NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Electronics and Telecommunications

Examination paper for TTT4120 Digital Signal Processing

Academic contact during examination:: Torbjørn Svendsen
Phone: 930 80 477

Examination date: Wednesday, December 4, 2013

Examination time (from - to): 09.00 - 13.00

Permitted examination support material: D – No printed or handwritten material allowed. Specified, simple calculator allowed

Other information:

- The examination consists of 4 problems where
  - problem 1 concerns structures and implementation of digital filters
  - problem 2 concerns linear, time invariant systems
  - problem 3 concerns design of digital filters
  - problem 4 concerns stochastic processes
- Weighting of each sub-problem is given in parenthesis at the start of each problem.
- All problems are to be answered
- Grades will be announced 3 weeks after the examination date.

Language: English

Total number of pages: 9

Of this, number of enclosure pages: 3

Checked by:

Date Signature

Problem 1 (30 points = 4+4+5+7+5+5 (34%))

We have a causal filter which is described by the following difference equation:

$$y(n) = -1.8y(n-1) - 0.81y(n-2) + x(n) - x(n-1)$$

1a) Show that the transfer function of the filter can be expressed as

$$H(z) = Y(z)/X(z) = \frac{1 - z^{-1}}{1 + 1.8z^{-1} + 0.81z^{-2}}$$

.

1b) Find the poles and zeros of the filter.
Sketch the position of poles and zeros in the z-plane and use this to determine which type of filter this is (LP, HP, BP etc.).

1c) Draw the filter when it is realized as direct form I (DF I) and direct form II (DF II) structures, respectively.
What can you generally say about the properties of the direct form structures?

1d) Find the impulse response of the filter in Figure 1.
Assume that a zero mean white signal $x(n)$ with power σ_x^2=1 is the filter input. What is the power, σ_y^2 of the filtered signal $y(n)$?

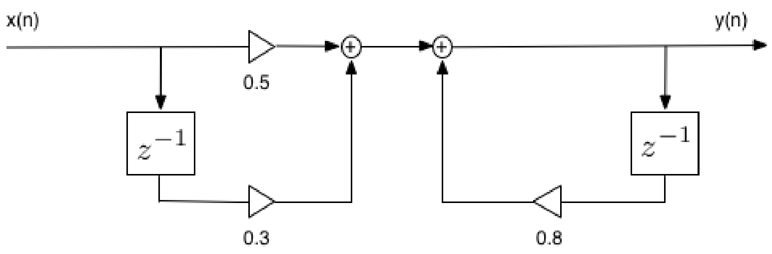

Figure 1: Digital filter

1e) The filter in Figure 1 is to be realized using binary fixed-point representation with B bits. Rounding is performed after every multiplication. Assume that the round-off error after every multiplication can be modeled as a zero mean white noise source with power σ_e^2. Find the power of the round-off error at the output of the filter, expressed by σ_e^2.

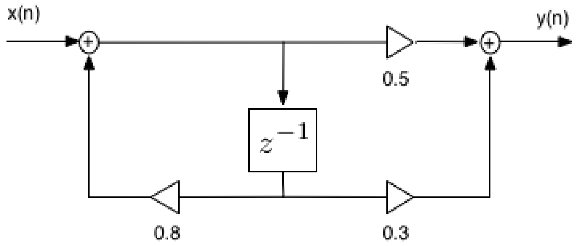

Figure 2: Equivalent digital filter

1f) Show that the filter in Figure 2 is equivalent to the filter in Figure 1.
This filter is to be realized with the same multipliers and the same binary representation as in the previous sub-problem. Find the power of the round-off error at the output of the filter in this case. Comment.

Problem 2 (18 points = 4+8+6 (20%))

A linear, time invariant system is specified by

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - \frac{5}{2}z^{-1} + z^{-2}}$$

2a) Show that the system can be represented in the z-plane as

$$H(z) = \frac{\frac{2}{3}}{1 - 2z^{-1}} - \frac{\frac{2}{3}}{1 - \frac{1}{2}z^{-1}}$$

2b) Find the region of convergence (ROC) and the impulse response for the system when it is causal, anti-causal and non-causal.
For which case is the system stable?

2c) A linear, time invariant system has poles and zeros as depicted in Figure 3.
Find the z-transform of the system.
What is the requirement for the system to be stable?

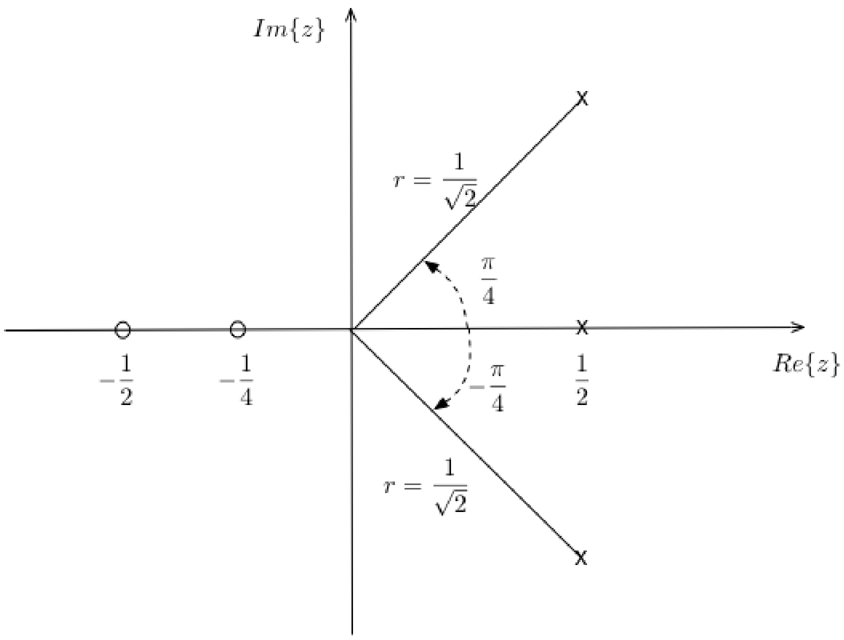

Figure 3: Placement of poles (x) and zeros (○)

Problem 3 (19 points = 5+8+6 (21%))

3a) We can design an IIR filter $H(z)$ by using a known analog filter, $H_a(s)$ and the bilinear transform

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

Use the relations $s = j\Omega$ og $z = e^{j\omega}$ to show that the mapping between analog and digital frequency is given by

$$\Omega = \frac{2}{T}\tan(\frac{\omega}{2})$$

3b) We start with the following analog, causal filter

$$H_a(s) = 2\frac{s + \Omega_c}{s + 4\Omega_c}$$

The filter is to be transformed to a time discrete filter by applying the bilinear transform, and we specify that Ω_c is to be mapped to ω_c which is given by $\tan(\frac{\omega_c}{2}) = 1/2$.

Find the relationship between T and Ω_c, and show that the resulting transfer function is given by

$$H(z) = \frac{1 - \frac{1}{3}z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

3c) Show that the unit pulse response (impulse response) of $H(z)$ is given by

$$h(n) = \begin{cases} 0 & n < 0 \\ 1 & n = 0 \\ 2\left(-\frac{1}{3}\right)^n & n > 0 \end{cases}$$

Problem 4 (22 points = 3+3+7+9 (25%))

A filter is given by its unit pulse response (impulse response)

$$h(n) = \begin{cases} 3 & n = 0 \\ 2 & n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

A stochastic process $x(n)$ is generated by passing zero mean white noise $e(n)$ with power $\sigma_e^2 = 1$ through the filter.

4a) What type of process is $x(n)$?
What is the order of the process? State the reasons for your answers.

4b) Give expressions for the autocorrelation function $\gamma_{ee}(l)$ and power spectral density $\Gamma_{ee}(\omega)$ of the white noise process $e(n)$.

4c) Show that the autocorrelation function of the process $x(n)$ is given as

$$\gamma_{xx}(l) = \begin{cases} 13 & l = 0 \\ 6 & l = \pm 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find the power spectral density $\Gamma_{xx}(\omega)$ of the process $x(n)$.

4d) Give a time domain expression for a general first order linear predictor.
Give the corresponding expression for the prediction error power when the predictor is applied to a stochastic process.

Derive a general expression for the optimal predictor coefficient by minimizing the prediction error power.

Find the optimal predictor coefficient and the corresponding prediction error power for the process $x(n)$ which has an autocorrelation function specified in problem 4c.

Appendix: Some basic equations and formulas.

A. Sequences:

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

$$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \quad \text{and} \quad -\sum_{n=-1}^{-\infty} \alpha^n = \frac{1}{1-\alpha}$$

B. Linear convolution:

$$y(n) = h(n) * x(n) = \sum_k h(k)x(n-k) = \sum_k x(k)h(n-k)$$

$$Y(z) = H(z)X(z) \Rightarrow Y(f) = H(f)X(f) \Rightarrow$$

$$Y(f_k) = H(f_k)X(f_k) \quad f_k = k/N \quad \text{for} \quad k = 0, \ldots, N-1 \quad \text{where we write} \quad Y(k) = Y(f_k)$$

C. Transforms:

$$\text{Z} \quad : H(z) = \sum_n h(n)z^{-n} \Rightarrow H(f) = \sum_n h(n)\ e^{-j2\pi nf}$$

$$\text{DFT} : H(k) = \sum_{n=0}^{L-1} h(n)\ e^{-j2\pi nk/N} \qquad k = 0, ..., N-1$$

$$\text{IDFT:}\ h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)\ e^{j2\pi nk/N} \qquad n = 0, ..., L-1$$

D. The sampling (Nyquist) theorem:

Given an analog signal $x_a(t)$ with bandwidth $\pm B$ which is sampled by $F_s = 1/T_s$:

$$x(n) = x(nT_s) = x_a(t)|_{t=nT_s} \qquad n = -\infty, ..., \infty$$

$$X(f) = X(F/F_s) = F_s \sum_k X_a[(f-k)F_s]$$

$x_a(t)$ can be recovered from $x(n) \quad \Leftrightarrow \quad F_s \geq 2B$

E. Autocorrelation, energy spectrum and Parseval's theorem:

Given a sequence $h(n)$ with finite energy E_h :

Autocorrelation: $\quad r_{hh}(m) = \sum_n h(n)h(n+m) \quad m = -\infty, ..., \infty$

Energy spectrum: $\quad S_{hh}(z) = H(z)H(z^{-1}) \Rightarrow S_{hh}(f) = |H(f)|^2$

Parseval's theorem: $\quad E_h = r_{hh}(0) = \sum_n h^2(n) = \int_0^{2\pi} |H(f)|^2 df$

F. Multirate formulae:

Decimation where $T_{sy} = DT_{sx}$:

$$v(mT_{sy}) = \sum_k h[(mD-k)T_{sx}]\; x(kT_{sx}) \quad m = -\infty, ..., \infty$$

Upsampling where $T_{sx} = UT_{sy}$:

$$y(lT_{sy}) = \sum_n h[(l-nU)T_{sy}]\; x(nT_{sx}) \quad l = -\infty, ..., \infty$$

Interpolation where $T_{sy} = DT_{sv} = \frac{D}{U}T_{sx}$:

$$y(lT_{sy}) = \sum_m h[(lD-mU)T_{sv}]\; x(mT_{sx}) \quad l = -\infty, ..., \infty$$

G. Autocorrelation, power spectrum and Wiener-Khintchin theorem:

Given a stationary, ergodic sequence $x(n)$ with infinite energy :

Autocorrelation: $\gamma_{xx}(m) = E[x(n)x(n+m)]\ m = -\infty, ..., \infty$

Power spectrum: $\Gamma_{xx}(z) = Z[\gamma_{xx}(m)] \quad \Rightarrow$

Wiener-Khintchin: $\Gamma_{xx}(f) = DTFT[\gamma_{xx}(m)] = \sum_{m} \gamma_{xx}(m)\ e^{-j2\pi mf}$

H. The Yule-Walker and Normal equations where $a_0 = 1$:

Yule-Walker equations: $$\sum_{k=0}^{p} a_k \gamma_{xx}(m-k) = \sigma_f^2\ \delta(m) \quad m = 0, ..., p$$

Normal equations: $$\sum_{k=1}^{p} a_k \gamma_{xx}(m-k) = -\gamma_{xx}(m) \quad m = 1, ..., p$$