NORWEGIAN UNIVERSITY
OF SCIENCE AND TECHNOLOGY
DEPARTMENT OF TELECOMMUNICATIONS

Contact during examination:
Name: Magne H. Johnsen
Tel.: 73 59 26 78/930 25 534

EXAMINATION IN COURSE TTT4120 DIGITAL SIGNAL PROCESSING

Date: Monday Desember 10th, 2012
Time: 09.00 - 13.00

Permitted aids: D–No printed or handwritten material allowed.
Specified, simple calculator allowed..

INFORMATION

- The examination consists of 4 problems.
  - Problem 1 concerns analysis of digital filters.
  - Problem 2 concerns rational process models.
  - Problem 3 concerns fixed point implementation.
  - Problem 4 concerns multirate systems
  - A list of formulas can be found in the appendix.
  - Task weighting is given in parenthesis. Total amount of points is 70.
- All answers should be justified!
- The teacher will visit you twice, the first time around 10.00 and the second time around 11.45.

Problem 1 **(3+3+5+4 = 15 points)**

1a) A stable and causal LTI system is given by the following transfer function :

$$\begin{aligned} H(z) &= H_0(z)H_1(z)H_2(z) \qquad \text{where} && (1) \\ H_0(z) &= 1 + \frac{5}{3}z^{-1} \\ H_1(z) &= \frac{1}{(1+\frac{1}{2}z^{-1})} \\ H_2(z) &= \frac{1}{(1-\frac{2}{3}z^{-1})} \end{aligned}$$

Show that the difference equation for the filter is given by :

$$y(n) - \frac{1}{6}y(n-1) - \frac{1}{3}y(n-2) = x(n) + \frac{5}{3}x(n-1), \; n = -\infty, \infty \qquad (2)$$

Answer:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1+\frac{5}{3}z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{2}{3}z^{-1})} = \frac{1+\frac{5}{3}z^{-1}}{1-\frac{1}{6}z^{-1}-\frac{1}{3}z{-2}} \Rightarrow$$

$$Y(z) - \frac{1}{6}Y(z)z^{-1} - \frac{1}{3}Y(z)z^{-2} = X(z) + \frac{5}{3}X(z)z^{-1}$$

Eq. 2 is easily derived by the inverse Z-transform.

1b) State the reason for your answers on the following:

- What is the region of convergence (ROC) for the filter in subtask 1a?
- Does the filter have linear phase?

- Does the filter have minimum phase?

Answer:

- $\max[|p_1|, |p_2|] = max[\frac{1}{2}, \frac{2}{3}] = \frac{2}{3} \quad \Rightarrow$ ROC : $|z| > \frac{2}{3}$
- The filter does not have linear phase since it has poles.
- Minimum phase is only possible if both poles and zeros are inside the unit circle. Here we have a zero at $|z| = |-\frac{5}{3}| > 1$, i.e. not minimum phase.

1c) Show that the unit pulse response of the filter is given by :

$$h(n) = -h_1(n) + 2h_2(n) \tag{3}$$

where

$$\begin{aligned} h_1(n) &= \begin{cases} (-\frac{1}{2})^n & n \geq 0 \\ 0 & n < 0 \end{cases} \\ h_2(n) &= \begin{cases} (\frac{2}{3})^n & n \geq 0 \\ 0 & n < 0 \end{cases} \end{aligned}$$

Answer :

$$\begin{aligned} H(z) &= \frac{1 + \frac{5}{3}z^{-1}}{(1 + \frac{1}{3}z^{-1})(1 - \frac{2}{3}z^{-1})} = \frac{A}{1 + \frac{1}{2}z^{-1}} + \frac{B}{1 - \frac{2}{3}z^{-1}} \quad \Rightarrow \\ H(z) &= \frac{A + B - (\frac{2A}{3} - \frac{B}{2})z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})} \end{aligned}$$

$$\begin{aligned} A + B &= 1 \\ (\frac{2A}{3} - \frac{B}{2})z^{-1} &= -\frac{5}{3}z^{-1} \quad \Rightarrow \quad 4A - 3B = -10 \\ 4(1 - B) - 3B &= 4 - 7B = -10 \ \Rightarrow \ B = 14/7 = 2 \quad \text{og} \quad A = 1 - B = -1 \end{aligned}$$

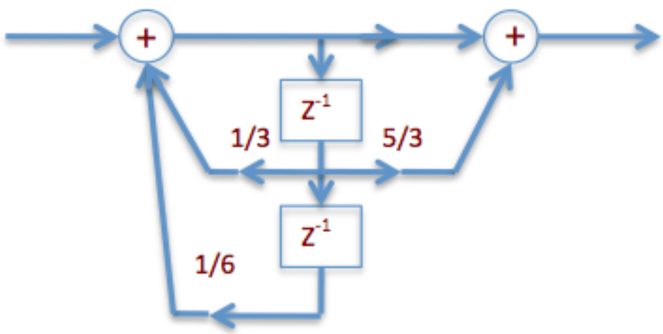
Thus we have :

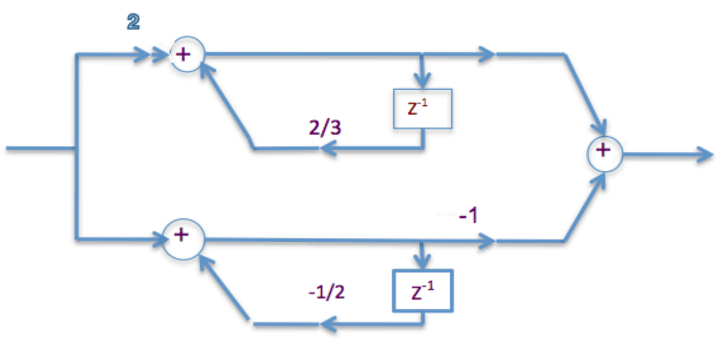
$$
\begin{aligned}
H(z) &= -H_1(z) + 2H_2(z) = -\frac{1}{1+\frac{1}{2}z^{-1}} + 2\frac{1}{1-\frac{2}{3}z^{-1}} \quad \Rightarrow \\
h(n) &= -h_1(n) + 2h_2(n) = \begin{cases} -(-\frac{1}{2})^n + 2(\frac{2}{3})^n & n \geq 0 \\ 0 & n < 0 \end{cases}
\end{aligned}
$$

1d) Sketch the following two structures of the filter $H(z)$:

- Direct form 2 (DF2)
- Parallel structure where the branch gain $G_2 = 2$ in equation 2 is placed *before* the feedback.

Answer :





Parallel

Problem 2 (4+7+3+4 = 18 points)

2a) The crosscorrelation sequence of two sequences $y(n)$ and $x(n)$, both with finite energy, is given by

$$\begin{aligned} r_{yx}(m) &= \sum_{n=-\infty}^{\infty} y(n+m)x(n) \quad m \geq 0 \\ r_{yx}(m) &= r_{xy}(-m) \quad\quad m < 0 \end{aligned}$$

Show that the crosscorrelation sequence of $h_1(n)$ and $h_2(n)$ in subtask 1c is given by

$$r_{h_1h_2}(m) = \begin{cases} \frac{3}{4}(-\frac{1}{2})^{|m|} = \frac{3}{4}(-\frac{1}{2})^m & m \geq 0 \\ \frac{3}{4}(\frac{2}{3})^{|m|} = \frac{3}{4}(\frac{3}{2})^m & m < 0 \end{cases} \quad (4)$$

Answer:

$$\begin{aligned} r_{h_1h_2}(m) &= \sum_{n=-\infty}^{\infty} h_1(n+m)h_2(n) \quad m \geq 0 \\ r_{h_1h_2}(m) &= \sum_{n=0}^{\infty} (-\frac{1}{2})^{n+m}(\frac{2}{3})^n \quad m \geq 0 \\ r_{h_1h_2}(m) &= (-\frac{1}{2})^m \sum_{n=0}^{\infty} (-\frac{1}{3})^n \quad m \geq 0 \\ r_{h_1h_2}(m) &= (-\frac{1}{2})^m \frac{1}{1+\frac{1}{3}} = \frac{3}{4}(-\frac{1}{2})^m \end{aligned}$$

Correspondingly for $r_{h_2h_1}(m)$:

$$\begin{aligned} r_{h_2h_1}(m) &= \sum_{n=0}^{\infty} (\frac{2}{3})^{n+m}(-\frac{1}{2})^n \quad m \geq 0 \\ x &= y \\ r_{h_2h_1}(m) &= (\frac{2}{3})^m \frac{1}{1+\frac{1}{3}} = \frac{3}{4}(\frac{2}{3})^m \end{aligned}$$

As $r_{h_1h_2}(m) = r_{h_2h_1}(-m)$ for $m < 0$ eq 4 is proved!

2b) Show that the unit pulse responses $h_1(n), h_2(n)$ and $h(n)$ in task 1 has the following autocorrelation sequences for $m \geq 0$:

$$\begin{aligned} r_{h_1h_1}(m) &= \frac{4}{3}(-\frac{1}{2})^m \\ r_{h_2h_2}(m) &= \frac{9}{5}(\frac{2}{3})^m \\ r_{hh}(m) &= -\frac{1}{6}(-\frac{1}{2})^m + \frac{57}{10}(\frac{2}{3})^m \end{aligned}$$

In addition the autocorrelation sequences are symmetric around $m = 0$.

Answer :
The two autocorrelation sequences for respectively $h_1(n)$ and $h_2(n)$ are :

$$\begin{aligned} r_{h_1h_1}(m) &= \sum_{n=0}^{\infty}(-\frac{1}{2})^{n+m}(-\frac{1}{2})^n \qquad m \geq 0 \\ r_{h_1h_1}(m) &= (-\frac{1}{2})^m \sum_{n=0}^{\infty}(\frac{1}{4})^n \qquad m \geq 0 \\ r_{h_1h_1}(m) &= (-\frac{1}{2})^m \frac{1}{1-\frac{1}{4}} = \frac{4}{3}(-\frac{1}{2})^m \qquad m \geq 0 \\ r_{h_2h_2}(m) &= \sum_{n=0}^{\infty}(\frac{2}{3})^{n+m}(\frac{2}{3})^n \qquad m \geq 0 \\ r_{h_2h_2}(m) &= (\frac{2}{3})^m \sum_{n=0}^{\infty}(\frac{4}{9})^n \qquad m \geq 0 \\ r_{h_2h_2}(m) &= (\frac{2}{3})^m \frac{1}{1-\frac{4}{9}} = \frac{9}{5}(\frac{2}{3})^m \qquad m \geq 0 \end{aligned} \tag{5}$$

Inserting $h(n) = -h_1(n) + 2h_2(n)$ we get

$$\begin{aligned} r_{hh}(m) &= \sum_{n=-\infty}^{\infty} h(n+m)h(n) \qquad \text{thus} \\ r_{hh}(m) &= r_{h_1h_1}(m) - 2r_{h_1h_2}(m) - 2r_{h_2h_1}(m) + 4r_{h_2h_2}(m) \end{aligned}$$

From subtask 2a we use the expressions for the cross correlations and arrange terms with identical exponents

$$
\begin{aligned}
(\frac{4}{3}-2\frac{3}{4})(-\frac{1}{2})^m &= \frac{(8-9)}{6}(-\frac{1}{2})^m = -\frac{1}{6}(-\frac{1}{2})^m \\
(4\frac{9}{5}-2\frac{3}{4})(\frac{2}{3})^m &= \frac{(144-30)}{20}(\frac{2}{3})^m = \frac{57}{10}(\frac{2}{3})^m
\end{aligned}
$$

which correspond to the two terms in $r_{hh}(m)$..

Figure 1 shows a chosen cascade structure for $H(z)$

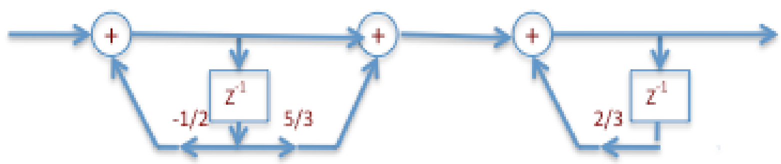


Figure 1: *Chosen cascade structure*

2c) White noise $w(n)$ with power $\sigma_w^2 = 1$ is input to the cascade structure.

Which kind of parametric processes are respectively the output signal and the internal signals in the structure? State the reason for your answer!

Answer :
We have two internal signals in the cascade structure, namely the outputs of the two first summation nodes. Delays do not modify the statistical properties of signals The first node has the transfer function $H_1(z)$. The filter is a first order allpole type (single pole), which gives an AR[1]-process. The second summation nodehas the transfer function $H_0(z)H_1(z)$, corresponding to an ARMA[1,1]-process. The filter output is of course described by the transfer function $H(z)$. From subtask 1a we see that the filter has a zero and two poles, i.e an ARMA[1,2]-process.

2d) White noise $w(n)$ with power $\sigma_w^2 = 1$ is input to the filter $H(z)$.

Find, by using linear prediction (i.e. the Yule-Walker or Normal equations), the filter coefficient a_1 of the best AR[1]-model for the filter output signal $y(n)$.

Show that the prediction error power, σ_f^2, always fulfills : $\sigma_f^2 \leq \gamma_{yy}(0)$ where $\sigma_y^2 = \gamma_{yy}(0)$ is the signal power of $y(n)$.

Answer:
For time lag $m = 1$ and the prediction error power $(m = 0)$ we have :

$$\begin{aligned} a_1\gamma_{yy}(0) &= -\gamma_{yy}(1) & m = 1 \\ \sigma_f^2 &= \gamma_{yy}(0) + a_1\gamma_{yy}(1) & m = 0 \end{aligned}$$

Further $\gamma_{yy}(m) = \sigma_w^2 r_{hh}(m)$, i.e. given $\sigma_w^2 = 1$ we get

$$\begin{aligned} \gamma_{yy}(0) &= -\frac{1}{6}(-\frac{1}{2})^0 + \frac{57}{10}(\frac{2}{3})^0 = -\frac{1}{6} + \frac{57}{10} = \frac{57*6-10}{60} = \frac{332}{60} \\ \gamma_{yy}(1) &= -\frac{1}{6}(-\frac{1}{2})^1 + \frac{57}{10}(\frac{2}{3})^1 = \frac{1}{12} + \frac{19}{5} = \frac{19*12+5}{60} = \frac{233}{60} \end{aligned}$$

Thus $a_1 = -\gamma_{yy}(1)/\gamma_{yy}(0) = -233/332 \approx -0.7$

We can further rewrite theprediction error power : $\sigma_f^2 = \gamma_{yy}(0)(1 + a_1\gamma_{yy}(1)/\gamma_{yy}(0))$. The quotient in the last term corresponds to the filter coefficient, thus we have $\sigma_f^2 = \gamma_{yy}(0)(1 - a_1^2)$. For all stable filters we have $|a_1| < 1$, thus $\sigma_f^2 \leq \gamma_{yy}(0)$. The quotient $\gamma_{yy}(0)/\sigma_f^2 = 1/(1 - a_1^2) \approx 1/(1 - (0.7)^2) \approx 2$ is called the prediction gain.

Problem 3 (4+4+7+4 = 19 points)

The discrete filter in task 1 is to be implemented in fixed point representation using $B+1$ bits and dynamic range $[-1,1)$. Rounding (quantization) is performed after each multiplication and the corresponding rounding error, $e(n)$, can be regarded as white noise with power $\sigma_e^2 = \frac{2^{-2B}}{12}$. Together, all the rounding noise sources result in a noise signal $z(n)$ at the output with a total power of σ_z^2.

3a) Find the resulting noise power σ_z^2 at the output of the cascade structure as a function of σ_e^2 .

Answer :
In the cascade structure we have three multiplications/roundings and thus three white noise sources. The sources at respectively 5/3 and 2/3 can both be moved in front of the last summation node. The corresponding unit pulse response is $h_2(n)$ for both sources. The source at $-1/2$ can be moved in front of the first summation node and sees the unit pulse response $h(n)$.
Thus we get: $\sigma_z^2 = (r_{hh}(0) + 2 * r_{h_2h_2}(0))\sigma_e^2 =$
$(332/60 + 2 * 9/5)\sigma_e^2 = ((332 + 216)/60)\sigma_e^2 = (548/60)\sigma_e^2 \approx 9\sigma_e^2$

3b) Also find the resulting noise power σ_z^2 at the output of the parallel structure in subtask 1d.
Note that the minus sign in front of $h_1(n)$ in equation 2 is implemented as an arithmetic operation (negation) and thus not as a multiplication!!

Answer :
We can ignore the gain of $G_1 = -1$. The sources at respectively 2 and 2/3belong to branch nr. 2. Both can be moved in front of the corresponding summation node and sees therefore $h_2(n)$. The source at $-1/2$ can correspondingly be moved in front of its summation node; thus it sees $h_1(n)$
Thus we get: $\sigma_z^2 = (2 * r_{h_2h_2}(0) + r_{h_1h_1}(0))\sigma_e^2 = (2 * 9/5 + 4/3)\sigma_e^2 = ((54 + 20)15)\sigma_e^2 =$
$(74/15)\sigma_e^2 \approx 5\sigma_e^2$
Consequently the (unscaled) parallel structure is superior wrt. noise power at the output. The filter input $x(n)$ has full dynamic range, i.e $x_{max} = \max_n |x(n)| = 1$.

3c) Show that one has to scale the input by 3/16 (downscaling by 16/3) in order to avoid overflow in the cascade structure used in subtask 3a.

Further show that the parallel structure used in subtask 3b requires a scaling by 1/6 (downscaling by 6).

Answer:
The cascade structure has two internal summation nodes and one at the output. The structure in subtask 3a is given by figure 1. The first node is described by the unit pulse response $h_1(n)$. The other internal node sees $h_3(n) = h_0(n) * h_1(n) = (-1/2)^n u(n) + 5/3(-1/2)^n u(n-1)$. The third node corresponds to the output, i.e. $h(n)$. Note that $h(n) \geq 0$ for all n.
Thus we get :

$$
\begin{aligned}
\sum_n |h(n)| &= \sum_{n=0}^{\infty}(2(\frac{2}{3})^n - (-\frac{1}{2})^n) = 2\frac{1}{1-\frac{2}{3}} - \frac{1}{1+\frac{1}{2}} = 2*3 - 2/3 = 16/3 \\
\sum_n |h_1(n)| &= \sum_{n=0}^{\infty}(\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2 \\
h_3(n) &= (-\frac{1}{2})^n u(n) + \frac{5}{3}(-\frac{1}{2})^{n-1}u(n-1) \quad \Rightarrow \\
h_3(0) &= 1 \\
h_3(n) &= (-\frac{1}{2})^n + \frac{5}{3}(-\frac{1}{2})^{n-1} \quad n > 0 \quad \Rightarrow \\
h_3(n) &= (-\frac{1}{2})^n - \frac{10}{3}(-\frac{1}{2})^n = -\frac{7}{3}(-\frac{1}{2})^n \quad n > 0 \quad \Rightarrow \\
\sum_n |h_3(n)| &= 1 + \frac{7}{3}\sum_{n=1}^{\infty}(\frac{1}{2})^n = 1 - \frac{7}{3} + \frac{7}{3}\sum_{n=0}^{\infty}(\frac{1}{2})^n \quad \Rightarrow \\
\sum_n |h_3(n)| &= -\frac{4}{3} + \frac{7}{3}(\frac{1}{1-\frac{1}{2}}) = -\frac{4}{3} + 2\frac{7}{3} = \frac{10}{3}
\end{aligned}
$$

As $16/3 > 10/3 > 2$ we must choose a scaling by 3/16.

The parallel structure also has two internal nodes (one in each branch) and the output node. The unit pulse responses of the output node and the lower branch node $h_1(n)$ are derived for the cascade structure. The upper branch has a unit pulse response from input equal to $2h_2(n)$.

$$
\sum_n |2h_2(n)| = \sum_{n=0}^{\infty}(2(\frac{2}{3})^n = \frac{2}{1-\frac{2}{3}} = 3*2 = 6. \tag{6}
$$

As $6 > 7/3 > 2$ we must choose a scaling of 1/6.

3d) Which of the two scaled structures have the best signal_to_noise_ratio ($SNR = \sigma_y^2/\sigma_z^2$) at the output ?

Answer :

Let us name the signal power at the output without scaling for σ_y^2. After scaling the signal output power is $S^2\sigma_y^2$ while the noise power at the output is unchanged (all the roundings are after the scaling). Thus the SNR of the two scaled structures are given by $S^2\sigma_y^2/\sigma_z^2$. As σ_y^2 is the same for the two structures the quotient S^2/σ_z^2 can be used . Thus we get:

$$\begin{aligned} SNR_{kask} &\approx (\frac{3}{16})^2/(9\sigma_e^2) = \frac{9}{256}\frac{1}{\sigma_e^2} \\ SNR_{par} &\approx (\frac{1}{6})^2/(5\sigma_e^2) \approx \frac{1}{180}\frac{1}{\sigma_e^2} \end{aligned}$$

Thus after scaling the cascade structure is the best choice.

Problem 4 **(5+4+4+5 = 18 points)**

Figure 2 shows a system for conversion of sampling rate from F_1 to F_2 where I and D are integers.

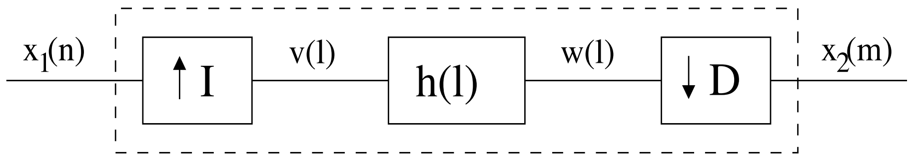


Figure 2: *System for sampling rate conversion*

4a) Shortly describe the three parts of and give the bandwidth and sampling rate of the internal signals $v(l)$ og $w(l)$.

Answer :
The first block inserts $I-1$ zeros between each sample. this will increase the sampling rate to $F_v = IF_1$and produce $I-1$ repetitions of the spectrum in the range $F_1/2$ til $F_v/2 = IF_1/2$. The ideal LP-filter $h(l)$ operates on the sampling rate of F_v and has a cutoff frequency of $F_g = F_v/max[D, I]$. This corresponds to half the sampling rate of the lowest of F_1 og F_2. The last block pick every D sample which corresponds to reduce the sampling rate to $F_2 = F_v/D = F_1 * (I/D)$.

4b) Derive a time domain expression for the output signal $x_2(m)$ as a function of the input signal $x_1(n)$, the filter $h(l)$ and I og D.

Answer:
The two signals $w(l)$ og $v(l)$ are related by linear convolution, i.e. $w(l) = \sum_k v(k)h(l-k)$. The output is thus given by $x_2(m) = w(mD) = \sum_k v(k)h(mD-k)$. But only every L sample of $v(k)$ is different from zero, thus $v(kI) = x_1(k)$ for $k = -\infty, \ldots, \infty$. This results in $x_2(m) = \sum_k x_1(k)h(mD-kI)$

4c) Discuss the operation of the system when $I > D$ and vice versa.

Answer :
When $I > D$ the final sampling rate is increased. Thus no frequency content is lost, but the range over $F_1/2$, dvs. $[F_1/2, F_2/2]$ is without content. The cut off frequency of the filter is given by $F_g = F_1/2$.
When $I < D$ the final sampling rate is decreased. The filter must then remove the frequency content over $F_2/2$, i.e.. $[F_2/2, F_1/2]$ to avoid aliasing. The cut off frequency must therefore be chosen as $f_g = F_2/2$.

4d) Given an analogue signal $x(t) = s(t) + sin(2\pi F_0 t)$ where $s(t)$ has bandwidth $\pm B = 5000Hz$ and $F_0 = 4000Hz$. The signal is sampled by a rate $F_1 = 10000Hz$, i.e. $x_1(n) = x_a(nT_1)$ hvor $T_1 = 1/F_1$.
Further, we have given a notch filter with a zero of $f_n = 0.25$.

How can one use the system in figure 2 together with the notch filter to remove the harmonic component (F_0) in $x_1(n)$?

Answer :
Using the original sampling rate results in that the harmonic component is placed at $f_{01} = F_0/F_1 = 4000/10000 = 0.4$. This is another normalized frequency than the zero of $f_n = 0.25$ of the notch filter. The sampling rate must therefore be changed such that $f_{02} = F_0/F_2 = f_n$, i.e. $F_2 = F_0/f_n = 4000/0.25 = 16000$ This corresponds to a sampe rate change of $16000/10000 = 8/5$. This is implemented by using the system in figure 2 with $I = 8$ og $D = 5$ and thereafter use the notch filter.

Some basic equations and formulas.

A. Sequences :

$$\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \Longleftrightarrow |\alpha| < 1$$
$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

B. Linear convolution :

$$y(n) = h(n) * x(n) = \sum_k h(k)x(n-k) = \sum_k x(k)h(n-k)$$

$$Y(z) = H(z)X(z) \Rightarrow Y(f) = H(f)X(f) \Rightarrow$$

$$Y(f_k) = H(f_k)X(f_k) \quad f_k = k/N \text{ for } k = 0, \ldots, N-1 \text{ where we write } Y(k) = Y(f_k)$$

C. Transforms :

$$H(z) = \sum_n h(n)z^{-n} \Rightarrow H(f) = \sum_n h(n)\ e^{-j2\pi nf}$$

$$\text{DFT} : H(k) = \sum_{n=0}^{L-1} h(n)\ e^{-j2\pi nk/N} \qquad k = 0, ..., N-1$$

$$\text{IDFT} : h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)\ e^{j2\pi nk/N} \qquad n = 0, ..., L-1$$

D. The sampling (Nyquist) theorem :

Given an analog signal $x_a(t)$ with bandwidth $\pm B$ which is sampled by $F_s = 1/T_s$:

$$x(n) = x(nT_s) = x_a(t)|_{t=nT_s} \qquad n = -\infty, ..., \infty$$

$$X(f) = X(F/F_s) = F_s \sum_k X_a[(f-k)F_s]$$

$$x_a(t) \text{ can be recovered from } x(n) \quad \Leftrightarrow \quad F_s \geq 2B \tag{7}$$

E. Autocorrelation, energy spectrum and Parsevals theorem :

Given a sequence $h(n)$ with finite energy E_h :

$$\text{Autocorrelation :} \quad r_{hh}(m) = \sum_n h(n)h(n+m) \quad m = -\infty, ..., \infty$$

$$\text{Energy spectrum :} \quad S_{hh}(z) = H(z)H(z^{-1}) \Rightarrow S_{hh}(f) = |H(f)|^2$$

$$\text{Parsevals theorem:} \quad E_h = r_{hh}(0) = \sum_n h^2(n) = \int_0^{2\pi} |H(f)|^2 df$$

F. Multirate formulaes :

Decimation where $T_{sy} = DT_{sx}$:

$$v(mT_{sy}) = \sum_k h[(mD-k)T_{sx}]\ x(kT_{sx}) \quad m = -\infty, ..., \infty$$

Upsampling where $T_{sx} = UT_{sy}$:

$$y(lT_{sy}) = \sum_n h[(l-nU)T_{sy}]\ x(nT_{sx}) \quad l = -\infty, ..., \infty$$

Interpolation where $T_{sy} = DT_{sv} = \frac{D}{U}T_{sx}$:

$$y(lT_{sy}) = \sum_m h[(lD-mU)T_{sv}]\ x(mT_{sx}) \quad l = -\infty, ..., \infty$$

G. Autocorrelation, power spectrum and Wiener-Khintchin theorem :

Given a stationary, ergodic sequence $x(n)$ with infinite energy :

Autocorrelation : $\gamma_{xx}(m) = E[x(n)x(n+m)]$ $m = -\infty, ..., \infty$

Power spectrum: $\Gamma_{xx}(z) = Z[\gamma_{xx}(m)] \quad \Rightarrow$

Wiener-Khintchin : $\Gamma_{xx}(f) = DTFT[\gamma_{xx}(m)] = \sum_{m} \gamma_{xx}(m)\, e^{-j2\pi mf}$

H. The Yule-Walker and Normal equations where $a_0 = 1$:

Yule-Walker equations : $\sum_{k=0}^{P} a_k \gamma_{xx}(m-k) = \sigma_f^2\, \delta(m) \quad m = 0, ..., P$

Normal equations: $\sum_{k=1}^{P} a_k \gamma_{xx}(m-k) = -\gamma_{xx}(m) \quad m = 1, ..., P$