EXAM 12.12.2017

SOLUTION PROBLEM 1

$$y[n] = a\,y[n-1] - a\,x[n] + x[n-1]$$

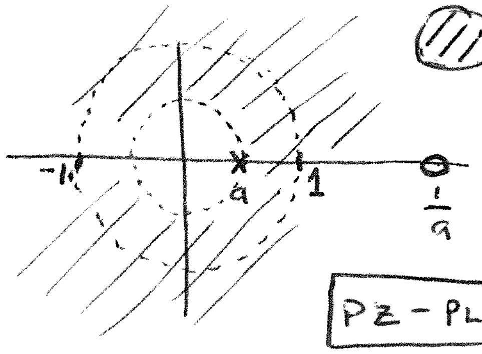
1a) $H_1(z) = \dfrac{Y(z)}{X(z)} \Rightarrow$

$$\mathcal{Z}\{y[n]\} = \mathcal{Z}\{a\,y[n-1]\} - \mathcal{Z}\{a\,x[n]\} + \mathcal{Z}\{x[n-1]\}$$

$$\Leftrightarrow \quad Y(z) = a z^{-1} Y(z) - a X(z) + z^{-1} X(z)$$

$$\Rightarrow \quad H_1(z) = \frac{z^{-1} - a}{1 - a z^{-1}}. \quad \text{(IIR filter)}.$$

1b)
- A zero in $z_1 = \frac{1}{a}$ and a pole in $p_1 = a$
- Causal filter $\Rightarrow$ ROC is exterior of the circle associated with the pole
- Stable filter $\Rightarrow$ pole inside the unit circle $\Rightarrow |a| < 1$



$\sim$ ROC, $\forall z : |z| > a$

PZ-PLOT

- Filter cannot have minimum phase, since this requires both poles and zeros to be inside the unit circle.

1c) $$H_1(z) = \frac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}}$$

Allpass filter has $|H(f)|^2 = \text{const.}$

$$|H(f)|^2 = H(z)H(z^{-1})\Big|_{z=e^{j2\pi f}}$$

$$H(z)H(z^{-1}) = \frac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}} \cdot \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z} = 1$$

1d) $$H(z) = \frac{z^{-1} - \frac{1}{2}}{1 - \frac{1}{2}z^{-1}} \cdot \frac{1}{1 + \frac{1}{2}z^{-1}}$$

$$= \frac{A}{1 - \frac{1}{2}z^{-1}} + \frac{B}{1 + \frac{1}{2}z^{-1}}$$

Residue calculus $\Rightarrow$ $A = (1 - \frac{1}{2}z^{-1})H(z)\Big|_{z=\frac{1}{2}} = \frac{3}{4}$

$$B = (1 + \frac{1}{2}z^{-1})H(z)\Big|_{z=-\frac{1}{2}} = -\frac{5}{4}$$

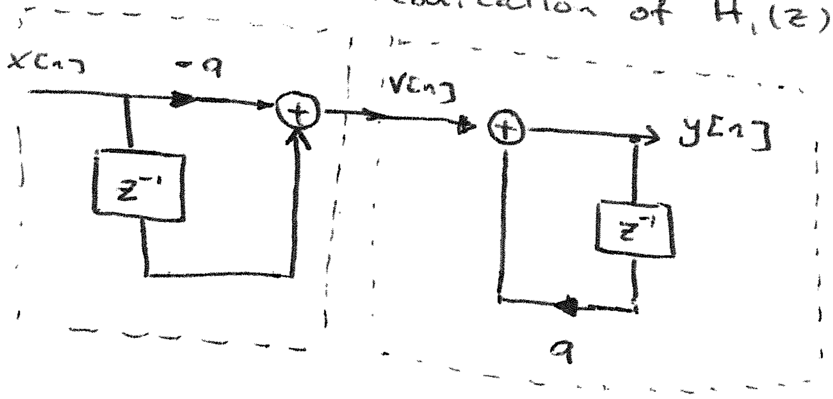
$$h[n] = \mathcal{Z}^{-1}\{H(z)\} = \mathcal{Z}^{-1}\left\{\frac{3}{4} \cdot \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{5}{4}\frac{1}{1 + \frac{1}{2}z^{-1}}\right\}$$

$$= \frac{3}{4}\mathcal{Z}^{-1}\left\{\frac{1}{1 - \frac{1}{2}z^{-1}}\right\} - \frac{5}{4}\mathcal{Z}^{-1}\left\{\frac{1}{1 + \frac{1}{2}z^{-1}}\right\}$$

$$= \frac{3}{4}\left(\frac{1}{2}\right)^n u[n] - \frac{5}{4}\left(-\frac{1}{2}\right)^n u[n].$$

1e) See lecture slides or course notes.

2 a) Direct form I realization of $H_1(z)$



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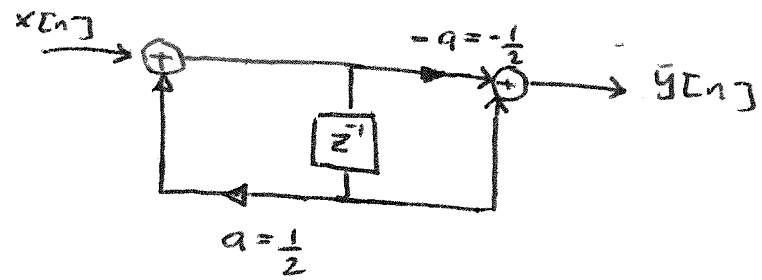
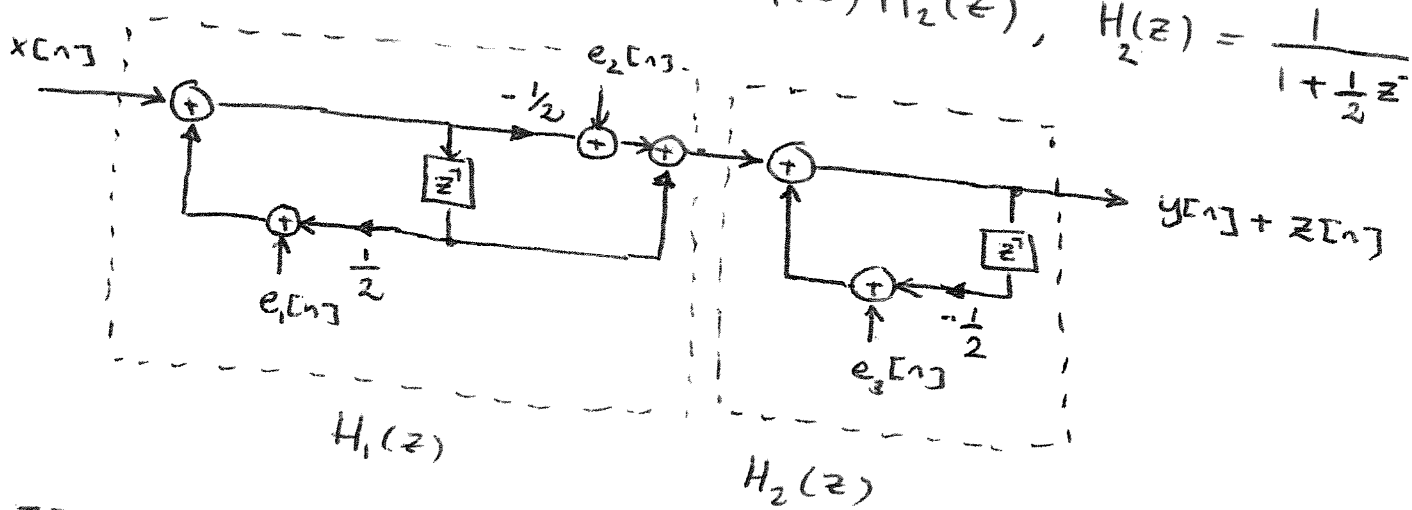
Direct form II realization of $H_1(z)$



Cascade structure $H(z) = H_1(z) H_2(z)$, $H_2(z) = \frac{1}{1+\frac{1}{2}z^{-1}}$



$$z[n] = e_1[n] * h[n] + e_2[n] * h_2[n] + e_3[n] * h_2[n]$$

$$\Rightarrow \sigma_z^2 = \sigma_e^2 \, r_{hh}[0] + 2\sigma_e^2 \, r_{h_2h_2}[0]$$

$$r_{hh}[0] = \sigma_e^2 \sum_{n=0}^{\infty} h^2[n] = \sigma_e^2 \int_0^1 |H(f)|^2 df$$

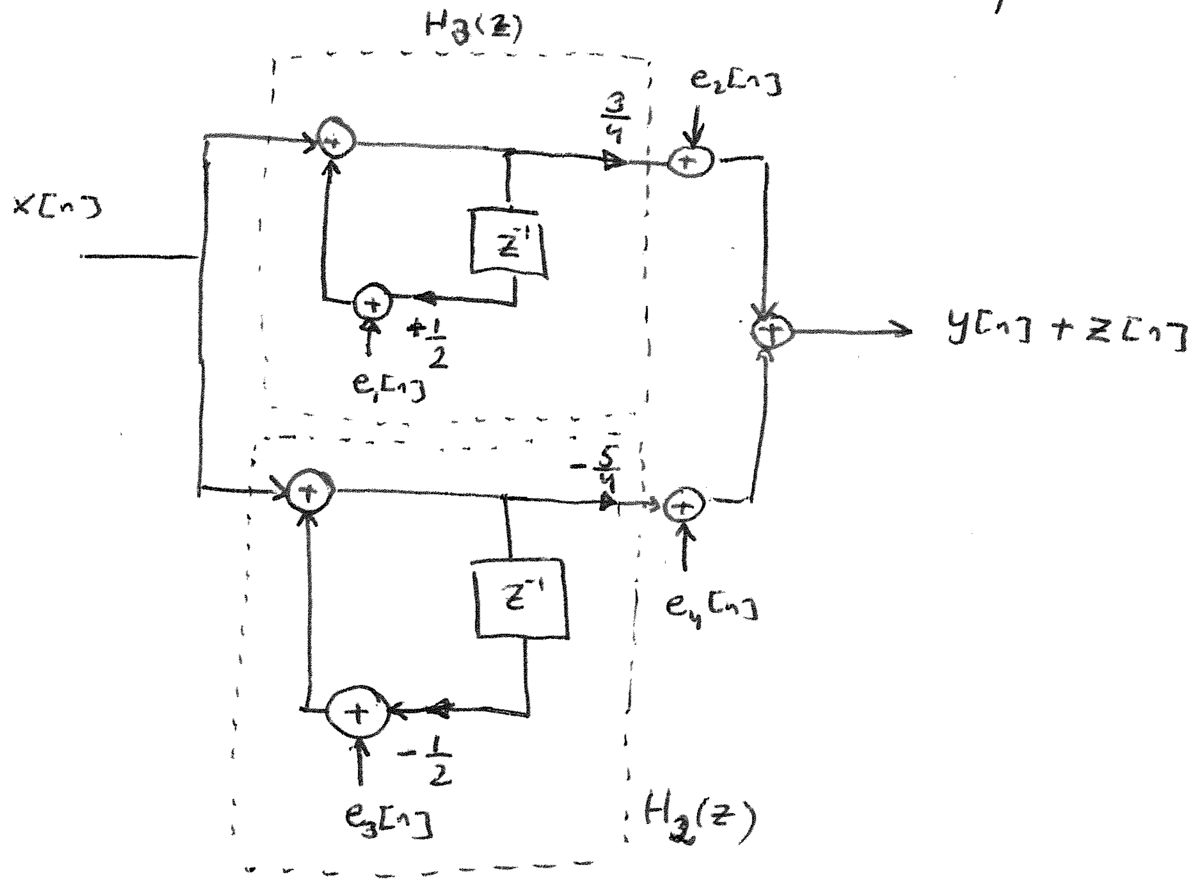
$$= \sigma_e^2 \int_0^1 |H_1(f) H_2(f)|^2 df = \sigma_e^2 \int_0^1 |H_1(f)|^2 |H_2(f)|^2 df$$

$$= \left\{ |H_1(f)|^2 = 1 \right\} = \sigma_e^2 \int_0^1 |H_2(f)|^2 df = \sigma_e^2 \sum_{k=0}^{\infty} h_2^2[n]$$

$$\Rightarrow \sigma_z^2 = 3\sigma_e^2 r_{h_2h_2}[0] = 3\sigma_e^2 \sum_{k=0}^{\infty} \left(-\frac{1}{2}\right)^{2n}$$

$$= 3\sigma_e^2 \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^n = 3\sigma_e^2 \cdot \frac{1}{1-\frac{1}{4}} = 4\sigma_e^2$$

2b)



$$z[n] = e_1[n] * h_4[n] + e_2[n] + e_3[n] * h_3[n] + e_4[n]$$

$$\sigma_z^2 = \sigma_e^2 r_{h_4h_4}[0] + \sigma_e^2 r_{h_3h_3}[0] + 2\sigma_e^2 \ldots$$

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$$\sigma_z^2 = \sigma_e^2\left(\frac{3}{4}\right)^2 \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^n + \sigma_e^2\left(\frac{5}{4}\right)^2 \sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^n + 2\sigma_e^2$$

$$= \sigma_e^2 \frac{4}{3}\left(\frac{9}{16}+\frac{25}{16}\right) + 2\sigma_e^2 = \sigma_e^2\left(\frac{9+25+24}{12}\right) = \underline{\underline{\frac{29}{6}\sigma_e^2}}$$

2c) The parallel structure suffers the most from rounding error

2d) Overflow may occur at the output of summation points. We, therefore, need to find the maximum values of the signals at the output of summation points

$$y[n] = x[n] * h[n] = x[n] * h_3[n] + x[n] * h_4[n]$$

$$|y[n]| = \left| \sum_{k=0}^{\infty} h_3[k]x[n-k] + \sum_{k=0}^{\infty} h_4[k]x[n-k] \right|$$

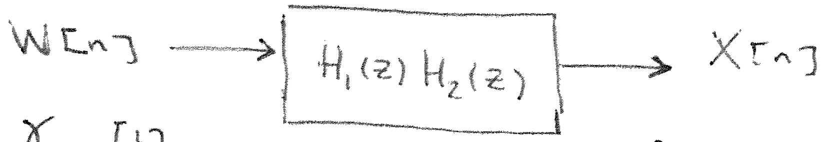
$$\leq \left| \sum_{k=0}^{\infty} h_3[k]x[n-k] \right| + \left| \sum_{k=0}^{\infty} h_4[k]x[n-k] \right|$$

$$\leq |x_{MAX}| \left(\sum_{k=0}^{\infty} |h_3[k]| + \sum_{k=0}^{\infty} |h_4[k]| \right)$$

$$= |x_{MAX}| \left(\frac{3}{4}\cdot\frac{1}{1-\frac{1}{2}} + \frac{5}{4}\cdot\frac{1}{1-\frac{1}{2}} \right) = |x_{MAX}| \cdot 4 \leq 1$$

$\Rightarrow$ To make sure that the signal at output remains in range $[-1, 1)$, $|x_{MAX}|$ cannot exceed $\frac{1}{4}$

3)



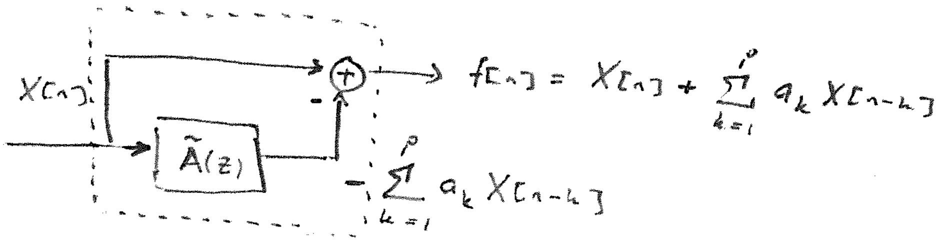
$\gamma_{ww}[l]$ $\qquad\qquad$ $\gamma_{xx}[l]$

$\Gamma_{ww}(f)$ $\qquad\qquad$ $\Gamma_{xx}(f)$

a)
$$\Gamma_{xx}(f) = |H_1(f)\,H_2(f)|^2\,\Gamma_{ww}(f) = \underbrace{|H_1(f)|^2}_{=1}|H_2(f)|^2\,\Gamma_{ww}(f)$$
$$= |H_2(f)|^2\,\Gamma_{ww}(f) = \frac{\sigma_w^2}{|1+\frac{1}{2}e^{j2\pi f}|^2}$$

(Output) spectrum of an AR(1) process.

b)



X[n] is an AR(1) process generated by filtering W[n] through linear filter $H(z) = \frac{1}{A(z)} = \frac{1}{1+\frac{1}{2}z^{-1}}$

$\Rightarrow$ Prediction error variance minimized when X[n] filtered through $A(z) = \frac{1}{H(z)}$. Optimal model order is $p = 1$

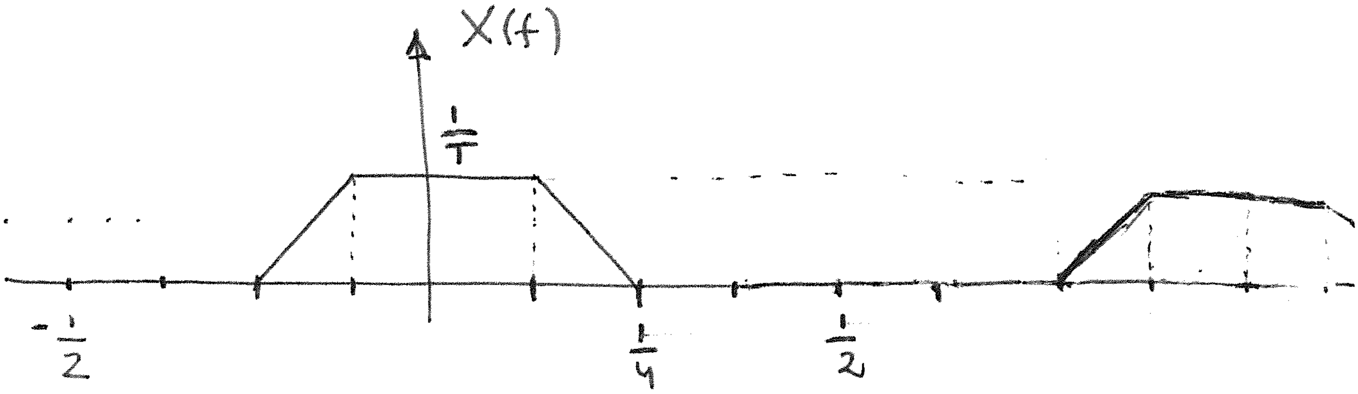
Yule-Walker: $\quad \gamma_{xx}[0]\cdot a_1 = -\gamma_{xx}[1]$

$$\Rightarrow \quad a_1 = -(-0{,}5) = \frac{1}{2}$$

c) See lecture slides.

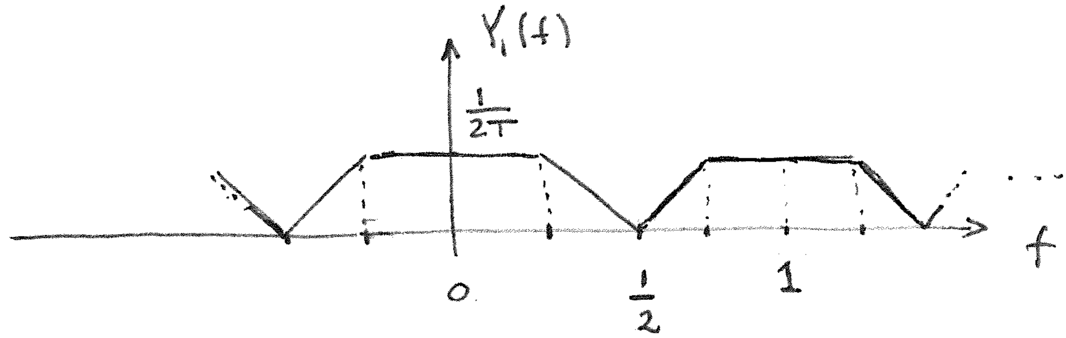
4) $T_x = 0{,}125 \cdot 10^{-3}\,s = \frac{1}{8000}\,s \Rightarrow F_x = 8\,kHz$

a)

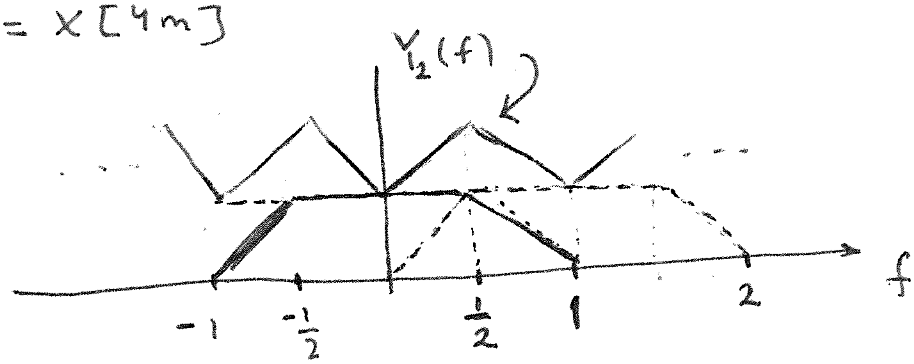


$$X(f+k) = X(f), \quad k \in \mathbb{Z}$$

b) $y_1[m] = x[2m]$

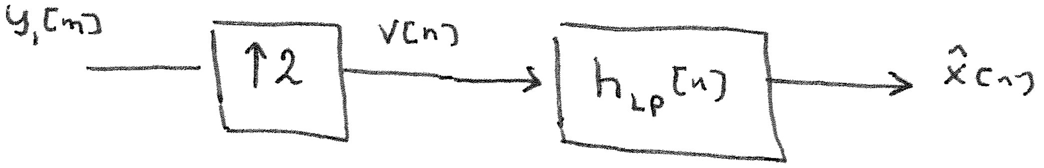


$y_2[m] = x[4m]$



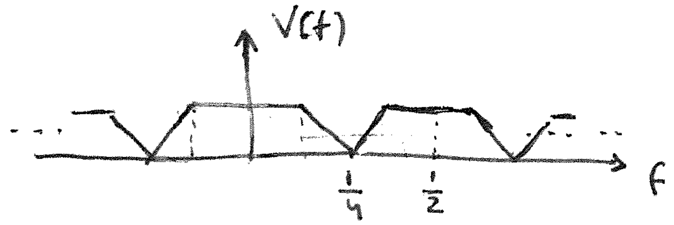
- $y_1[m]$ can be used as replacement, there is no aliasing
- $y_2[m]$ cannot be used due to severe aliasing.
  If we like to downsample $x[n]$ with a factor $D=4$ we need to prefilter $x[n]$ using a lowpass filter removing signal above $f_c = \frac{1}{8}$ to limit distortion due to aliasing.

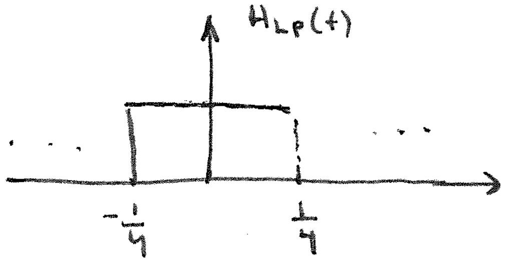
4 c)



$$H_{LP}(f) = \begin{cases} \text{const}, & |f| \le \frac{1}{4} \\ 0, & \text{elsewhere} \end{cases}$$

- $v[n]$ obtained by adding 1 zero between samples of $y_1[m]$





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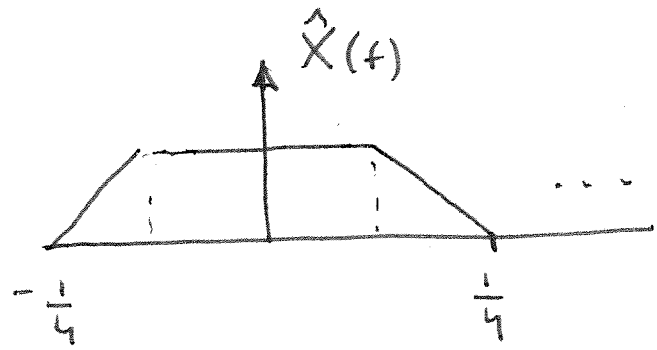

4d) The requested DFTs are:

$$X(k) = \mathrm{DFT}_{N_x}\{x[n]\}, \quad k = 0, 1, \ldots, N_x - 1$$

$$H(k) = \mathrm{DFT}_{N_h}\{h[n]\}, \quad k = 0, 1, \ldots, N_h - 1$$

$$Y(k) = \mathrm{DFT}_{N_y}\{y[n]\}, \quad k = 0, 1, \ldots, N_y - 1$$

$$N_x \geq M = 1090$$

$$N_h \geq L = 52$$

$$N_y \geq M + L - 1 = 1141$$

Using radix-2 FFT for computing $Y(k)$, the most suitable length is $N_y = 2048 = 2^{11}$