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OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF TELECOMMUNICATIONS

Contact during examination:
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EXAMINATION IN COURSE TTT4120 DIGITAL SIGNAL PROCESSING

Date: Monday Desember 10th, 2012
Time: 09.00 - 13.00

Permitted aids: D–No printed or handwritten material allowed.
Specified, simple calculator allowed..

INFORMATION

- The examination consists of 4 problems.
  - Problem 1 concerns analysis of digital filters.
  - Problem 2 concerns rational process models.
  - Problem 3 concerns fixed point implementation.
  - Problem 4 concerns multirate systems
  - A list of formulas can be found in the appendix.
  - Task weighting is given in parenthesis. Total amount of points is 70.
- All answers should be justified!
- The teacher will visit you twice, the first time around 10.00 and the second time around 11.45.

Problem 1 (3+3+5+4 = 15 points)

1a) A stable and causal LTI system is given by the following transfer function :

$$
\begin{aligned}
H(z) &= H_0(z)H_1(z)H_2(z) \qquad \text{where} \\
H_0(z) &= 1 + \frac{5}{3}z^{-1} \\
H_1(z) &= \frac{1}{(1 + \frac{1}{2}z^{-1})} \\
H_2(z) &= \frac{1}{(1 - \frac{2}{3}z^{-1})}
\end{aligned}
$$

Show that the difference equation for the filter is given by :

$$
y(n) - \frac{1}{6}y(n-1) - \frac{1}{3}y(n-2) = x(n) + \frac{5}{3}x(n-1), \; n = -\infty, \infty \qquad (1)
$$

1b) State the reason for your answers on the following:

- What is the region of convergence (ROC) for the filter in subtask 1a?
- Does the filter have linear phase?
- Does the filter have minimum phase?

1c) Show that the unit pulse response of the filter is given by :

$$h(n) = -h_1(n) + 2h_2(n) \tag{2}$$

where

$$h_1(n) = \begin{cases} (-\frac{1}{2})^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$h_2(n) = \begin{cases} (\frac{2}{3})^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

1d) Sketch the following two structures of the filter $H(z)$:

- Direct form 2 (DF2)
- Parallel structure where the branch gain $G_2 = 2$ in equation 2 is placed *before* the feedback.

Problem 2 **(4+7+3+4 = 18 points)**

2a) The crosscorrelation sequence of two sequences $y(n)$ and $x(n)$, both with finite energy, is given by

$$\begin{aligned} r_{yx}(m) &= \sum_{n=-\infty}^{\infty} y(n+m)x(n) \quad m \geq 0 \\ r_{yx}(m) &= r_{xy}(-m) \qquad\qquad\quad m < 0 \end{aligned}$$

Show that the crosscorrelation sequence of $h_1(n)$ and $h_2(n)$ in subtask 1c is given by

$$r_{h_1h_2}(m) = \begin{cases} \frac{3}{4}(-\frac{1}{2})^{|m|} = \frac{3}{4}(-\frac{1}{2})^m & m \geq 0 \\ \\ \frac{3}{4}(\frac{2}{3})^{|m|} = \frac{3}{4}(\frac{3}{2})^m & m < 0 \end{cases} \tag{3}$$

2b) Show that the unit pulse responses $h_1(n), h_2(n)$ and $h(n)$ in task 1 has the following autocorrelation sequences for $m \geq 0$:

$$\begin{aligned} r_{h_1h_1}(m) &= \frac{4}{3}(-\frac{1}{2})^m \\ r_{h_2h_2}(m) &= \frac{9}{5}(\frac{2}{3})^m \\ r_{hh}(m) &= -\frac{1}{6}(-\frac{1}{2})^m + \frac{57}{10}(\frac{2}{3})^m \end{aligned}$$

In addition the autocorrelation sequences are symmetric around $m = 0$.

Figure 1 shows a chosen cascade structure for $H(z)$

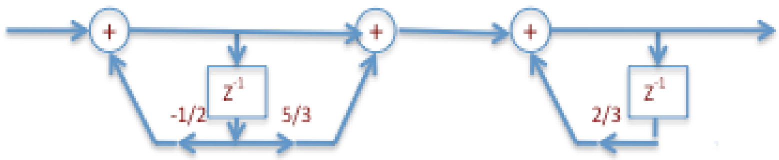

Figure 1: *Chosen cascade structure*

2c) White noise $w(n)$ with power $\sigma_w^2 = 1$ is input to the cascade structure.

Which kind of parametric processes are respectively the output signal and the internal signals in the structure? State the reason for your answer!

2d) White noise $w(n)$ with power $\sigma_w^2 = 1$ is input to the filter $H(z)$.

Find, by using linear prediction (i.e. the Yule-Walker or Normal equations), the filter coefficient a_1 of the best AR[1]-model for the filter output signal $y(n)$.

Show that the prediction error power, σ_f^2, always fulfills : $\sigma_f^2 \leq \gamma_{yy}(0)$ where $\sigma_y^2 = \gamma_{yy}(0)$ is the signal power of $y(n)$.

Problem 3 **(4+4+7+4 = 19 points)**

The discrete filter in task 1 is to be implemented in fixed point representation using $B + 1$ bits and dynamic range $[-1, 1)$. Rounding (quantization) is performed after each multiplication and the corresponding rounding error, $e(n)$, can be regarded as white noise with power $\sigma_e^2 = \frac{2^{-2B}}{12}$. Together, all the rounding noise sources result in a noise signal $z(n)$ at the output with a total power of σ_z^2.

3a) Find the resulting noise power σ_z^2 at the output of the cascade structure as a function of σ_e^2 .

3b) Also find the resulting noise power σ_z^2 at the output of the parallel structure in subtask 1d.
Note that the minus sign in front of $h_1(n)$ in equation 2 is implemented as an arithmetic operation (negation) and thus not as a multiplication!!

The filter input $x(n)$ has full dynamic range, i.e $x_{max} = \max_n |x(n)| = 1$.

3c) Show that one has to scale the input by 3/16 (downscaling by 16/3) in order to avoid overflow in the cascade structure used in subtask 3a.

Further show that the parallel structure used in subtask 3b requires a scaling by 1/6 (downscaling by 6).

3d) Which of the two scaled structures have the best signal_to_noise_ratio ($SNR = \sigma_y^2/\sigma_z^2$) at the output ?

Problem 4 (5+4+4+5 = 18 points)

Figure 2 shows a system for conversion of sampling rate from F_1 to F_2 where I and D are integers.

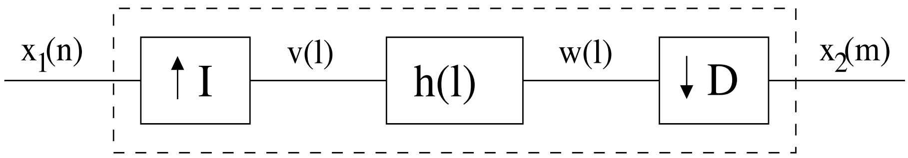

Figure 2: *System for sampling rate conversion*

4a) Shortly describe the three parts of and give the bandwidth and sampling rate of the internal signals $v(l)$ og $w(l)$.

4b) Derive a time domain expression for the output signal $x_2(m)$ as a function of the input signal $x_1(n)$, the filter $h(l)$ and I og D.

4c) Discuss the operation of the system when $I > D$ and vice versa.

4d) Given an analogue signal $x(t) = s(t) + sin(2\pi F_0 t)$ where $s(t)$ has bandwidth $\pm B = 5000Hz$ and $F_0 = 4000Hz$. The signal is sampled by a rate $F_1 = 10000Hz$, i.e. $x_1(n) = x_a(nT_1)$ hvor $T_1 = 1/F_1$.
Further, we have given a notch filter with a zero of $f_n = 0.25$.

How can one use the system in figure 2 together with the notch filter to remove the harmonic component (F_0) in $x_1(n)$?

Some basic equations and formulas.

A. Sequences :

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \Longleftrightarrow |\alpha| < 1$$
$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

B. Linear convolution :

$$\begin{aligned} y(n) &= h(n) * x(n) = \sum_k h(k)x(n-k) = \sum_k x(k)h(n-k) \\ Y(z) &= H(z)X(z) \Rightarrow Y(f) = H(f)X(f) \Rightarrow \\ Y(f_k) &= H(f_k)X(f_k) \quad f_k = k/N \;\; \text{for} \;\; k = 0, \ldots, N-1 \;\; \text{where we write} \;\; Y(k) = Y(f_k) \end{aligned}$$

C. Transforms :

$$\begin{aligned} H(z) &= \sum_n h(n) z^{-n} \Rightarrow H(f) = \sum_n h(n)\; e^{-j2\pi nf} \\ \text{DFT} \;: H(k) &= \sum_{n=0}^{L-1} h(n)\; e^{-j2\pi nk/N} \qquad k = 0, ..., N-1 \\ \text{IDFT} : h(n) &= \frac{1}{N} \sum_{k=0}^{N-1} H(k)\; e^{j2\pi nk/N} \qquad n = 0, ..., L-1 \end{aligned}$$

D. The sampling (Nyquist) theorem :

Given an analog signal $x_a(t)$ with bandwidth $\pm B$ which is sampled by $F_s = 1/T_s$:

$$x(n) = x(nT_s) = x_a(t)|_{t=nT_s} \qquad n = -\infty, ..., \infty$$

$$X(f) = X(F/F_s) = F_s \sum_k X_a[(f-k)F_s]$$

$$x_a(t) \text{ can be recovered from } x(n) \quad \Leftrightarrow \quad F_s \geq 2B \qquad (4)$$

E. Autocorrelation, energy spectrum and Parsevals theorem :

Given a sequence $h(n)$ with finite energy E_h :

$$\text{Autocorrelation :} \quad r_{hh}(m) = \sum_n h(n)h(n+m) \quad m = -\infty, ..., \infty$$

$$\text{Energy spectrum :} \quad S_{hh}(z) = H(z)H(z^{-1}) \Rightarrow S_{hh}(f) = |H(f)|^2$$

$$\text{Parsevals theorem:} \quad E_h = r_{hh}(0) = \sum_n h^2(n) = \int_0^{2\pi} |H(f)|^2 df$$

F. Multirate formulaes :

Decimation where $T_{sy} = DT_{sx}$:

$$v(mT_{sy}) = \sum_k h[(mD-k)T_{sx}]\; x(kT_{sx}) \quad m = -\infty, ..., \infty$$

Upsampling where $T_{sx} = UT_{sy}$:

$$y(lT_{sy}) = \sum_n h[(l-nU)T_{sy}]\; x(nT_{sx}) \quad l = -\infty, ..., \infty$$

Interpolation where $T_{sy} = DT_{sv} = \frac{D}{U}T_{sx}$:

$$y(lT_{sy}) = \sum_m h[(lD-mU)T_{sv}]\; x(mT_{sx}) \quad l = -\infty, ..., \infty$$

G. Autocorrelation, power spectrum and Wiener-Khintchin theorem :

Given a stationary, ergodic sequence $x(n)$ with infinite energy :

Autocorrelation : $\gamma_{xx}(m) = E[x(n)x(n+m)]\ m = -\infty, ..., \infty$

Power spectrum: $\Gamma_{xx}(z) = Z[\gamma_{xx}(m)] \quad \Rightarrow$

Wiener-Khintchin : $\Gamma_{xx}(f) = DTFT[\gamma_{xx}(m)] = \sum_{m} \gamma_{xx}(m)\ e^{-j2\pi mf}$

H. The Yule-Walker and Normal equations where $a_0 = 1$:

Yule-Walker equations : $\sum_{k=0}^{P} a_k \gamma_{xx}(m-k) = \sigma_f^2\ \delta(m) \quad m = 0, ..., P$

Normal equations: $\sum_{k=1}^{P} a_k \gamma_{xx}(m-k) = -\gamma_{xx}(m) \quad m = 1, ..., P$