NTNU
Norwegian University of Science and Technology

Department of Electronic Systems

# Examination paper for TTT4120 Digital Signal Processing

**Academic contact during examination:** Stefan Werner

**Phone:** +358401424948

**Examination date:** Tuesday, December 12, 2017

**Examination time (from-to):** 15:00-19:00

**Permitted examination support material:**

D - Basic calculator allowed

No printed or handwritten materials allowed

**Other information:**

- Exam consists of four (4) problems
- A few basic formulas are provided in the Appendix

**Language:** English.

**Number of pages (front page excluded):** 6

**Number of pages enclosed:** 2

**Informasjon om trykking av eksamensoppgave**

**Originalen er:**

**1-sidig** ☐ **2-sidig** ☐

**sort/hvit** ☐ **farger** ☐

**skal ha flervalgskjema** ☐

**Checked by:**

Date Signature

**Problem 1 (3+4+5+5+3=20): Basics of filter theory and design**

A causal filter is given by the following difference equation

$$y[n] = ay[n-1] - ax[n] + x[n-1]$$

where $a$ is a finite real-valued constant.

**1a)** Provide the system function corresponding to $y[n]$ in the form

$$H_1(z) = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}}$$

and justify whether the filter is of type FIR or IIR

**1b)** Draw the pole-zero plot and sketch the region of convergence (ROC). For which values of the coefficient $a$ is the filter stable? Can the filter have minimum phase?

**1c)** Set $a = \frac{1}{2}$ and show that $H_1(z)$ is an allpass filter, i.e., $|H_1(f)|^2 = 1$.

**1d)** Given a stable causal filter of the form $H(z) = H_1(z) \cdot H_2(z) = H_3(z) + H_4(z)$, where $H_1(z)$ is given above, with $a = \frac{1}{2}$, and $H_2(z)$ is given as

$$H_2(z) = \frac{1}{1 + \frac{1}{2} z^{-1}}$$

Show that the unit impulse response $h[n] = \mathcal{Z}^{-1}\{H(z)\}$ can be expressed as

$$h[n] = \frac{3}{4}\left(\frac{1}{2}\right)^n u[n] - \frac{5}{4}\left(-\frac{1}{2}\right)^n u[n]$$

**1e)** Define the following properties of a system $h[n]$ as a function of the input signal $x[n]$ and the output signal $y[n]$:

- Stability
- Causality
- Time invariance

**Problem 2 (6+6+6=18): Filter structures and implementations**

The filter in **Problem 1d)** is implemented using fixed-point representation with $B+1$ bits and dynamic range $[-1,1)$. Rounding is performed after each multiplication and the rounding error $e[n]$ can be modeled as white noise with variance $\sigma_e^2$. Consequently, each multiplier in the fixed-point implementation is modeled as

$$Q(ay[n-k]) = ay[n-k] + e[n]$$

which is equivalent to adding noise sources after multipliers in the infinite-precision realization. Rounding noise sources combine into an equivalent noise signal $z[n]$ at the filter output with variance $\sigma_z^2$ (see hint below).

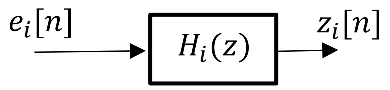
**2a)** Draw the cascade-structure, $H(z) = H_1(z)H_2(z)$, with noise sources due to rounding included. Determine the variance of the round-off noise at the filter output. Filter $H_1(z)$ is implemented in direct-form structure II (DF-II). Remember that $H_1(z)$ is allpass and use hint below.

**2b)** Draw the parallel-structure, $H(z) = H_3(z) + H_4(z)$, with noise sources due to rounding included. Determine the variance of the round-off noise at the filter output. You may put the multipliers, obtained from the residue calculus in Problem 1d, at the output of the structure.

**2c)** Find the necessary scaling factor at the input of the parallel structure in 2b) so that overflow is avoided. You may put the multipliers, obtained from the residue calculus in Problem 1d, wherever you may find suitable to simplify the problem (e.g., at the input of the filters).

**[Hint:]** Assuming noise source $e_i[n]$ with variance $\sigma_{ei}^2$ acts as input to (sub-)filter $h_i[n]$ that terminates at the output. The variance of the noise signal $z_i[n]$, due to $e_i[n]$, is given by

$$\sigma_{zi}^2 = \sigma_{ei}^2 r_{h_i h_i}[0] = \sigma_{ei}^2 \sum_k h_i^2[k] = \sigma_{ei}^2 \int_0^1 |H_i(f)|^2 df$$




$\sigma_{ei}^2 = E\{e_i^2[n]\}$ $\qquad$ $\sigma_{zi}^2 = E\{z_i^2[n]\}$

**Problem 3 (6+6+6=18): Parametric modeling and Wiener filtering**

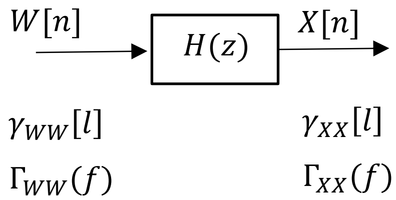



Fig. 1: Filtering of stochastic processes

A wide-sense stationary (WSS) stochastic process $X[n]$ is generated by filtering a white noise process $W[n]$, with autocorrelation sequence $\gamma_{WW}[l] = \sigma_W^2 \delta[l]$, through the causal and stable filter $H(z) = H_1(z)H_2(z)$ from Problem 1d), as depicted in Fig. 1. The autocorrelation sequence and spectrum of $X[n]$ is obtained from

$$\gamma_{XX}[l] = \begin{cases} \sigma_W^2 \sum_{n=0}^{\infty} h[n]h[n+l], & l \geq 0 \\ \gamma_{XX}[-l], & l < 0 \end{cases}$$

$$\Gamma_{XX}(f) = |H(f)|^2 \Gamma_{WW}(f)$$

**3a)** Provide answers (with motivations) to the following two questions:

- <u>What type of process</u>, AR($p$), MA($q$), or ARMA($p, q$), can $X[n]$ be modeled as when the noise is filtered by $H(z)$ in **Problem 1d** for the case of $\boldsymbol{a} = \frac{\mathbf{1}}{\mathbf{2}}$? Provide the <u>model order</u> and the <u>spectrum</u> of $X[n]$.

  **Hint:** Remember that $H_1(z)$ is an allpass filter and make use of the above formula for $\Gamma_{XX}(f)$.

**3b)** You are to model the output signal $X[n]$ above as an AR($p$) process using linear prediction. <u>Find the best AR(1) model and the corresponding prediction error power $\sigma_f^2$</u>. You can use the following result for the autocorrelation sequence of $X[n]$

$$\gamma_{XX}[l] = \sigma_W^2 \frac{(-0.5)^{|l|}}{1 - 0.5^2}$$

with $\sigma_W^2 = 1$. Can you also say <u>which model order that is optimal</u> for the problem at hand?

**3c)** Explain the general principle behind Wiener filtering, preferably by using a block diagram. You may consider the case of signal estimation (noise reduction) wherein the input signal to the filter is $X[n] = S[n] + W[n]$, with $S[n]$ being the signal of interest and $W[n]$ is uncorrelated white noise with power $\sigma_W^2$.

**Problem 4 (2+6+6+4=18): Sampling, rate-conversion and DFT**

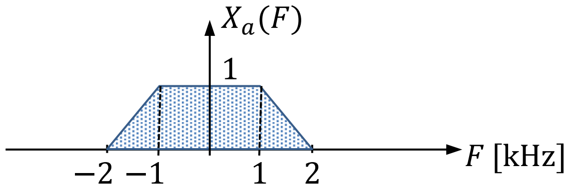



Fig. 2: Spectrum $X_a(F)$ of continuous-time signal $x_a(t)$

Let $x_a(t)$ be a bandlimited continuous-time signal whose spectrum $X_a(F)$ is shown in Fig. 2. Signal $x_a(t)$ is sampled at every $T_x = 0.125$ ms to generate sequence $x[n] = x_a(t)|_{t\ =nT_x}$.

**4a)** Sketch spectrum of the discrete-time sequence $x[n]$, i.e., $X(f) = \sum_{n=-\infty}^{\infty} x[n]e^{-j2\pi fn}$ with $f = FT_x$ being the normalized frequency.

**4b)** You have stored the sampled sequence $x[n]$ in a file on your computer, but would also like to send a copy of it to your email account. However, the file is too large so you decide to create two new sequences $y_1[m] = x[2m]$ and $y_2[m] = x[4m]$. Sketch the spectra $Y_1(f)$ and $Y_2(f)$ and justify whether one or both can serve as a replacement for the original sequence $x[n]$? Which sequence will you send as backup to your email account?

**4c)** You face the awful, but not uncommon, reality that your computer crashes and your precious sequence $x[n]$ from Problem 4a is lost forever. Luckily you have a copy of sequence $y_1[m] = x[2m]$ in your email. Is it possible to restore the original sequence $x[n]$ from $y_1[m]$? Explain how to solve this interpolation problem using upsampling and filtering.

**4d)** Finally, $M = 1090$ samples of $x[n]$ are passed through a causal lowpass FIR filter, $h_{\mathrm{LP}}[n]$, having $L = 52$ coefficients, to obtain output sequence $y[n]$. What are the minimum sizes required of the discrete Fourier transforms (DFTs) to uniquely represent $x[n]$, $h[n]$, and $y[n]$ in frequency domain (i.e., $N_x$, $N_h$, and $N_y$)? Also provide the most suitable length for computing the radix-2 FFT of $y[n]$.

**Appendix: TTT4120 Table of formulas, 2017**

**A. Sequences:**

$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ and $-\sum_{n=-1}^{-\infty} \alpha^n = \frac{1}{1-\alpha}$

$\sum_{n=0}^{N-1}(n+1)\alpha^n = \frac{1-\alpha^N}{(1-\alpha)^2} - \frac{N\alpha^N}{1-\alpha}$ ; $\alpha \neq 1$

$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty}(n+1)\alpha^n = \frac{1}{(1-\alpha)^2}$

**B. Linear convolution:**

$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$Y(z) = H(z)X(z)$

$Y(f) = H(f)X(f)$

$Y(k) = H(k)X(k)$, $k = 0,1,\dots,N-1$ where $Y(k) = Y(f_k)$ with $f_k = k/N$

**C. Transforms:**

Z-transform: $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$

DTFT: $H(f) = \sum_{n=-\infty}^{\infty} h[n]e^{-j2\pi fn}$

DFT: $H(k) = \sum_{n=0}^{N-1} h[n]e^{-j2\pi fnk/N}$ $k = 0,1,\dots,N-1$

IDFT: $h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi fnk/N}$ $n = 0,1,\dots,N-1$

**D. Sampling theorem:**

Given an analog signal $x_a(t)$ sampled at $F_s = 1/T$. The DTFT of the resulting discrete-time sequence $x[n] = x_a(t)|_{t=nT}$ is given by

$$X(f) = X(F/Fs) = F_s \sum_{k=-\infty}^{\infty} X([f-k]F_s)$$

**E. Autocorrelation, energy spectrum and Parseval:**

Given a sequence $h[n]$ with finite energy $E_h$

Autocorrelation: $r_{hh}[l] = \sum_{n=-\infty}^{\infty} h[n]h[n+l]$ $l \in \mathbb{Z}$

Energy spectrum: $S_{hh}(z) = H(z)H(z^{-1}) \Rightarrow S_{hh}(f) = |H(f)|^2$

Parseval's theorem: $E_h = r_{hh}[0] = \sum_{n=-\infty}^{\infty} h^2[n] = \int_0^1 |H(f)|^2 df$

**F. Multirate:**

Decimation (downsampling) where $T_y = DT_x$

$v(mT_y) = \sum_{k=-\infty}^{\infty} h[(mD-k)T_x]x(kT_x) \quad m \in \mathbb{Z}$

Interpolation (upsampling) where $T_y = T_x/I$

$y(lT_y) = \sum_{n=-\infty}^{\infty} h\left[(l-nI)T_y\right]x(nT_x) \quad l \in \mathbb{Z}$

Rate conversion where $T_y = DT_v = \frac{D}{I}T_x$

$y(lT_y) = \sum_{m=-\infty}^{\infty} h[(lD-mI)T_v]x(mT_x) \quad l \in \mathbb{Z}$

**G. Autocorrelation, power density spectrum and Wiener-Khintchin:**

Given a wide-sense stationary and ergodic sequence $X[n]$ with infinite energy

Autocorrelation: $\gamma_{XX}[l] = E\{X[n]X[n+l]\} \quad l \in \mathbb{Z}$

Power spectrum: $\Gamma_{XX}(z) = \mathcal{Z}\{\gamma_{XX}[l]\} \Rightarrow$

Wiener-Khintchin: $\Gamma_{XX}(f) = \text{DTFT}\{\gamma_{XX}[l]\} = \sum_{l=-\infty}^{\infty} \gamma_{XX}[l]e^{-j2\pi fl}$

**H. Yule-Walker and Normal equations where $a_0 = 1$:**

Autocorrelation: $\sum_{k=0}^{P} a_k \gamma_{XX}[n-k] = \sigma_f^2 \delta[n] \quad n = 0, \dots, p$

Normal equations: $\sum_{k=1}^{P} a_k \gamma_{XX}[n-k] = -\gamma_{XX}[n] \quad n = 1, \dots, p$

**I. Some common z-transform pairs:**

| Sequence | Transform | ROC |
|---|---|---|
| $\delta[n]$ | $1$ | $\forall z$ |
| $a^n u[n]$ | $\frac{1}{1-az^{-1}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $-b^n u[-n-1]$ | $\frac{1}{1-bz^{-1}}$ | $\lvert z\rvert < \lvert b\rvert$ |
| $(a^n \sin\omega_0 n)u[n]$ | $\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $(a^n \cos\omega_0 n)u[n]$ | $\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $na^n u[n]$ | $\frac{az^{-1}}{(1-az^{-1})^2}$ | $\lvert z\rvert > \lvert a\rvert$ |
| $-nb^n u[-n-1]$ | $\frac{bz^{-1}}{(1-bz^{-1})^2}$ | $\lvert z\rvert < \lvert b\rvert$ |