NTNU – Trondheim
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Department of Electronics and Telecommunications

Examination paper for TTT4120 Digital Signal Processing

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Examination date: Thursday, December 18, 2014

Examination time (from - to): 09.00 - 13.00

Permitted examination support material: **D** – No printed or handwritten material allowed. Specified, simple calculator allowed

Other information:

- The examination consists of 4 problems where
  - problem 1 concerns LTI systems
  - problem 2 concerns digital filters
  - problem 3 concerns filter realization
  - problem 4 concerns signal generation
- Weighting of each sub-problem is given in parenthesis at the start of each problem.
- All problems are to be answered
- Grades will be announced 3 weeks after the examination date.

Language: English

Total number of pages: 9

Of this, number of enclosure pages: 1

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Date	Signature

Problem 1 (3+3+4+2+3+4=19)

1a) Which properties need to be fulfilled in order to describe a time discrete system by its unit pulse response $h(n)$?
Define the properties *stability* and *causality* using $h(n)$.

1b) A time discrete signal $y(n)$ is formed by processing another time discrete signal $x(n)$. The relationship between the signals is expressed by

$$y(n) = x(n) + 2x(n-1) + x(n-2)$$

Find the transfer function, $H(z)$, of an LTI system that recovers $x(n)$ from $y(n)$.

1c) Find the causal unit pulse response, $h(n)$, of the system in problem 1b.

1d) We have a stable and causal LTI system with real coefficients and transfer function

$$H(z) = \frac{1 - \beta z^{-1}}{1 - \alpha z^{-1}}$$

Define the legal region for poles and zeros in the z plane.
Sketch the region of convergence (ROC) for the system.

1e) Show that the time discrete Fourier transform (DTFT), $X(f)$, of a sequence $x(n)$ has the following properties

i) $X(-f) = X^*(f)$ when $x(n)$ is real
ii) $X(f) = -X(-f)$ when $x(n) = -x(-n)$

1f) Suppose you have an FIR filter with real coefficients, $h(n)$, and a finite duration real sequence, $x(n)$ We shall use this filter to produce a filtered version of $x(n)$ that has the same phase as the original signal.

1. We first filter $x(n)$ with $h(n)$ to form $s(n) = x(n) * h(n)$.
2. Thereafter, we filter the time reversed sequence using the same filter, i.e., $v(n) = s(N-1-n) * h(n)$, where N is the length of $s(n)$.
3. Finally, the filtered output is obtained by time reversing $v(n)$, $y(n) = v(K-1-n)$. K is the length of $v(n)$

Show that the DTFT of $y(n)$, $Y(f)$, has the same phase as $X(f)$.

Problem 2 (2+4+2+2+3=13)

We have a linear, time invariant time discrete system defined by the difference equation:

$$y(n) = \sum_{k=0}^{\infty}(-\frac{1}{2})^k \; x(n-2k); \quad n \geq 0$$

2a) Calculate the 8 first values of the unit pulse response of this system.

2b) Show that the system can be implemented with the filter depicted in Figure 1.
(Hint: Start by finding $y(n)$ for the first values of n)

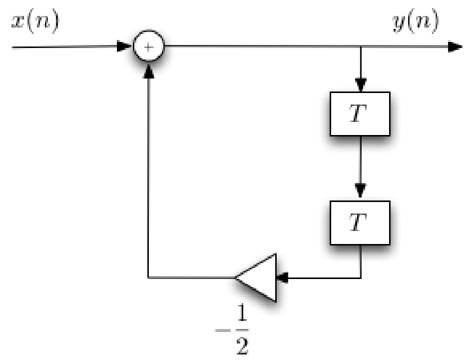

Figure 1: Filter implementation

2c) Find the z transform $H(z)$ for this filter?

2d) What are the locations of the poles and zeros of $H(z)$?

2e) Suppose we input a signal $x(n) = cos(2\pi fn)$ to the filter.
For which frequency will the amplitude of the output signal $y(n)$ be the largest?
How big is the maximum amplitude?

Problem 3 (3+2+4+4+4=17)

3a) A causal digital filter is given by the block diagram in i Figure 2.

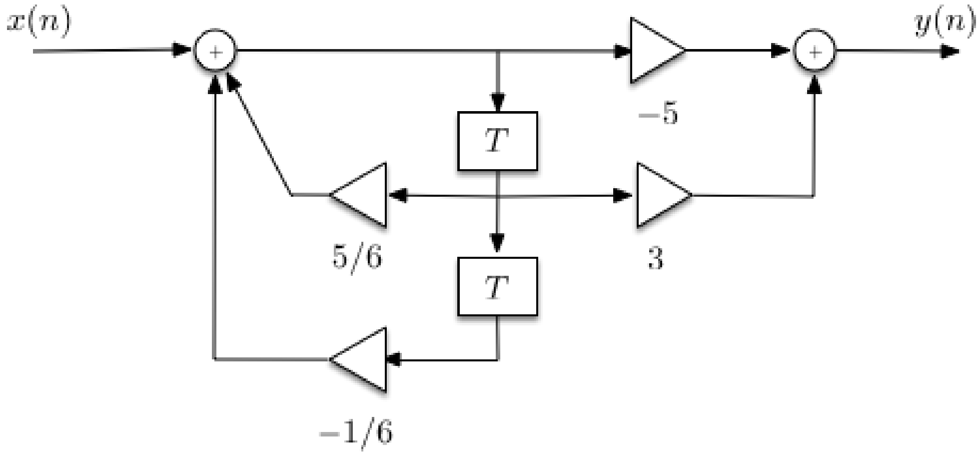

Figure 2: Digital filter

Find the transfer function, $H(z)$ of the filter.
Sketch the location of poles and zeros in the z plane.

3b) The filter in Figure 2 can be realized using other structures.
Draw a Direct Form I realization of the filter (DF I).

3c) Show that the filter in Figure 3 is equivalent to the filter in Figure 2. Based on the filter structure in Figure 3, find the unit pulse response of the filter, $h(n)$.

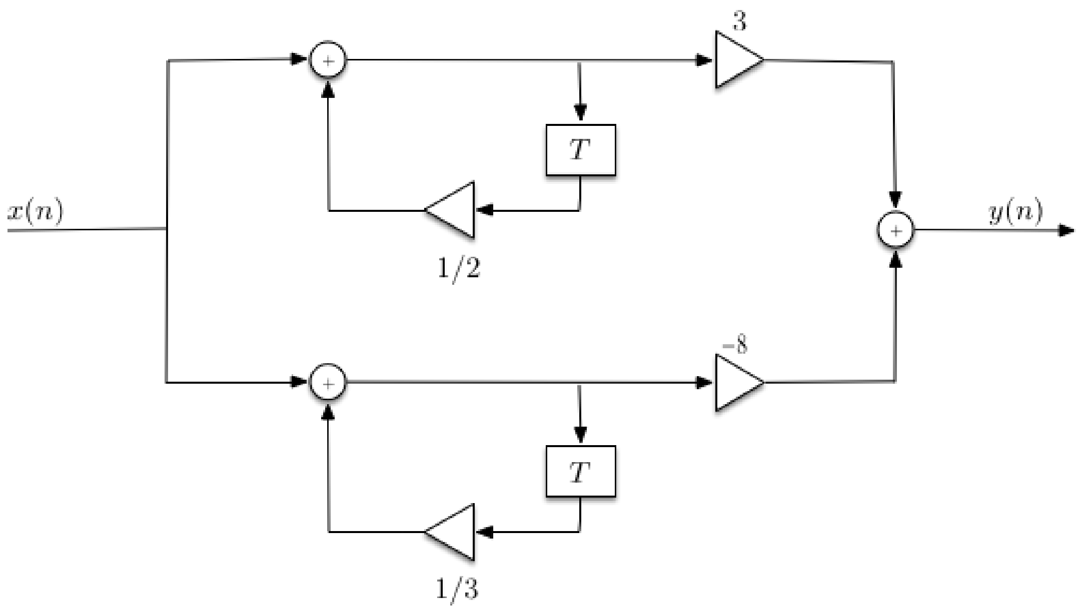

Figure 3: Alternative filter structure

Our filter is to be implemented in fixed point arithmetic, and we wish to ensure that we will not have overflow in the summations, and that we have control of the round-off errors from the multiplications.

3d) Find how we need to scale the input signal in order to avoid summation overflow in the two filter structures of Figures 2 and 3.
Hint: If a sequence, $x(n)$, increases monotonously, such that $x(n) < 0$ for $n < K$ and $x(n) \geq 0$ for $n \geq K$, then $\sum_{n=0}^{\infty} |x(n)| = \sum_{n=0}^{\infty} x(n) - 2\sum_{n=0}^{K-1} x(n)$

3e) Assume that the round-off error after each multiplication can be modeled as an additive noise source with zero mean and variance σ_q^2.
Find the power of the total round-off noise at the output of the two filters expressed by σ_q^2.

Problem 4 (2+4+2+2=10)

We are to make a causal 2nd order IIR filter with an unit pulse response that is a pure sinusoid. For simplicity, we let the unit pulse response be a zero phase cosine function. The filter will operate on a sampling rate of $F_s = 48kHz$, and the sinusoid is to have a frequency of $8kHz$.

4a) Find an expression for the unit pulse response, $h(n)$, of the filter expressed as a sum of exponential functions. (Remember that $e^{j\omega} = \cos\omega + j\sin\omega$)

4b) Show that the z transform of the filter's unit pulse response is

$$H(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 - z^{-1} + z^{-2}}$$

Draw the location of poles and zeros in the z plane.

4c) What is the expression for the difference equation describing the system?

4d) Explain how a filter like this can be used as a computationally efficient signal generator for a digital sinusoid with normalized frequency $f = \frac{1}{6}$.

Appendix: Some basic equations and formulas.

A. Sequences:

$$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$$

$$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \quad \text{and} \quad -\sum_{n=-1}^{-\infty} \alpha^n = \frac{1}{1-\alpha}$$

$$\sum_{n=0}^{N-1} (n+1)\alpha^n = \frac{1-\alpha^N}{(1-\alpha)^2} - \frac{N\alpha^N}{1-\alpha} \ ; \quad \alpha \neq 1$$

$$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty} (n+1)\alpha^n = \frac{1}{(1-\alpha)^2}$$

B. Linear convolution:

$$y(n) = h(n) * x(n) = \sum_k h(k)x(n-k) = \sum_k x(k)h(n-k)$$

$$Y(z) = H(z)X(z) \Rightarrow Y(f) = H(f)X(f) \Rightarrow$$

$$Y(f_k) = H(f_k)X(f_k) \quad f_k = k/N \quad \text{for} \quad k = 0, \ldots, N-1 \quad \text{where we write} \quad Y(k) = Y(f_k)$$

C. Transforms:

$$\text{Z} \quad : H(z) = \sum_n h(n)z^{-n} \Rightarrow H(f) = \sum_n h(n)\ e^{-j2\pi nf}$$

$$\text{DFT} : H(k) = \sum_{n=0}^{L-1} h(n)\ e^{-j2\pi nk/N} \qquad k = 0, ..., N-1$$

$$\text{IDFT:}\ h(n) = \frac{1}{N} \sum_{k=0}^{N-1} H(k)\ e^{j2\pi nk/N} \qquad n = 0, ..., L-1$$

D. The sampling (Nyquist) theorem:

Given an analog signal $x_a(t)$ with bandwidth $\pm B$ which is sampled by $F_s = 1/T_s$:

$$x(n) = x(nT_s) = x_a(t)|_{t=nT_s} \qquad n = -\infty, ..., \infty$$

$$X(f) = X(F/F_s) = F_s \sum_k X_a[(f-k)F_s]$$

$x_a(t)$ can be recovered from $x(n)$ $\quad \Leftrightarrow \quad F_s \geq 2B$

E. Autocorrelation, energy spectrum and Parseval's theorem:

Given a sequence $h(n)$ with finite energy E_h :

Autocorrelation: $\quad r_{hh}(m) = \sum_n h(n)h(n+m) \quad m = -\infty, ..., \infty$

Energy spectrum: $\quad S_{hh}(z) = H(z)H(z^{-1}) \Rightarrow S_{hh}(f) = |H(f)|^2$

Parseval's theorem: $\quad E_h = r_{hh}(0) = \sum_n h^2(n) = \int_0^{2\pi} |H(f)|^2 df$

F. Multirate formulae:

Decimation where $T_{sy} = DT_{sx}$:

$$v(mT_{sy}) = \sum_k h[(mD-k)T_{sx}]\; x(kT_{sx}) \quad m = -\infty, ..., \infty$$

Upsampling where $T_{sx} = UT_{sy}$:

$$y(lT_{sy}) = \sum_n h[(l-nU)T_{sy}]\; x(nT_{sx}) \quad l = -\infty, ..., \infty$$

Interpolation where $T_{sy} = DT_{sv} = \frac{D}{U}T_{sx}$:

$$y(lT_{sy}) = \sum_m h[(lD-mU)T_{sv}]\; x(mT_{sx}) \quad l = -\infty, ..., \infty$$

G. Autocorrelation, power spectrum and Wiener-Khintchin theorem:

Given a stationary, ergodic sequence $x(n)$ with infinite energy :

Autocorrelation: $\gamma_{xx}(m) = E[x(n)x(n+m)]$ $m = -\infty, ..., \infty$

Power spectrum: $\Gamma_{xx}(z) = Z[\gamma_{xx}(m)] \quad \Rightarrow$

Wiener-Khintchin: $\Gamma_{xx}(f) = DTFT[\gamma_{xx}(m)] = \sum_m \gamma_{xx}(m)\ e^{-j2\pi mf}$

H. The Yule-Walker and Normal equations where $a_0 = 1$:

Yule-Walker equations: $$\sum_{k=0}^{p} a_k \gamma_{xx}(m-k) = \sigma_f^2\ \delta(m) \quad m = 0, ..., p$$

Normal equations: $$\sum_{k=1}^{p} a_k \gamma_{xx}(m-k) = -\gamma_{xx}(m) \quad m = 1, ..., p$$

I. Some common z-transform pairs

	Signal, $x(n)$	$X(z)$	ROC
1	$\delta(n)$	1	Alle z
2	$u(n)$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
3	$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
4	$na^n u(n)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
5	$-a^n u(-n-1)$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
6	$-na^n u(-n-1)$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$