NTNU

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Department of Electronic Systems

Examination paper for TTT4120 Digital Signal Processing

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Examination time (from-to): 09:00-13:00

Permitted examination support material:

D - Basic calculator allowed

No printed or handwritten materials allowed

Other information:

- Exam consists of four (4) problems.
- A few basic formulas are provided in the Appendix

Language: English.

Number of pages (front page excluded): 6

Number of pages enclosed: 2

Informasjon om trykking av eksamensoppgave

Originalen er:

1-sidig ☐ **2-sidig** ☐

sort/hvit ☐ **farger** ☐

skal ha flervalgskjema ☐

Checked by:

Date Signature

Merk! Studentane finn sensur i Studentweb. Har du spørsmål om sensuren må du kontakte instituttet ditt. Eksamenskontoret vil ikkje kunne svare på slike spørsmål.

Problem 1 (2+6+5+5=18): Basics of filter theory and design

A causal filter is given by the following difference equation

$$y[n] = x[n] + \alpha^2 y[n-2]$$

where α is a finite real-valued constant.

1a) Provide the system function corresponding to $y[n]$ in the form

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

1b) Express the filter as a cascade of two filters, i.e.

$$H(z) = H_1(z) \cdot H_2(z) = \frac{1}{(1 - p_1 z^{-1})} \cdot \frac{1}{(1 - p_2 z^{-1})}$$

where p_1 and p_2 denote the poles of the filter. Based on your findings, determine the range of α that yields a stable filter. Draw the pole-zero plot and sketch the region of convergence (ROC) for a stable filter realization.

1c) Express the filter in its parallel form

$$H(z) = H_3(z) + H_4(z) = \frac{A_1}{(1 - p_1 z^{-1})} + \frac{A_2}{(1 - p_2 z^{-1})}$$

and provide the unit impulse response $h[n] = \mathcal{Z}^{-1}\{H(z)\}$.

1d) Provide answers to the following questions (justify the answers):

- Find the causal inverse filter $H_I(z)$ such that $H(z)H_I(z) = 1$.
- Find the ROC for $H_I(z)$ and sketch the pole-zero plot.
- For $\alpha = 1$, discuss what type of filter $H_I(z)$ is (lowpass, highpass, bandpass, allpass).
- Determine the range for α for which $H_I(z)$ is a minimum-phase filter.
- Does filter $H_I(z)$ have linear phase?

Problem 2 (6+6+6+2=20): Filter structures and implementations

The filter in **Problem 1a) with $\boldsymbol{\alpha} = \mathbf{0.5}$** is implemented using fixed-point representation with $B+1$ bits and dynamic range $[-1,1)$. Rounding is performed after each multiplication and the rounding error $e[n]$ can be modeled as white noise with variance $\sigma_e^2 = 2^{-2B}/12$. Consequently, each multiplier in the fixed-point implementation is modeled as

$$Q(ay[n-k]) = ay[n-k] + e[n]$$

which is equivalent to adding noise sources after multipliers in the infinite-precision realization. Rounding noise sources combine into an equivalent noise signal $z[n]$ at the filter output with variance σ_z^2 (see hint below).

2a) Draw the direct-form structure II (DF-II) of $H(z)$ with noise sources due to rounding included. Determine the variance of the round-off noise at the filter output.

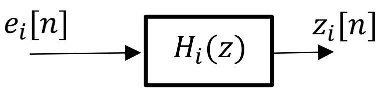
2b) Draw the cascade-structure, $H_1(z)H_2(z)$, with noise sources due to rounding included. Determine the variance of the round-off noise at the filter output.

2c) Draw the parallel-structure, $H_3(z) + H_4(z)$, with noise sources due to rounding included. Determine the variance of the round-off noise at the filter output. You may put the multiplier(s) A_1 and A_2, obtained from the residue calculus in Problem 1c, at the output of the structure.

2d) Which of the three implementations above suffers the most from rounding noise? Which implementation suffers the least? Justify your answers.

[Hint:] Assuming noise source $e_i[n]$ with variance σ_{ei}^2 acts as input to (sub-)filter $h_i[n]$ that terminates at the output. The variance of the noise signal $z_i[n]$, due to $e_i[n]$, is given by

$$\sigma_{zi}^2 = \sigma_{ei}^2 r_{h_i h_i}[0] = \sigma_{ei}^2 \sum_k h_i^2[k]$$



$\sigma_{ei}^2 = E\{e_i^2[n]\}$ $\qquad$ $\sigma_{zi}^2 = E\{z_i^2[n]\}$

Problem 3 (2+6+6=14): Parametric modeling

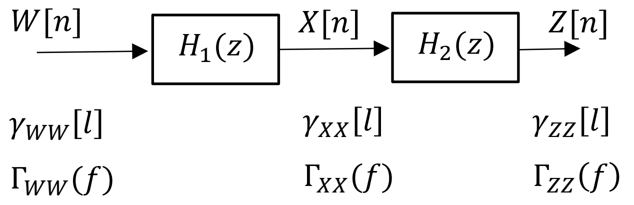


Fig. 1: Filtering of stochastic processes

A wide-sense stationary (WSS) stochastic process $X[n]$ is generated by filtering a white noise process $W[n]$, with autocorrelation sequence $\gamma_{WW}[l] = \sigma_W^2\delta[l]$, through a causal and stable filter $H(z)$, as depicted in Fig. 1. The autocorrelation sequence and spectrum of $X[n]$ is obtained from

$$\gamma_{XX}[l] = \begin{cases} \sigma_W^2 \sum_{n=0}^{\infty} h[n]h[n+l], & l \geq 0 \\ \gamma_{XX}[-l], & l < 0 \end{cases}$$

$$\Gamma_{XX}(f) = |H(f)|^2\Gamma_{WW}(f)$$

3a) Provide answers (with motivations) to the following two questions:

- What type of process, AR(p), MA(q), or ARMA(p,q), is $X[n]$ when the noise is filtered by $H(z)$ in **Problem 1a** for the case of $\boldsymbol{\alpha = 0.5}$? Provide the model order.

3b) For filter $H(z)$ specified in **3a)** and $\boldsymbol{\sigma_W^2 = 1}$, show that the respective autocorrelation sequence and spectrum of $X[n]$ are given by

$$\gamma_{XX}[l] = \begin{cases} \frac{16}{15} \cdot \frac{1}{2^{|l|}}, & l \text{ even} \\ 0, & l \text{ odd} \end{cases}$$

$$\Gamma_{XX}(f) = \frac{1}{\frac{17}{16} - \frac{1}{2}\cos 4\pi f}$$

3c) You are given the task to design a second-order predictor to model $X[n]$. That is, you form an estimate of $X[n]$, denoted $\hat{X}[n]$, through the following linear combination $\hat{X}[n] = a_1X[n-1] + a_2X[n-2]$.

- Find the optimal values of a_1 and $a_2$that minimize the prediction error power. **Hint:** Use the values $\gamma_{XX}[-2]$ through $\gamma_{XX}[2]$ from **3b)** together with the Normal equations, see Section H in the Table of formulas attached to the exam.
- Find the resulting prediction error power σ_f^2 when using the optimal coefficients.
- Comment on your results.

Problem 4 (6+2+6=14): Sampling and rate-conversion

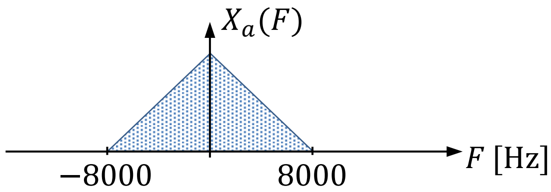


Fig. 2: Spectrum $X_a(F)$ of continuous-time signal $x_a(t)$

Let $x_a(t)$ be a continuous-time signal whose spectrum $X_a(F)$ is shown in Fig. 2. Signal $x_a(t)$ is sampled at rate $F_x = \frac{1}{T_x} = 16$ kHz to generate sequence $x[n] = x_a(t)|_{t\ =nT_x}$. We would now like to design a system that changes the sampling frequency of signal $x[n]$ in digital domain from F_x to $F_y = \frac{1}{T_y} = 12$ kHz. Let $y[m]$ be the resulting output signal. The conversion should not introduce any distortion due to aliasing.

4a) Sketch the block diagram of the digital system that implements the sampling rate conversion. Explain the function of each individual block along with the specifications that arise from the rate conversion at hand (e.g., upsampling/downsampling factors, filter bandwidths, etc.).

4b) Will the rate conversion above incur any loss of information? That is, can $x_a(t)$ be perfectly reconstructed from $y[m]$?

4c) Sketch the spectra of all the signals in the rate-conversion system.

Appendix: TTT4120 Table of formulas, 2017

A. Sequences:

$\sum_{n=0}^{N-1} \alpha^n = \frac{1-\alpha^N}{1-\alpha}$

$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$ and $-\sum_{n=-1}^{-\infty} \alpha^n = \frac{1}{1-\alpha}$

$\sum_{n=0}^{N-1} (n+1)\alpha^n = \frac{1-\alpha^N}{(1-\alpha)^2} - \frac{N\alpha^N}{1-\alpha}$; $\alpha \neq 1$

$|\alpha| < 1 \Rightarrow \sum_{n=0}^{\infty} (n+1)\alpha^n = \frac{1}{(1-\alpha)^2}$

B. Linear convolution:

$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$

$Y(z) = H(z)X(z)$

$Y(f) = H(f)X(f)$

$Y(k) = H(k)X(k),\, k = 0,1,\dots,N-1$ where $Y(k) = Y(f_k)$ with $f_k = k/N$

C. Transforms:

Z-transform: $H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n}$

DTFT: $H(f) = \sum_{n=-\infty}^{\infty} h[n]e^{-j2\pi fn}$

DFT: $H(k) = \sum_{n=0}^{N-1} h[n]e^{-j2\pi fnk/N}$ $\quad k = 0,1,\dots,N-1$

IDFT: $h[n] = \frac{1}{N}\sum_{k=0}^{N-1} H(k)e^{j2\pi fnk/N}$ $\quad n = 0,1,\dots,N-1$

D. Sampling theorem:

Given an analog signal $x_a(t)$ sampled at $F_s = 1/T$. The DTFT of the resulting discrete-time sequence $x[n] = x_a(t)|_{t=nT}$ is given by

$$X(f) = X(F/Fs) = F_s \sum_{k=-\infty}^{\infty} X([f-k]F_s)$$

E. Autocorrelation, energy spectrum and Parseval:

Given a sequence $h[n]$ with finite energy E_h

Autocorrelation: $r_{hh}[l] = \sum_{n=-\infty}^{\infty} h[n]h[n+l]$ $\quad l \in \mathbb{Z}$

Energy spectrum: $S_{hh}(z) = H(z)H(z^{-1}) \Rightarrow S_{hh}(f) = |H(f)|^2$

Parseval's theorem: $E_h = r_{hh}[0] = \sum_{n=-\infty}^{\infty} h^2[n] = \int_0^{2\pi} |H(f)|^2 df$

F. Multirate:

Decimation (downsampling) where $T_y = DT_x$

$v(mT_y) = \sum_{k=-\infty}^{\infty} h[(mD-k)T_x]x(kT_x) \quad m \in \mathbb{Z}$

Interpolation (upsampling) where $T_y = T_x/I$

$y(lT_y) = \sum_{n=-\infty}^{\infty} h\left[(l-nI)T_y\right]x(nT_x) \quad l \in \mathbb{Z}$

Rate coversion where $T_y = DT_v = \frac{D}{I}T_x$

$y(lT_y) = \sum_{m=-\infty}^{\infty} h[(lD-mI)T_v]x(mT_x) \quad l \in \mathbb{Z}$

G. Autocorrelation, power density spectrum and Wiener-Khintchin:

Given a wide-sense stationary and ergodic sequence $X[n]$ with infinite energy

Autocorrelation: $\gamma_{XX}[l] = E\{X[n]X[n+l]\} \quad l \in \mathbb{Z}$

Power spectrum: $\Gamma_{XX}(z) = \mathcal{Z}\{\gamma_{XX}[l]\} \Rightarrow$

Wiener-Khintchin: $\Gamma_{XX}(f) = \mathrm{DTFT}\{\gamma_{XX}[l]\} = \sum_{l=-\infty}^{\infty} \gamma_{XX}[l]e^{-j2\pi fl}$

H. Yule-Walker and Normal equations where $a_0 = 1$:

Autocorrelation: $\sum_{k=0}^{P} a_k \gamma_{XX}[n-k] = \sigma_f^2 \delta[n] \quad n = 0, \dots, p$

Normal equations: $\sum_{k=1}^{P} a_k \gamma_{XX}[n-k] = -\gamma_{XX}[n] \quad n = 1, \dots, p$

I. Some common z-transform pairs:

Sequence	Transform	ROC
$\delta[n]$	1	$\forall z$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-b^n u[-n-1]$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$
$(a^n \sin\omega_0 n)u[n]$	$\frac{(a\sin\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$(a^n \cos\omega_0 n)u[n]$	$\frac{1-(a\cos\omega_0)z^{-1}}{1-(2a\cos\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-nb^n u[-n-1]$	$\frac{bz^{-1}}{(1-bz^{-1})^2}$	$\lvert z\rvert < \lvert b\rvert$