j2\pi mf}$

H. The Yule-Walker and Normal equations where $a_0 = 1$:

The Yule-Walker equations :

$$\sum_{k=0}^{P} a_k \gamma_{xx}(m-k) = \sigma_f^2 \; \delta(m) \quad m = 0, ..., P \tag{2}$$

Or reformulated as the Normal equations :

$$\sum_{k=1}^{P} a_k \gamma_{xx}(m-k) = -\gamma_{xx}(m) \quad m = 1, ..., P$$

$$\sigma_f^2 = \sum_{k=0}^{P} a_k \gamma_{xx}(k) \qquad \text{(corresponds to } \; m = 0)$$