NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Electronics and Telecommunications

Examination paper for TTT4120 Digital Signal Processing

Solutions

Academic contact during examination:: Torbjørn Svendsen
Phone: 930 80 477

Examination date: Thursday December 18, 2014

Examination time (from - to): 09.00 - 13.00
Permitted examination support material: **D** – No printed or handwritten material allowed. Specified, simple calculator allowed

Other information:

- The examination consists of 4 problems where
  - problem 1 concerns LTI systems
  - problem 2 concerns digital filters
  - problem 3 concerns filter realization
  - problem 4 concerns signal generation
- Weighting of each sub-problem is given in parenthesis at the start of each problem.
- All problems are to be answered
- Grades will be announced 3 weeks after the examination date.

Language: English

Total number of pages: 12

Of this, number of enclosure pages: 1

Checked by:

Date Signature

Problem 1 (3+3+4+2+3+4=19)

1a) The system must be linear and time invariant.

Causality: $h(n) = 0$ for $n < 0$

Stability (BIBO-stability): $\sum_n |h(n)| < \infty$

1b) Take the z-transform of both sides:

$$Y(z) = X(z) + 2z^{-1}X(z) + z^{-2}X(z) = (1 + 2z^{-1} + z^{-2})X(z)$$

In order to recover $x(n)$ from $y(n)$, we need a filter $H(z)$ that fulfils:

$$X(z) = H(z)Y(z) = H(z)(1 + 2z^{-1} + z^{-2})X(z) \Leftrightarrow H(z) = \frac{1}{1 + 2z^{-1} + z^{-2}}$$

1c) The causal impulse response can be found in (at lest) two different ways:

1. We factorize $H(z)$:
$$H(z) = \frac{1}{1+z^{-1}}\frac{1}{1+z^{-1}} = H_1(z)H_1(z)$$
Noting that multiplication in the frequency domain is equivalent to convolution in the time domain, we se that
$$h(n) = h_1(n) * h_1(n)$$
We know that for a causal LTI system
$$\mathcal{Z}^{-1}\left\{\frac{\beta}{1-\alpha z^{-1}}\right\} = \beta\alpha^n u(n)$$
The inverse z-transform of $H_1(z)$ for a causal system is thus $h_1(n) = (-1)^n u(n)$ and we can find $h(n)$:
$$\begin{aligned} h(n) &= \sum_{k=0}^{\infty} h_1(k)h_1(n-k) = \sum_{k=0}^{\infty}(-1)^k u(k)(-1)^{n-k}u(n-k) \\ &= (-1)^n \sum_{k=0}^{\infty} u(n-k)u(k) = (-1)^n \sum_{k=0}^{n} 1 = (-1)^n(n+1) \ ; \quad n \geq 0 \end{aligned}$$
or
$$h(n) = (-1)^n(n+1)u(n)$$

2. Noting that $\sum_{n=0}^{\infty}(n+1)\alpha^n = \frac{\alpha}{(1-\alpha)^2}$, we here have $\alpha = -z^{-1}$, thus
$$\sum_{n=0}^{\infty}(n+1)(-z^{-1})^n = \frac{-z^{-1}}{(1-(-z^{-1}))^2} = \frac{-z^{-1}}{(1+z^{-1})^2} = H(z)$$
and the corresponding causal impulse response is
$$h(n) = (n+1)(-1)^n u(n)$$

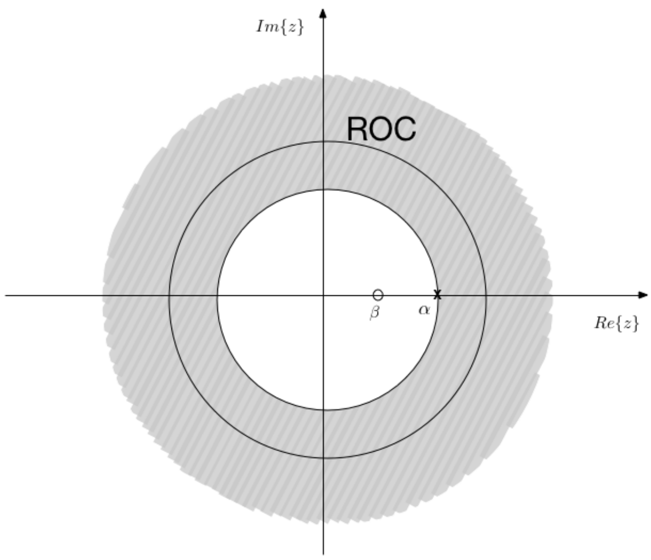


Figure 1: ROC and singularities

1d) The system has a zero in $z = \beta$ and a pole in $z = \alpha$.
For a stable causal system, all poles must be inside the unit circle, and the ROC is the area outside the largest pole. There are no requirements on the zeros. Thus:

$$\begin{aligned} |\alpha| &< 1 \\ \beta &\in Z \\ \text{ROC:}|z| &> \alpha \end{aligned}$$

1e) When $x(n)$ is real:

$$X^*(f) = \left(\sum_{-\infty}^{\infty} x(n) e^{-j2\pi f n} \right)^* = \sum_{-\infty}^{\infty} x(n) e^{j2\pi f n} = X(-f)$$

When $x(n)$ is anti-symmetric:

$$X(-f) = \sum_{-\infty}^{\infty} x(n) e^{j2\pi f n} = \sum_{-\infty}^{\infty} -x(-n) e^{-j2\pi f n} = -\sum_{-\infty}^{\infty} x(m) e^{j2\pi f m} = -X(f)$$

1f) The effect of the first filter operation:

$$s(n) = h(n) * x(n) \Leftrightarrow S(f) = H(f)X(f)$$

The DTFT of the time reversed sequence is:

$$DTFT\{s(N-n)\} = \sum_{n=0}^{N-1} s(N-n) e^{-j2\pi f n} = \sum_{m=0}^{N-1} s(m) e^{j2\pi f m} \underbrace{e^{j2\pi(N-1)}}_{=1} = S(-f)$$

where $N = M + N - 1$ is the length of the filtered sequence.

Filtering the time reversed sequence, we obtain:

$$V(f) = DTFT\left\{h(n) * s(-n)\right\} = H(f)S(-f) = H(f)H(-f)X(-f)$$

Finally, time reversing $v(n)$, we obtain

$$Y(f) = V(-f) = H(-f)H(f)X(f) = |H(f)|^2 X(f)$$

since $h(n)$ is real (see problem 1e).

Problem 2 (2+4+2+2+3=13)

2a) Expanding the difference equation $y(n) = \sum_{k=0}^{\infty}(-\frac{1}{2})^k x(n-2k)$ for $x(n) = \delta(n)$:

$$
\begin{aligned}
y(0) &= -(\frac{1}{2})^0\delta(0) = 1\\
y(1) &= (-\frac{1}{2})^0\delta(1) = 0\\
y(2) &= (-\frac{1}{2})^0\delta(2) + (-\frac{1}{2})^1\delta(0) = -\frac{1}{2}\\
y(3) &= (-\frac{1}{2})^0\delta(3) + (-\frac{1}{2})^1\delta(1) = 0\\
y(4) &= (-\frac{1}{2})^0\delta(4) + (-\frac{1}{2})^1\delta(2) + (-\frac{1}{2})^2\delta(0) = \frac{1}{4}\\
y(5) &= 0\\
y(6) &= (-\frac{1}{2})^3\delta(0) = -\frac{1}{8}\\
y(7) &= 0
\end{aligned}
$$

2b) We start by inspecting the output, then use induction to prove the relationship.

$$
\begin{aligned}
&y(0) = x(0)\\
&y(1) = x(1)\\
&y(2) = x(2) - \frac{1}{2}x(0) = x(2) - \frac{1}{2}y(0)\\
&y(3) = x(3) - \frac{1}{2}x(1) = x(3) - \frac{1}{2}y(1)\\
&y(4) = x(4) - \frac{1}{2}x(2) + \frac{1}{4}x(0) = x(4) - \frac{1}{2}y(2)\\
&\quad \ldots
\end{aligned}
$$

Inspecting the output we form the assumption that the output can be expressed as

$$y(n) = x(n) - \frac{1}{2}y(n-2)$$

This is obviously true for $n = 0, 1, 2$ (assuming $x(n)$ is causal), as shown above.
For n:

$$
\begin{aligned}
y(n-2) &= x(n-2) - \frac{1}{2}x(n-4) + \frac{1}{4}x(n-6) - \frac{1}{8}x(n-8)\ldots\\
y(n) &= x(n) - \frac{1}{2}x(n-2) + \frac{1}{4}x(n-4) - \frac{1}{8}x(n-6)\ldots\\
&= x(n) - \frac{1}{2}(x(n-2) - \frac{1}{2}x(n-4) + \frac{1}{4}x(n-6) - \frac{1}{8}x(n-8)\ldots) \quad = x(n) - \frac{1}{2}y(n-2)
\end{aligned}
$$

The difference equation $y(n) = x(n) - \frac{1}{2}y(n-2)$ is exactly the difference equation of the system in Figure 1.

2c) The z-transform:

$$y(n) = x(n) - \frac{1}{2}y(n-2) \Leftrightarrow H(z) = \frac{1}{1 + \frac{1}{2}z^{-2}}$$

2d) Pole and zero locations:

$$H(z) = \frac{1}{1 + \frac{1}{2}z^2} = \frac{z^2}{(z^2 + \frac{1}{2})} = \frac{z^2}{(z + j\frac{1}{\sqrt{2}})(z - j\frac{1}{\sqrt{2}})}$$

i.e. the system has a double zero at $z_n = 0$ and complex conjugate poles at $z_{p_1} = -j\frac{1}{\sqrt{2}}$ and $z_{p_1} = j\frac{1}{\sqrt{2}}$

2e) The spectrum of a zero-phase cosine with unit amplitude and frequency f is $\delta(\pm f)$. Thus, the output of the filter will be maximum at the frequency for which the filter has the highest amplification. We see from the analysis above that the poles are on the imaginary axis, i.e. the frequency for which $y(n)$ has the highest amplitude will be $f = \pi/2$.

The max amplitude will then be $|X(f)H(f)| = |\delta(\pi/2)H(f)| = |H(\pi/2)|$:

$$|H(\pi/2)| = \frac{1}{1 + \frac{1}{2}(e^{j\pi/2})^{-2}} = \frac{1}{1 + \frac{1}{2}e^{-j\pi}} = \frac{1}{1 - \frac{1}{2}} = 2$$

Problem 3 (3+2+4+4+4=17)

3a) The transfer function of the filter is

$$H(z) = \frac{-5 + 3z^{-1}}{1 - \frac{5}{6}z^{-1} + \frac{1}{6}z^{-2}}$$

Rewriting to find poles and zeros:

$$H(z) = \frac{-5 + 3z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})} = -5\frac{1 - \frac{3}{5}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})} = -5\frac{z(z - \frac{3}{5})}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

I.e. the filter has zeros in $z_{n_1} = 0$ and $z_{n_2} = \frac{3}{5}$ and poles in $z_{p_1} = \frac{1}{2}$ and $z_{p_2} = -\frac{1}{3}$

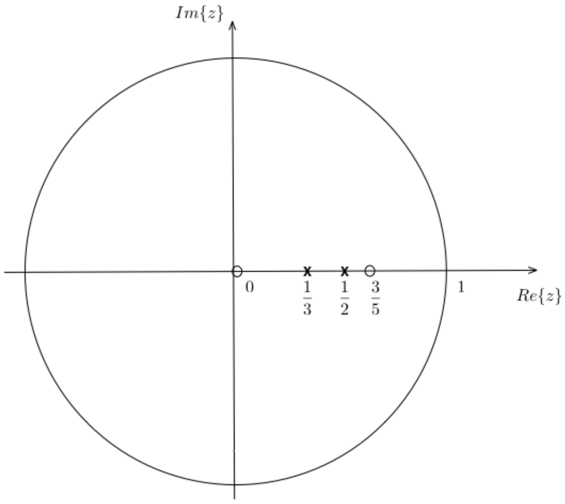
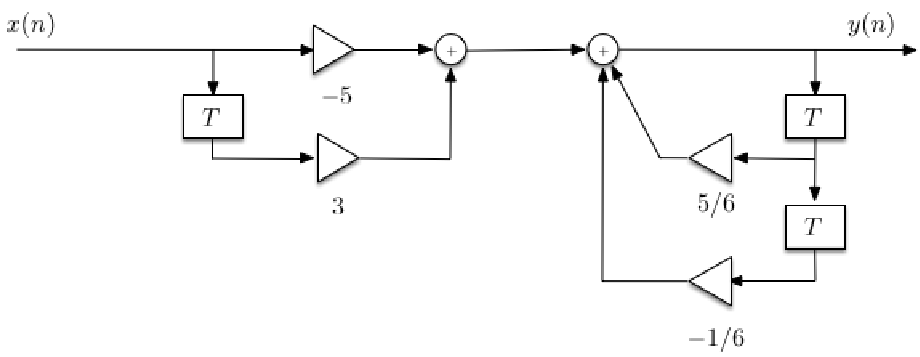


Figure 2: Location of poles and zeros

3b) The DF 1 implementation:



3c) The transfer function of the filter in Figure 3 is:

$$G(z) = \frac{3}{1 - \frac{1}{2}z^{-1}} - \frac{8}{1 - \frac{1}{3}z^{-1}}$$

Organizing the expression we get:

$$G(z) = \frac{3(1-\frac{1}{3}z^{-1}) - 8(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{-5+3z^{-1}}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{-5+3z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = H(z)$$

The unit pulse response:

$$\begin{aligned} h(n) &= \mathcal{Z}^{-1}\left\{\frac{3}{1-\frac{1}{2}z^{-1}}\right\} - \mathcal{Z}^{-1}\left\{\frac{8}{1-\frac{1}{3}z^{-1}}\right\} \\ &= \left(3(\frac{1}{2})^n - 8(\frac{1}{3})^n\right) u(n) \end{aligned}$$

3d) In order to avoid overflow, we must divide the filter input with a factor

$$\textbf{scale} = \max_i \sum_{n=0}^{\infty} |h_i(n)|$$

where $h_i(n)$ is the impulse response from the filter input to the output of summation node i in the filter.

DF2 filter structure In the filter in Figure 2, we have two summation nodes. The rightmost is at the filter output. The impulse response for this node is the same as the impulse response for the entire filter, $h_2(n) = h(n)$. The impulse response for the leftmost summation corresponds to the transfer function of the recursive part of the filter.

$$H_1(z) = \frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1-\frac{1}{3}z^{-1}}$$

We solve for A and B (can also be done by residual calculus) by combining the two fractions to one and comparing the numerators:

$$\frac{A(1-\frac{1}{3}z^{-1}) + B(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{2}z^{-1})(1-\frac{1}{3}z^{-1})} = \frac{A+B-(\frac{1}{3}A+\frac{1}{2}B)z^{-1}}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = \frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}$$

We get two equations:

$$\begin{aligned} A + B &= 1 \\ \frac{1}{3}A + \frac{1}{2}B &= 0 \end{aligned}$$

with the solution $A = 3$ and $B = -2$. The impulse response is then:

$$\begin{aligned} h_1(n) &= \mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}}\right\} = \mathcal{Z}^{-1}\left\{\frac{A}{1-\frac{1}{2}z^{-1}}\right\} + \mathcal{Z}^{-1}\left\{\frac{B}{1-\frac{1}{3}z^{-1}}\right\} \\ &= \mathcal{Z}^{-1}\left\{\frac{3}{1-\frac{1}{2}z^{-1}}\right\} - \mathcal{Z}^{-1}\left\{\frac{2}{1-\frac{1}{3}z^{-1}}\right\} \\ &= (3(\frac{1}{2})^n - 2(\frac{1}{3})^n)u(n) \end{aligned}$$

We can easily see that $h_1(n) > 0$ for $n \geq 0$. Thus, the required scaling for this summation node is

$$\sum_{n=0}^{\infty} |h_1(n)| = 3\sum_{n=0}^{\infty}(\frac{1}{2})^n - 2\sum_{n=0}^{\infty}(\frac{1}{3})^n = 3\frac{1}{1-\frac{1}{2}} - 2\frac{1}{1-\frac{1}{3}} = 3$$

The impulse response for the entire filter was found in problem 3c. For our rightmost summation node we thus have

$$h_2(n) = h(n) = (3(\frac{1}{2})^n - 8(\frac{1}{3})^n)u(n)$$

Inspecting $h(n)$ see that $h(n) < 0$ for $0 \leq n \leq 2$. Consequently,

$$\sum_{n=0}^{\infty} |h(n)| = \sum_{n=0}^{\infty} h(n) - 2\sum_{n=0}^{2} h(n)$$

Then:

$$\sum_{n=0}^{2} h(n) = (3-8) + (\frac{3}{2} - \frac{8}{3}) + (\frac{3}{4} - \frac{8}{9}) = -\frac{227}{38} \approx -6.306$$

And

$$\sum_{n=0}^{\infty} h(n) = \sum_{n=0}^{\infty}(3(\frac{1}{2})^n - 8(\frac{1}{3})^n) = 3\frac{1}{1-\frac{1}{2}} - 8\frac{1}{1-\frac{1}{3}} = 6 - 8\frac{3}{2} = -6$$

So, finally,

$$\sum_{n=0}^{\infty} |h(n)| = -6 + 2 \cdot \frac{227}{38} \approx 6.61$$

We need to scale by the largest factor, so the input of the DF2 filter must be downscaled by a factor of 6.61.

Parallel filter structure We have three summation points. The rightmost has the same impulse response as the total filter. We examine the two branch points first:

$$h_1(n) = \mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{1}{2}}\right\} = (\frac{1}{2})^n u(n)$$

which is positive for all $n \geq 0$. Consequently:

$$\sum_{n=0}^{\infty} |h_1(n)| = \sum_{n=0}^{\infty}(\frac{1}{2})^n = \frac{1}{1-\frac{1}{2}} = 2$$

The second branch:

$$h_2(n) = \mathcal{Z}^{-1}\left\{\frac{1}{1-\frac{1}{3}}\right\} = (\frac{1}{3})^n u(n)$$

which is positive for all $n \geq 0$. Consequently:

$$\sum_{n=0}^{\infty} |h_2(n)| = \sum_{n=0}^{\infty}(\frac{1}{3})^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$$

For the rightmost summation node, we have already calculated the scaling:

$$\sum_{n=0}^{\infty} |h_3(n)| = \sum_{n=0}^{\infty} |h(n)| = 6.61$$

Thus, the input to the parallel structure will also need to be downscaled with a factor of 6.61

3e) The power at the output of a zero mean white noise source with variance σ_q^2 is generally

$$\sigma_o^2 = \sigma_q^2 \sum_{n=0}^{\infty} h_i^2(n)$$

where $h_i(n)$ is the impulse response from the point of noise injection to the filter output.

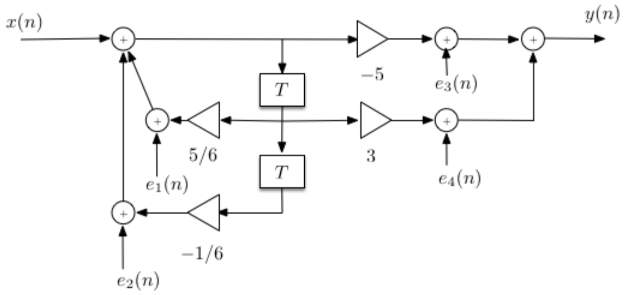


Figure 3: Model of roundoff noise for DF2 filter

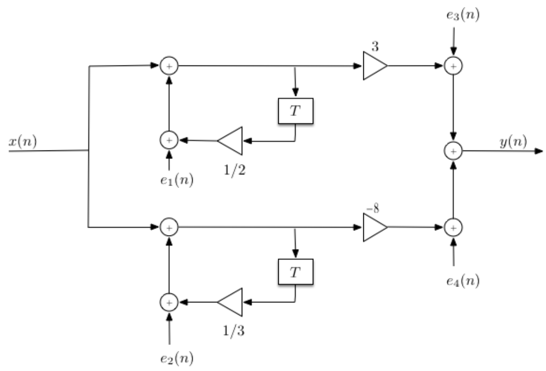


Figure 4: Model of roundoff noise for parallel structure

DF2 structure We have 4 multipliers, corresponding to 4 noise sources. The two rightmost are fed directly to the output, the two leftmost will both be convolved with the total impulse response of the filter. We thus need to calculate the sum of squares of the impulse response.

$$\begin{aligned}
\sum_{n=0}^{\infty} h^2(n) &= \sum_{n=0}^{\infty} (3(\frac{1}{2})^n - 8(\frac{1}{3})^n)^2 \\
&= 9\sum_{n=0}^{\infty}(\frac{1}{2})^{2n} - 48\sum_{n=0}^{\infty}(\frac{1}{2}\cdot\frac{1}{3})^n + 64\sum_{n=0}^{\infty}(\frac{1}{3})^{2n} \\
&= 9\sum_{n=0}^{\infty}(\frac{1}{4})^n - 48\sum_{n=0}^{\infty}(\frac{1}{6})^n + 64\sum_{n=0}^{\infty}(\frac{1}{9})^n \\
&= 9\frac{1}{1-\frac{1}{4}} - 48\frac{1}{1-\frac{1}{6}} + 64\frac{1}{1-\frac{1}{9}} \\
&= \frac{132}{5}
\end{aligned}$$

The total noise power at the filter output is then:

$$\sigma_e^2 = \sigma_q^2\left(2\sum_{n=0}^{\infty} h^2(n) + 2\right) = (2\frac{132}{5} + 2)\sigma_q^2 = \frac{272}{5}\sigma_q^2 = 54.8\sigma_q^2$$

Parallel structure The upper branch:

$$H_1(z) = \frac{3}{1 - \frac{1}{2}z^{-1}} \Rightarrow h_1(n) = 3(\frac{1}{2})^n u(n)$$

$$\sum_{n=0}^{\infty} |h_1(n)|^2 = 9 \sum_{n=0}^{\infty} (\frac{1}{2})^{2n} = \frac{9}{1 - \frac{1}{4}} = 12$$

The lower branch

$$H_2(z) = \frac{-8}{1 - \frac{1}{3} z^{-1}} \Rightarrow h_2(n) = -8(\frac{1}{3})^n u(n)$$

$$\sum_{n=0}^{\infty} |h_2(n)|^2 = 64 \sum_{n=0}^{\infty} (\frac{1}{3})^2 n = \frac{64}{1 - \frac{1}{9}} = 72$$

The total noise power at the filter output is then:

$$\sigma_e^2 = \sigma_q^2 \left(\sum_{n=0}^{\infty} h_1^2(n) + \sum_{n=0}^{\infty} h_2^2(n) + 2 \right) = 86 \sigma_q^2$$

Problem 4 (2+4+2+2=10))

4a) Normalized frequency of the sinusoid is $8/48 = 1/6$. The desired unit pulse response is thus

$$h(n) = \cos(2\pi \frac{1}{6} n) = \cos(\frac{\pi}{3} n) = \frac{1}{2}\left(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right)$$

4b)

$$\begin{aligned}
H(z) &= \mathcal{Z}^{-1}\{h(n)\} = \frac{1}{2}\sum_{n=0}^{\infty}\left[e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right] z^{-n} \\
&= \frac{1}{2}\left[\frac{1}{1-e^{j\frac{\pi}{3}}z^{-1}} + \frac{1}{1-e^{-j\frac{\pi}{3}}z^{-1}}\right] = \frac{1}{2}\frac{1-e^{j\frac{\pi}{3}}z^{-1}+1-e^{-j\frac{\pi}{3}}z^{-1}}{(1-e^{j\frac{\pi}{3}}z^{-1})(1-e^{-j\frac{\pi}{3}}z^{-1})} \\
&= \frac{1}{2}\frac{2-(e^{j\frac{\pi}{3}}+e^{-j\frac{\pi}{3}})z^{-1}}{1-(e^{j\frac{\pi}{3}}+e^{-j\frac{\pi}{3}})z^{-1}+z^{-2}} = \frac{1}{2}\frac{2-2\cos(\frac{\pi}{3})z^{-1}}{1-2\cos(\frac{\pi}{3})z^{-1}+z^{-2}} \\
&= \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+z^{-2}}
\end{aligned}$$

Finding the poles and zeros:

$$H(z) = \frac{1-\frac{1}{2}z^{-1}}{1-z^{-1}+z^{-2}} = \frac{z(z-\frac{1}{2})}{z^2-z+1} = \frac{z(z-\frac{1}{2})}{(z-e^{j\frac{\pi}{3}})(z-e^{-j\frac{\pi}{3}})}$$

I.e. we have zeros in $z_{n_1} = 0$ and $z_{n_2} = \frac{1}{2}$ and poles in $z_{p_1} = e^{j\frac{\pi}{3}}$ and $z_{p_2} = e^{-j\frac{\pi}{3}}$

4c) The difference equation is

$$y(n) = x(n) - \frac{1}{2}x(n-1) + y(n-1) - y(n-2)$$

4d) The impulse response of the filter is the desired sinusoid. Procedure:

1. Initialize and set $y(0) = 1$, and $y(1) = -\frac{1}{2} + 1 = \frac{1}{2}$
2. Run, using $y(n) = y(n-1) - y(n-2)$