NTNU – Trondheim
Norwegian University of
Science and Technology

Department of Electronics and Telecommunications

Examination paper for TTT4120 Digital signal processing

Solutions sketch

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Examination date: Tuesday August 5, 2014

Examination time (from - to): 09.00 - 13.00
Permitted examination support material: **D** – No printed or handwritten material allowed. Specified, simpe calculator allowed

Other information:
- The examination consists of 4 problems where
  - problem 1 concerns fundamental properties of structures/filters
  - problem 2 concerns digital filter design
  - problem 3 concerns stochastic processes
  - problem 4 concerns digital filter realization
- Weighting of each sub-problem is given in parenthesis at the start of each problem.
- All problems are to be answered
- Grades will be announced 3 weeks after the examination date.

Language: English

Total number of pages: 10

Of this, number of enclosure pages: 1

Checked by:

Date Signature

Problem 1 (3+3+5+5+3+3=22)

1a)

$$\mathcal{Z}\{x(n+k)\} = \sum_{n=-\infty}^{\infty} x(n+k)z^{-n} = \sum_{m=-\infty}^{\infty} x(m)z^{-(m-k)} = z^k \sum_{m=-\infty}^{\infty} x(m)z^{-m} = z^k X(z),$$

where we used the substitution $m = n + k$, and $X(z) = \mathcal{Z}\{x(n)\}$.

1b) We find the frequency response by evaluating the transfer function $H(z)$ along the unit circle. At $\omega = 0$ and $\omega = \pi$ the frequency response is zero due to the two zeroes on the unit circle. The amplitude response has its maximum at $\omega = \pi/2$ (and at $\omega = 3\pi/2$,) due to the pole close to the unit circle at this frequency. Thus, this is a bandpass filter.

1c) **The figure and the transfer function in the problem text do not match. For the complex conjugate poles in the figure, the denominator of the transfer function should be $1 + a^s?2z^{-2}$ Showing the transfer function to be as in the problem is thus not possible. Two solutions will be given here, one assuming the transfer function as given in the problem text, the second for poles as shown in the problem figure.**

Solution based on text:

The transfer function is

$$H(z) = \frac{(1 - z^{-1})(1 + z^{-1})}{(1 - az^{-1})(1 + az^{-1})} = \frac{1 - z^{-2}}{1 - a^2 z^{-2}}$$

The order of the numerator and denominator of $H(z)$ is equal, i.e. that direct partial fraction expansion cannot be done. We take a simpler route, by observing that the transfer function can be written as:

$$H(z) = \frac{1}{2}\left[\frac{1 + Az^{-1}}{1 - az^{-1}} + \frac{1 + Bz^{-1}}{1 + az^{-1}}\right] = \frac{1 - z^{-2}}{1 - a^2 z^{-2}}$$

We combine the terms in brackets:

$$\begin{aligned} H(z) &= \frac{1}{2}\left[\frac{1 + Az^{-1} + az^{-1} + aAz^{-2} + a + Bz^{-1} - az^{-1} - aBz^{-2}}{1 + az^{-1}}\right] \\ &= \frac{1}{2}\left[\frac{2 + (A + B)z^{-1} + a(A - B)z^{-2}}{1 + az^{-1}}\right] \end{aligned}$$

We easily see that in order to obtain equality, $A + B = 0$ and $a(A - B) = -2$, thus

$$H(z) = \frac{1}{2}\left[\frac{1 - \frac{1}{a}z^{-1}}{1 - az^{-1}} + \frac{1 + \frac{1}{a}z^{-1}}{1 + az^{-1}}\right]$$

i.e. that we can write the system equation as:

$$H(z) = \frac{1}{2}\left[\frac{1}{1 - az^{-1}} - \frac{\frac{1}{a}z^{-1}}{1 - az^{-1}} + \frac{1}{1 + az^{-1}} + \frac{\frac{1}{a}z^{-1}}{1 + az^{-1}}\right]$$

The unit pulse response can then be found to be

$$\frac{1}{2}\left[a^n + (-a)^n\right] u(n) + \frac{1}{2a}\left[(-a)^{(n-1)} - a^{(n-1)}\right] u(n-1)$$

or

$$h(n) = \begin{cases} \frac{1}{2}a^n - \frac{1}{2a}a^{n-1} + \frac{1}{2}(-a)^n + \frac{1}{2a}\,(-a)^{n-1} & n > 0 \\ 1 & n = 0 \\ 0 & n < 0 \end{cases}$$

Solution based on figure:

The transfer function is

$$H(z) = \frac{(1-z^{-1})(1+z^{-1})}{(1-jaz^{-1})(1+jaz^{-1})} = \frac{1-z^{-2}}{1+a^2z^{-2}}$$

Following the derivations above, we do the partial fraction expansion, finding $A = -\frac{1}{ja}$ and $B = \frac{1}{ja}$, and

$$H(z) = \frac{1}{2}\left[\frac{1-\frac{1}{ja}z^{-1}}{1-jaz^{-1}} + \frac{1+\frac{1}{ja}z^{-1}}{1+jaz^{-1}}\right]$$

The unit pulse response in this case is then:

$$\frac{1}{2}\left[(ja)^n + (-ja)^n\right] u(n) + \frac{1}{2ja}\left[(-ja)^{(n-1)} - (ja)^{(n-1)}\right] u(n-1)$$

or

$$h(n) = \begin{cases} \frac{1}{2}(ja)^n - \frac{1}{2ja}(ja)^{n-1} + \frac{1}{2}(-ja)^n + \frac{1}{2ja}\,(-ja)^{n-1} & n > 0 \\ 1 & n = 0 \\ 0 & n < 0 \end{cases}$$

1d) The ROC for a causal LTI system is defined as the area for where $|z| > \max\{|p_k|\}$ where p_k are the poles of the system. In our case this means that $ROC = |z| > |a|$.
The filter is stable when all poles are inside the unit circle, i.e. when $|a| < 1$.
The filter will have minimum phase when all poles and zeros are inside the unit circle. Since the zeros are on the unit circle, the filter is not minimum phase.

1e) We have a causal filter given by

$$H(z) = \frac{1-\frac{5}{4}z^{-1}}{(1+\frac{1}{2}z^{-1})(1+\frac{3}{4}z^{-1})} = \frac{z(z-\frac{5}{4})}{(z+\frac{1}{2})(z+\frac{3}{4})}.$$

The filter has zeros when the numerator is zero. In this case we have two zeros

$$n_1 = \frac{5}{4},\ n_2 = 0.$$

The poles of the filter are the roots of the denominator polynomial, i.e.

$$p_1 = -\frac{1}{2},\ p_2 = -\frac{3}{4}.$$

The transfer function of the inverse filter is given by

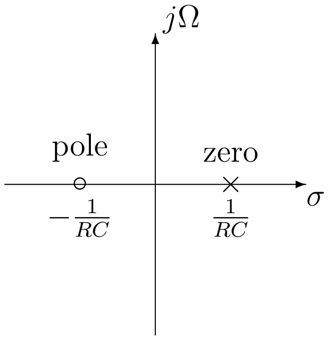
$$H_I(z) = \frac{1}{H(z)}$$

The poles of the inverse filter are thus equal to the zeros of $H(z)$. Since $H(z)$ has one zero outside the unit circle, the inverse filter will have a pole outside the unit circle. Therefore, the inverse filer is not causal and stable.

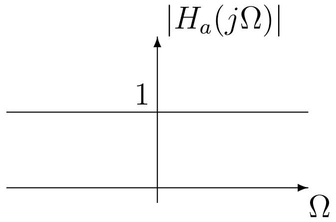
1f) The filter has zeros for $z = 5/4$ and the poles are on the negative real axis, i.e. most prominent for $\omega = \pi$. The filter is thus a high-pass filter.

Problem 2 (3+4+7+5+2=21)

2a) The poles and zeros in the s-plane show that they cancel out, and the result is an allpass filter.



Allpass filter



2b)

$$\frac{T}{2}j\Omega = \frac{1-e^{-j\omega}}{1+e^{-j\omega}} = \frac{e^{-j\omega/2}(e^{j\omega/2}-e^{-j\omega/2})}{e^{-j\omega/2}(e^{j\omega/2}+e^{-j\omega/2})}$$

$$\frac{T}{2}j\Omega = \frac{2j[e^{j\omega/2}-e^{-j\omega/2}]/2j}{2[e^{j\omega/2}+e^{-j\omega/2}]/2}$$

$$\frac{T}{2}j\Omega = j\tan\frac{\omega}{2} \Longrightarrow \Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

2c)

$$H_a(s) = \frac{s-\frac{1}{RC}}{s+\frac{1}{RC}} = \frac{s-\Omega_c}{s+\Omega_c} = \frac{s/\Omega_c - 1}{s/\Omega_c + 1}$$

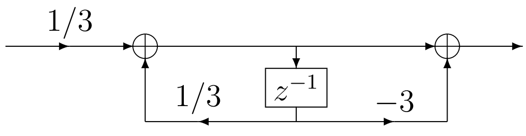
$$\Omega_c = \frac{2}{T}\tan\frac{\omega_c}{2} = \frac{1}{T} \Longrightarrow T = \frac{1}{\Omega_c} = RC$$

$$\Longrightarrow s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} = 2\Omega_c\frac{1-z^{-1}}{1+z^{-1}}$$

$$\Longrightarrow \frac{s}{\Omega_c} = 2\frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = \frac{\frac{2(1-z^{-1})}{1+z^{-1}} - 1}{\frac{2(1-z^{-1})}{1+z^{-1}} + 1} = \frac{2-2z^{-1}-1-z^{-1}}{2-2z^{-1}+1+z^{-1}}$$

$$= \frac{1-3z^{-1}}{3-z^{-1}} = \frac{1}{3}\cdot\frac{1-3z^{-1}}{1-\frac{1}{3}z^{-1}}$$



2d)

$$H(z) = \frac{1}{3} \cdot \frac{1 - 3z^{-1}}{1 - 1/3z^{-1}}$$

$$G(z) = \frac{1}{1 - 1/3z^{-1}} \longrightarrow g(n) = \begin{cases} (1/3)^n, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$H(z) = \frac{1}{3}G(z) - z^{-1}G(z)$$

$$\Longrightarrow h(n) = \frac{1}{3}g(n) - g(n-1)$$

$$h(0) = \frac{1}{3}g(0) = \frac{1}{3}$$

$$h(n) = \begin{cases} \frac{1}{3}(1/3)^n - (1/3)^{n-1}, & n \geq 1 \\ 1/3, & n = 0 \\ 0, & n < 0 \end{cases}$$

hvor $\frac{1}{3}(\frac{1}{3})^n - (\frac{1}{3})^{n-1} = [(\frac{1}{3})^2 - 1](\frac{1}{3})^{n-1} = -\frac{8}{9}(\frac{1}{3})^{n-1}$.

2e) The transfer function can easily be split up into:

$$H(z) = \frac{1}{3}\left(\frac{1}{1 - \frac{1}{3}z^{-1}} - \frac{3z^{-1}}{1 - \frac{1}{3}z^{-1}}\right) = \frac{\frac{1}{3}}{1 - \frac{1}{3}z^{-1}} + \frac{-z^{-1}}{1 - \frac{1}{3}z^{-1}}$$

Problem 3 (4+2+5+3+4=17)

3a) The output of $H_1(z)$ is an AR(1)-process, the output of $H_2(z)$ is an ARMA(1,1) process, and the output of $H(z)$ is an ARMA(1,1) process.

The output of $H_1(z)$ is $y_1(n) = \frac{1}{3}x(n) + \frac{1}{3}y_1(n-1)$ where $x(n)$ is a white noise process with variance σ_x^2. The autocorrelation function is

$$\gamma_{y_1}(k) = E\{y_1(n)y_1(n-k)\} = E\{\left(\frac{1}{3}x(n) + \frac{1}{3}y_1(n-1)\right) y_1(n-k)\}$$
$$= \frac{1}{3}\delta(k)\sigma_x^2 + \frac{1}{9}\gamma_{y_1}(|k|-1)$$

The acf can also be calculated analog to the solution in 3e. The result is then

$$\gamma_{y_1}(0) = \frac{\frac{1}{9}\sigma_x^2}{1-\frac{1}{9}} = \frac{\sigma_x^2}{8}$$
$$\gamma_{y_1}(k) = \frac{1}{9}\gamma_{y_1}(k) = (\frac{1}{3})^k\gamma_{y_1}(0) = (\frac{1}{3})^k\frac{\frac{1}{9}\sigma_x^2}{1-\frac{1}{9}}$$
$$= (\frac{1}{3})^k\frac{\sigma_x^2}{8}$$

3b) The sampling rate of 8000Hz is equivalent to a sampling interval of $1/8000 = 125\mu$s. The delay of the reflected signal is $(l_1 + l_2)/340$s $= (0.04 + 0.045)/340 = 0.000250$s $= 250\mu$s, which is equivalent to two sampling intervals. The reflected signal is thus $\alpha x(t-2T)$ and the sampled version is $\alpha x(n-2)$ so that $z(n) = x(n) + \alpha x(n-2)$

3c)

$$r_{zz}(k) = E\{(x(n) + \alpha x(n-2))(x(n-k) + \alpha x(n-2-k))\}$$
$$= E\{x(n)x(n-k)\} + \alpha E\{x(n)x(n-2-k)\} + \alpha E\{x(n-2)x(n-k)\} + \alpha^2 E\{x(n-2)x(n-2-k)$$
$$= R_{xx}(k) + \alpha R_{xx}(k+2) + \alpha R_{xx}(k-2) + \alpha^2 R_{xx}(k)$$
$$= (1+\alpha^2)R_{xx}(k) + \alpha R_{xx}(k+2) + \alpha R_{xx}(k-2)$$

3d) When $x(n)$ is white noise, $R_{xx}(k) = \delta(k)\sigma_x^2$. The acf of the sampled signal is then

$$r_{zz}(k) = \left[(1+\alpha^2)\delta(k) + \alpha\delta(k+2) + \alpha\delta(k-2\right]\sigma_x^2$$

I.e. the acf has non-zero values only for $k = -2$, $k = 0$ and $k = 2$

3e) $x(n)$ is AR(1), i.e. $x(n) = ax(n-1) + w(n)$. The autocorrelation function:

$$r_{zz}(0) = E\{(x(n)x(n)\} = E\{(ax(n-1) + w(n))(ax(n-1) + w(n))\}$$
$$= a^2 r_{zz}(0) + \sigma_w^2 \Longleftrightarrow$$
$$r_{zz}(0) = \frac{\sigma_w^2}{1-a^2}$$

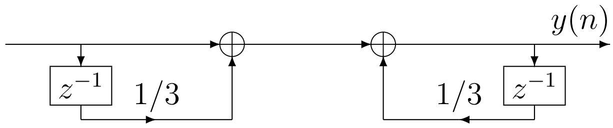
For $k \neq 0$:

$$r_{zz}(k) = E\{(x(n)x(n-k)\} = E\{(ax(n-1) + w(n))x(n-k)\}$$
$$= ar_{xx}(k-1) = a^k r_{zz}(0)$$
$$= a^k\frac{\sigma_w^2}{1-a^2}$$

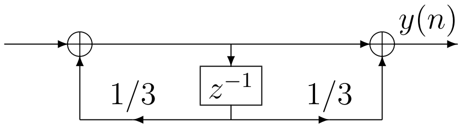
since $w(n)$ is uncorrelated with $x(n-k)$.

Problem 4 (4+5+6+6=21)

4a)

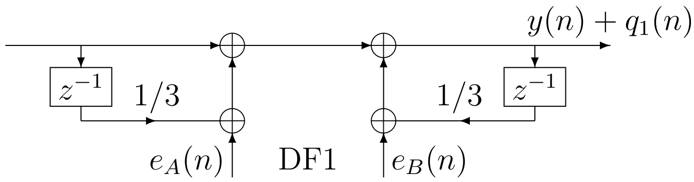


DF1



DF2

4b)



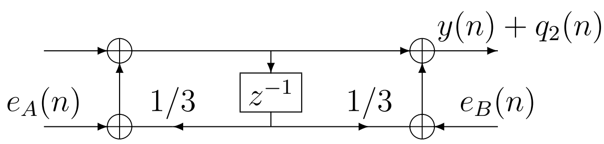
DF1

$$q_1(n) = \big(e_A(n) + e_B(n)\big) * g(n)$$

$$g(n) = \begin{cases} 0, & n < 0 \\ (1/3)^n, & n \geq 0 \end{cases}$$

$$r_{gg}(0) = \sum_{n=0}^{\infty} g^2(n) = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^{2n} = \frac{1}{1-\frac{1}{9}} = \frac{9}{8}$$

$$\sigma_{q_1}^2 = 2\sigma_e^2 r_{gg}(0) = \underline{\underline{\frac{9}{4}\sigma_e^2}}$$



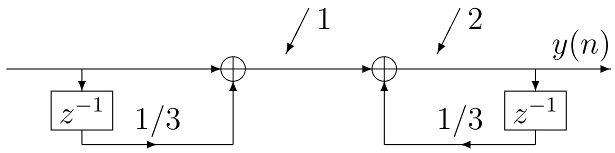
DF2

$$r_{hh}(0) = \sum_{n=0}^{\infty} h^2(n) = 1 + \sum_{n=1}^{\infty} 4\left[(\frac{1}{3})^2\right]^n = 1 + 4\frac{(\frac{1}{3})^2}{1-(\frac{1}{3})^2} = \frac{3}{2}$$

$$q_2(n) = e_B(n) + e_A(n) * h(n) \longrightarrow$$

$$\sigma_{q_2}^2 = \sigma_e^2 + \sigma_e^2 r_{hh}(0) = \sigma_e^2\left(1 + \frac{3}{2}\right) = \underline{\underline{\frac{5}{2}\sigma_e^2}}$$

4c)



DF1

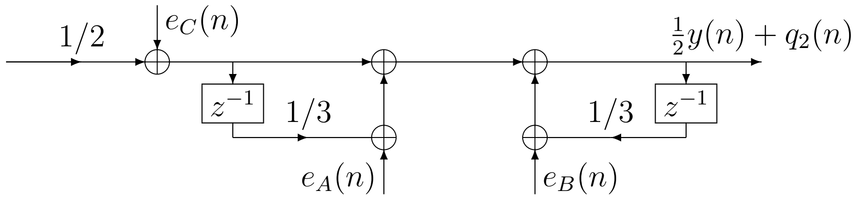
1 and 2 are candidate locations for overload.

$$h_1(n) = \begin{cases} 1, & n=0 \\ 1/3, & n=1 \\ 0, & \text{ellers} \end{cases} \implies \sum_n |h_1(n)| = \frac{4}{3}$$

$$h_2(n) = h(n) \implies$$

$$\sum_{n=0}^{\infty} |h(n)| = 1 + 2\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = 1 + \frac{2}{3}\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)^k = 1 + \frac{2/3}{1-1/3} = \underline{\underline{2}} > 4/3 \implies$$

downscale with 2.



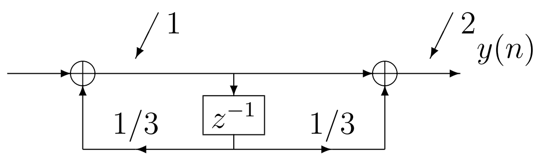
$$q_2(n) = e_C(n) * h(n) + \big(e_A(n) + e_B(n)\big) * g(n)$$

$$\sigma_{q_2}^2 = \sigma_e^2 \cdot \frac{3}{2} + \sigma_e^2 \cdot \frac{9}{4} = \frac{15}{4}\sigma_e^2$$

Signal power is reduced by $2^2 = 4$.

$\implies$ Total relative reduction in S/N: $\frac{SNR_{scaled}}{SNR_{unscaled}} = \frac{\left[\frac{\sigma_x^2}{\sigma_{q2}^2}\right]}{\left[\frac{\sigma_x^2}{\sigma_{q1}^2}\right]} = \frac{\sigma_{q1}^2}{4\sigma_{q2}} = \frac{\frac{9}{4}\sigma_e^2}{\frac{15}{4}\cdot 4\sigma_e^2} = \frac{3}{20} \approx 0.15.$

4d)

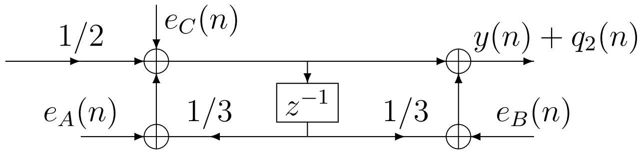


DF2

$$h_1(n) = g(n) \longrightarrow \sum_{n=0}^{\infty}|h_1(n)| = \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{1-1/3} = \frac{3}{2}$$

$$h_2(n) = h(n) \longrightarrow \sum |h_2(n)| = 2 > 3/2$$

$\implies$ also here downscale by 2.



$$q_2(n) = \big(e_B(n) + e_C(n)\big) * h(n) + e_B(n)$$

$$\sigma_q^2 = 2\sigma_e^2 \cdot \frac{3}{2} + \sigma_e^2 = 4\sigma_e^2$$

$$\Longrightarrow S/N \text{ reduction: } \frac{5/2}{4 \cdot 4} = \frac{5}{32} \approx \underline{0.156}$$