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Analysis and Design of Marine Structures subjected to Accidental Loads

Transient response of structural components to explosion loads

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Jørgen Amdahl, NTNU,
Dept. Marine Technology

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1 Introduction

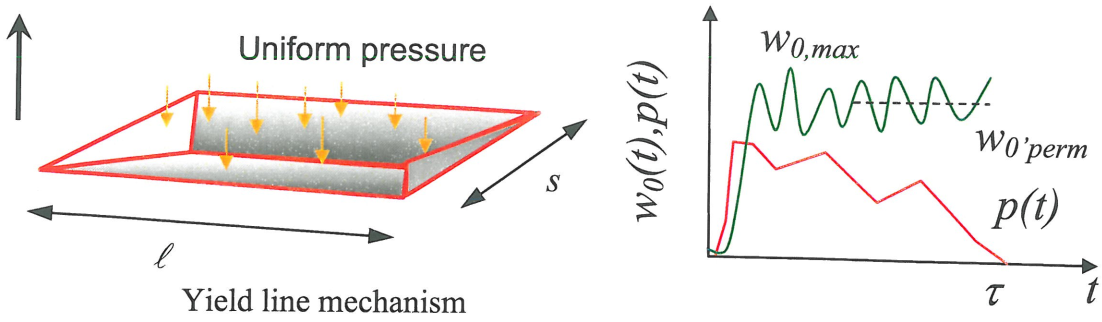


Figure 1.1 Plate subjected to pressure pulse

Consider the plate in Fig. 1 which is subjected to a pressure pulse $p(t)$ with duration τ. The plate deforms when subjected to this pulse. The maximum deflection of the plate is denoted by $w_0(t)$. Depending on the pulse duration and the dynamic characteristics of the plate the deflection history $w_0(t)$ may be as illustrated in Figure 1. It deforms to a maximum level and continues to vibrate afterwards. If the plate has undergone yielding and plastic deformation it will vibrate about a permanent set $w_{0,perm}$.

We are especially interested in the maximum deformation, $w_{0,\max}$, because this is the deformation creating the largest stresses and strains in the plate. The maximum stresses and strains have to be evaluated against acceptance criteria.

2 Transient response of a Single Degree of Freedom systems (SDOF)

2.1 Rectangular pulse

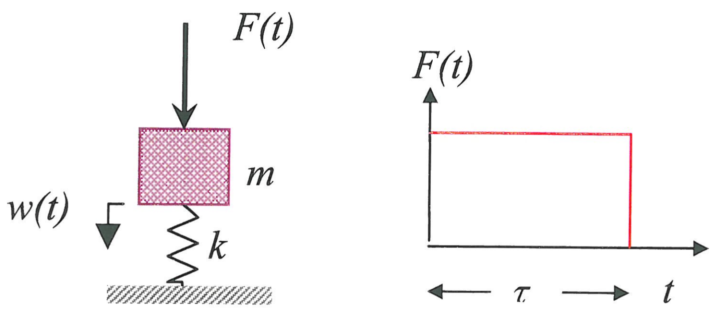
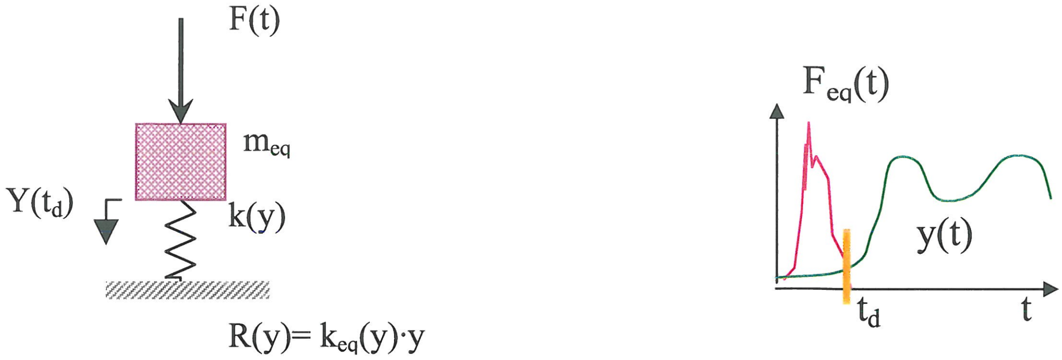


Figure 2.1 Single degree of freedom model



Consider the Single Degree of Freedom system in Figure 2. It consists of a mass, m, and a linear spring with stiffness k. Damping is disregarded because the pressure pulse and the response time are fairly small. It is subjected to a rectangular pulse load with duration τ. This means that the force is suddenly applied and remains constant during the pulse duration. The dynamic response $w_0(t)$ is sought.

We have to split the solution into two phases

i The pulse phase $0 < t < \tau$
ii After the pulse has ceased $\tau < t$ (1)

In the pulse phase dynamic equilibrium is given by

$$m\ddot{w} + kw = F \quad (2)$$

The solution consists of a homogeneous part $(F = 0)$ and a particular solution

$$w = w_n + w_n = A\sin\omega_0 t + B\cos\omega_0 t + \frac{F}{k} \quad (3)$$

The last term is the particular solution and is simply the static response

$$w_{stat} = \frac{F}{k} \quad (4)$$

$\omega_0 = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}$ is the eigenfrequency of the system for free vibration. T denotes the eigenperiod. The constants A and B must be determined such that the initial conditions are fulfilled.

$$w(0)=0 \qquad \text{and} \qquad \dot{w}(0)=0 \tag{5}$$

This gives

$$B+w_{stat}=0 \Rightarrow B=-w_{stat} \tag{6}$$

$$A\omega_0=0 \rightarrow A=0 \tag{7}$$

Hence the solution in the pulse phase is given by

$$w=w_{stat}\left(1-\cos\frac{2\pi\cdot t}{T}\right) \qquad \underline{0<t<\tau} \tag{8}$$

After the pulse has ceased the system is subjected to free vibration

$$m\ddot{w}+kw=0 \tag{9}$$

The solution is given by

$$w_1=A_1\sin w_0(t-\tau)+B_1\cos\cdot w_0(t-\tau) \qquad t\geq\tau \tag{10}$$

The constant A_1 and B_1 have to be determined such that continuity in deformation and speed is maintained

$$w_1(t=\tau)=w(\tau) \tag{11}$$

$$\dot{w}_1(t=\tau)=w(\tau) \tag{12}$$

$$\Rightarrow \qquad B_1=w_{stat}\left(1-\cos\cdot\frac{2\pi\cdot\tau}{T}\right) \tag{13}$$

$$A_1 w_0=w_{stat}\cdot\omega_0\sin\frac{2\pi\cdot\tau}{T} \tag{14}$$

Hence the solution for $t\geq T$ is given by:

$$w_1=w_{stat}\left\{\sin\frac{2\pi\cdot\tau}{T}\sin\frac{2\pi}{T}(t-\tau)+\left(1-\cos\frac{2\pi\cdot\tau}{T}\right)\cos\frac{2\pi}{T}(t-\tau)\right\} \tag{15}$$

Equations (8) and (15) provide the response history. It is observed that the duration, τ, of the pulse relative to the eigenperiod, T, is very important.

Normalized displacement histories w/w_{stat} are plotted versus normalized time t/τ in Figure 3. Note that the pulse lasts until $t/\tau = 1$. It is observed that the response depends on the τ/T-ratio. For small values the duration is too short to provide a significant response. When $\tau/T > 0.8$ the maximum response is 2, i.e. two times the static deformation. This is the maximum dynamic magnification of the system.

It is also observed that depending on the τ/T-ratio the maximum amplitude to the opposite side varies between 2 and zero.

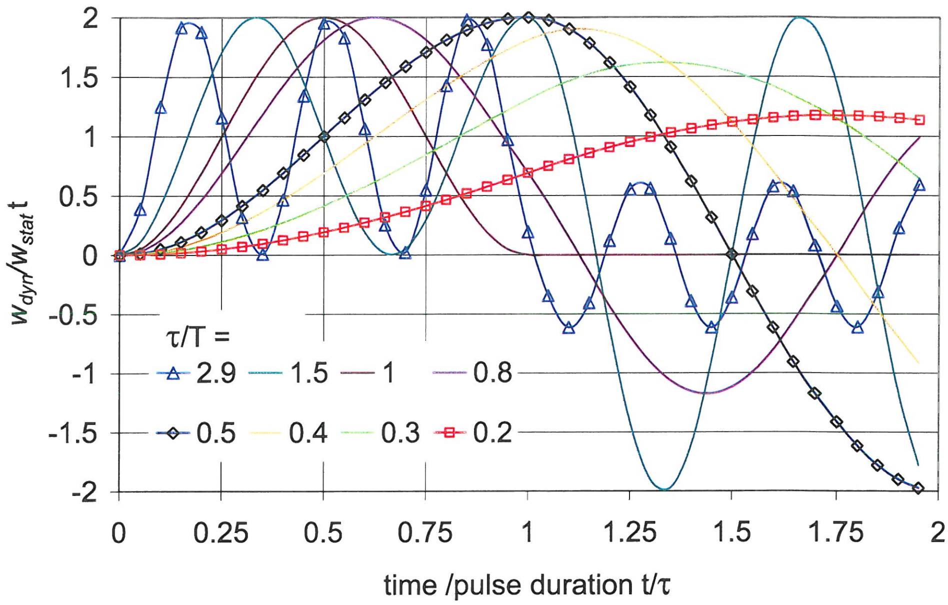


Figure 2.2 Dynamic response of an SDOF system to rectangular pulse

2.2 Sinusoidal pulse

If the pulse is sinusoidal, $F = F_0 \sin \omega t = F_0 \sin \frac{\pi t}{\tau}$ the same dynamic equation apply, namely Equations (2) and (9). The solution in the pulse phase is then

$$w = A \sin w_0 t + B \cos w_0 t + \frac{F_0}{k} \frac{1}{1 - \left(\frac{w}{w_0} \right)^2} \sin \omega t \tag{16}$$

where the last term signifies the particular solution.

Again $F_0 / k\, w_{stat}$ is the static solution. A and B is determined from the initial conditions

$$w(0) = 0 \Rightarrow B = 0 \tag{17}$$

$$\dot{w}(0) = 0 \Rightarrow A\omega_0 + w_{stat}\frac{\omega}{1-\left(\frac{\omega}{\omega_0}\right)^2} = 0 \tag{18}$$

$$\Rightarrow \qquad A = \div w_{stat}\frac{\omega/\omega_0}{1-\left(\frac{\omega}{\omega_0}\right)^2} \tag{19a}$$

$$w = \frac{w_{stat}}{1-\left(\frac{\omega}{\omega_0}\right)^2}\left\{-\frac{\omega}{\omega_0}\sin\omega_0 t + \sin\omega t\right\} \tag{19b}$$

The displacement and speed at the end of the pulse is given by $\omega t = \pi$.

$$w = w_{stat}\frac{1}{1-\left(\frac{\omega}{\omega_0}\right)^2}\left\{-\frac{\omega}{\omega_0}\sin\pi\frac{\omega_0}{\omega} + 0\right\} \tag{20}$$

$$\dot{w} = w_{stat}\frac{1}{1-\left(\frac{\omega}{\omega_0}\right)^2}\omega_0\left\{-\frac{\omega}{\omega_0}\cos\pi\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right\} \tag{21}$$

These values constitute the initial condition for the second phase, where the solution is given by Equation (10).

This yields

$$B_1 = w_{stat} \tag{22}$$

$$A_1\omega_0 = \dot{w} \tag{23}$$

The total solution becomes accordingly:

$$w_1 = w_{stat}\frac{\omega/\omega_0}{1-\left(\frac{\omega}{\omega_0}\right)^2}\left\{-\sin\pi\frac{\omega_0}{\omega} - \sin\omega_0\left(\tau-\tau\right) + \left(\cos\frac{\pi\omega_0}{\omega} - 1\right)\cos\omega_0(t-\tau)\right\} \tag{24}$$

The response histories plotted in Figure 4 for various ratios of pulse duration versus eigenperiod, τ / T. It is observed that the displacement history depends heavily on the τ / T-ratio. The maximum response seems to occur for an intermediate value.

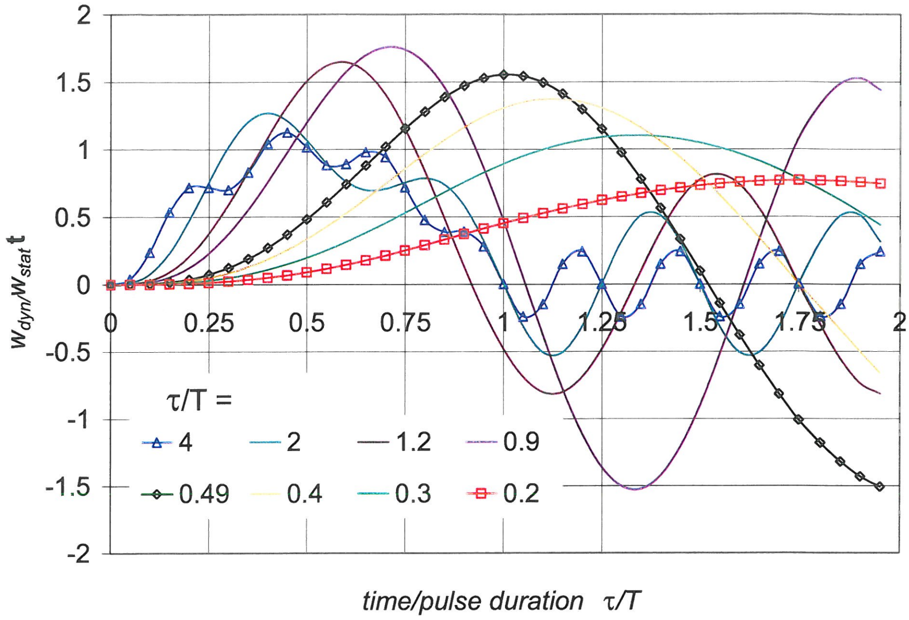


Figure 2.3 Dynamic response of an SDOF system to a sinusoidal pulse

3 Maximum response

3.1 Rectangular pulse

The maximum response will occur in the pulse phase for long duration pulses or in the free vibration phase for short duration pulses.

The maximum displacement is obtained by setting:

$$\dot{w} = 0 \qquad \text{pulse phase} \tag{25}$$

$$\dot{w}_1 = 0 \qquad \text{free vibration phase} \tag{26}$$

For a rectangular pulse there is obtained from Equation (8)

$$\dot{w} = w_{stat} \cdot \frac{2\pi}{T} \sin 2\pi \cdot \frac{t^1}{T} = 0 \tag{27}$$

which is true when

$$\frac{2\pi t^1}{T} = \pi \rightarrow \begin{matrix} t^1 = \frac{T}{2} & \text{when} & \frac{T}{2} < \tau \\ t^1 = \tau & \text{when} & \frac{T}{2} > \tau \end{matrix} \tag{28}$$

$$\frac{w_{\max}}{w_{stat}} = \begin{matrix} 2 & \text{when} & \frac{T}{2} < \tau \\ 1 - \cos\frac{2\pi \cdot \tau}{T} & \text{when} & \frac{T}{2} > \tau \end{matrix} \tag{29}$$

In the free vibration phase the maximum displacement is obtained from Equation (15)

$$\dot{w}_1 = w_{stat} \frac{2\pi}{T}\left\{\sin\frac{2\pi \cdot \tau}{T} \cos\frac{2\pi}{T}(t'-\tau) + \left(1 - \cos\frac{2\pi \cdot \tau}{T} \cdot \sin\frac{2\pi}{T}(t'-\tau)\right)\right\} = 0$$

$\Rightarrow$ this yields

$$tg\frac{2\pi}{T}(t'-\tau) = \frac{\sin\frac{2\pi \cdot \tau}{T}}{1 - \cos\left\{\frac{2\pi \cdot \tau}{T}\right\}}$$

This is easily solved for t' and then introduced into Equation (15).

The maximum deflection normalized versus the static deflection is plotted in Figure 5 for various ratios of pulse duration versus eigenperiod. It is observed that the solution increases almost linearly with τ / T initially and branches off to 2 for $\tau / T \geq 2$.

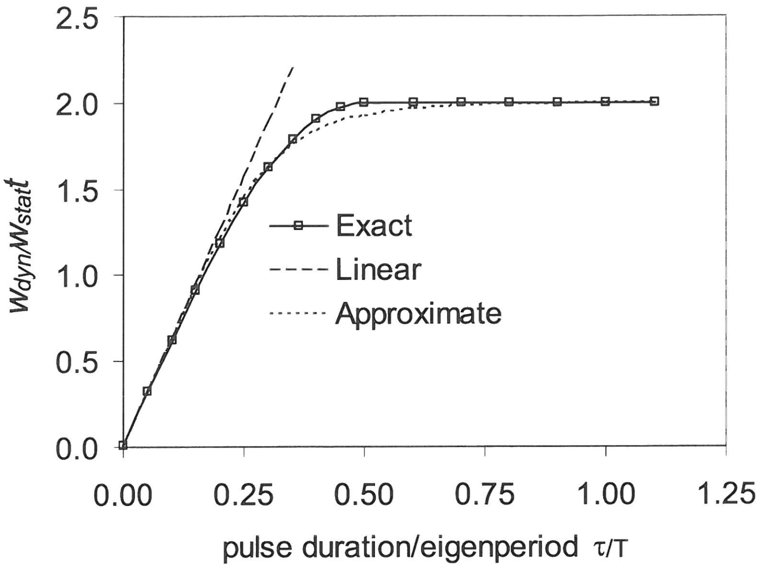


Figure 3.1 Maximum response versus pulse duration

The asymptotic values for short duration pulses and long duration pulses can be determined by simple energy considerations.

If the pulse is short, the mass has not been displaced noticeably before the pulse ends. This means that the stiffness term can be disregarded. Hence, the speed at the end of the pulse, *V*, is obtained by

$$V = \int_0^\tau a(t)dt = \frac{1}{m}\int F(t)dt = \frac{1}{m}I = \frac{F \cdot \tau}{m} \tag{30}$$

where $I = F\tau$ denotes the impulse.

The pulse has imparted a kinetic energy to the mass given by

$$E_{kin} = \frac{1}{2}mV^2 \frac{I^2}{m} = \frac{F^2\tau^2}{2m} \tag{31}$$

This energy is converted to strain energy in the spring, hence

$$E_{kin} = E_{strain} \tag{32}$$

$$\frac{F^2\tau^2}{2m} = \frac{1}{2}kw_{\max}^2 \tag{33}$$

$$\Rightarrow w_{\max}^2 = \frac{F^2\tau^2}{km} = \left(\frac{F}{k}\right)^2 \tau^2 \frac{k}{m} = w_{stat}^2 \cdot \left(2\pi\frac{\tau}{T}\right)^2$$
$$w_{\max} = w_{stat} \cdot \left(2\pi\frac{\tau}{T}\right) \tag{34}$$

When the pulse is long, the maximum response is obtained when the full force acts.

The external work is:

$$E_{ext} = F \cdot w_{\max} \tag{35}$$

The strain energy is:

$$E_{strain} = \frac{1}{2}kw_{\max}^2 \tag{36}$$

Equating external work and the strain energy there is obtained

$$w_{\max} = 2\frac{F}{k} = 2w_{stat} \tag{37}$$

The two solutions provide good estimates when τ / T is small and $\tau / T > 0.5$ as shown in Figure 4.

An approximate solution valid for the whole range may be derived by means of power interpolation between the two solutions:

$$\left(\frac{w_{\max} / w_{stat}}{2\pi\frac{\tau}{T}}\right)^p + \left(\frac{w_{\max} / w_{stat}}{2}\right)^p = 1 \tag{38}$$

where p is an exponent. E.g. $p = 4$ gives

$$\frac{w_{\max}}{w_{stat}} = \frac{1}{\sqrt[4]{1 + 1/\left(\pi\frac{\tau}{T}\right)^4}} \tag{39}$$

As seen in Figure 4 this curve provides a reasonable estimate also in the intermediate range of τ / T -ratios.

3.2 Sinusoidal pulse

For the sinusoidal pulse derivation of Equation 19b gives the following condition for the maximum displacement

$$\sin \omega_0 t = \sin \omega t \tag{40}$$

This is satisfied when

$$\omega_0 t = k2\pi - \omega t \qquad k = 1, 2, ... \tag{41}$$

when

$$\omega_0 > \omega \tag{42}$$

This can be written

$$(\omega_0 + \omega)t = \left(2\frac{\tau}{T} + 1\right)\frac{t}{\tau} = 2k \tag{43}$$

The solution has to be determined by trial and error, because several maxima may exist in the pulse range as shown in Figure 4. When $\tau / T > 0.5$ the maximum displacement occurs in the free vibration phase. The whole solution becomes a little complicated and is omitted here.

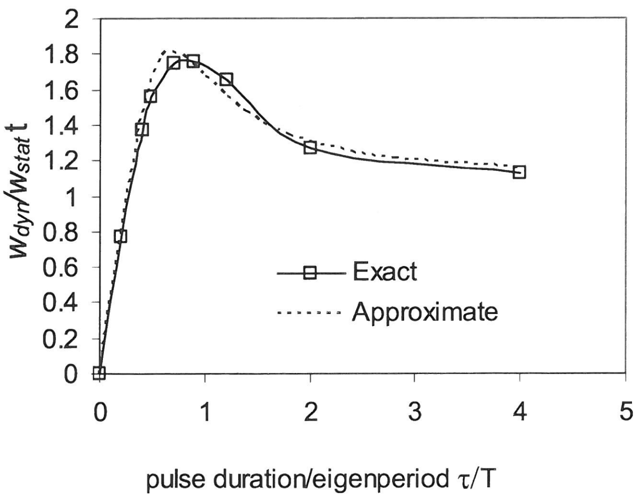


Figure 3.2 Maximum response versus pulse duration

4 Equivalent SDOF systems

4.1 Elastic beams

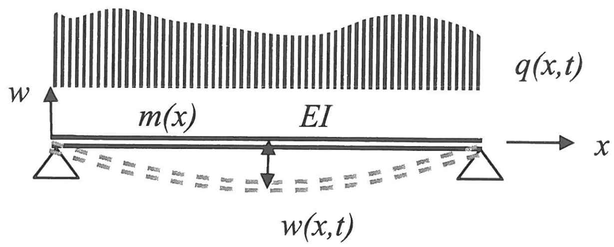


Figure 4.1 **Beam subjected to pressure pulse**

Consider the beam in Figure 5 subjected to a pulse load $q(x,t)$. The lateral deformation of the beam is denoted $w(x,t)$.

The beam has generally an infinite numbers of degrees of freedom for the deflection. In order to transform the problem into an SDOF system we will assume that the deflection is described by a shape function, $\varphi(x)$, so that

$$w(x,t) = \varphi(x) \cdot y(t) \tag{44}$$

In this way the problem is reduced to one unknown deflection parameter $y(t)$. The equivalent system reads:

$$\bar{m}\ddot{y} + \bar{k}y = \bar{f}(t) \tag{45}$$

where the generalized mass, $\bar{m}$, stiffness $\bar{k}$ and level $\bar{f}$ are given by

$$\bar{m} = \int_{\ell} m(x)\o(x)^2 dx \tag{46}$$

$$\bar{k} = \int_{\ell} EI(x)\o_{xx}^2(x)dx \tag{47}$$

$$\bar{f}(t) = \int q(x,t)\o(x)dx \tag{48}$$

Example: Simply supported beam

Assume that $m(x) = m$, $EI(x) = EI$ are constant and that the load is uniformly distributed, $q(x,t) = q(t)$. Assume that the beam deflects in the shape $\phi(x) = \sin\frac{\pi x}{\ell}$.

This yields:

$$\bar{m} = m\frac{\ell}{2} \tag{49}$$

$$\bar{k} = EI\left(\frac{\pi}{\ell}\right)^4 \cdot \frac{\ell}{2} \tag{50}$$

$$\bar{f} = q(t) \cdot \frac{2\ell}{\pi} \tag{51}$$

It is convenient to modify the generalized coefficients such that the right hand side is equal to the total load. This is obtained by multiplying the dynamic equilibrium equation by $\pi/2$. Then the following equilibrium equation is obtained.

$$\frac{\pi}{4}m\ell\ddot{y} + \frac{\pi^5}{4}\frac{EI}{\ell^3}y = q\ell \tag{52}$$

or

$$0.785M\ddot{y} + 76.5\frac{EI}{\ell^3}y = F \tag{53}$$

where M is the total mass and F is the total load. It is observed that the equivalent system has a mass equal to 0.785 of the beam total mass. It is less than one because all the mass is not subjected to the full displacement and acceleration.

If we had used the static deflection pattern as the shape function we would have obtained the following stiffness term

$$\frac{384}{5}\frac{EI}{\ell^3} = 76.8\frac{EI}{\ell^3}$$

and a mass factor of 0.78.

It is interesting to see that the sinusoidal ok shape gives almost the same equivalent system properties as using the static deflection shape. The eigenperiod of the system is

$$T = 2\pi\sqrt{\frac{\frac{\pi}{4}m\ell}{\pi^5\frac{EI}{\ell^3}}} = \frac{2\ell^2}{\pi}\sqrt{\frac{m}{EI}} \tag{54}$$

Using the properties in an SDOF model we can calculate the dynamic response for the beam with the methods described in Section 2.

Example: **Fully clamped beam**

If the beam is fully clamped at the ends the displacement shape function may be taken as

$$\phi(x) = \frac{1}{2}\left(1 - \cos\frac{2\pi x}{\ell}\right) \tag{55}$$

and equation (34) becomes

$$\frac{3m\ell}{8}\ddot{y} + \frac{8\pi^4}{\ell^3}EIy = \frac{q\ell}{2} \tag{56}$$

or

$$0.75M\ddot{y} + 389.6\frac{EI}{\ell^3}y = F \tag{57}$$

where M is total mass and F is total load. If the static displacement pattern is used as the displacement shape function the stiffness becomes $K = 385\frac{EI}{\ell^3}$ and the load-mass factor becomes 0.77, i.e. very similar results.

4.2 Elasto-plastic beams

If the beam is subjected to a sufficiently large deformation – plastic hinges will form. For the simply supported beam, a hinge forms at beam midspan once the action effect exceeds $R_{el} = q\ell = \frac{4M_p}{\ell}$ where M_p is the plastic bending moment of the beam. The beam may be considered to behave perfectly elastic (with stiffness $K = \frac{385}{5}\frac{EI}{\ell^3}$) until the resistance $R = Ky$ becomes equal to the plastic collapse resistance in bending $R_{el} = q\ell = \frac{8M_p}{\ell}$ where M_p is the

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plastic bending moment of the cross-section. The corresponding deformation $y_{el} = \frac{R_{e\ell}}{K}$ signifies the end of the elastic deformation range.

After a hinge has been formed the stiffness of the beam vanishes. The deformation field in the mechanism phase is often assumed to be triangular as shown in Figure 6

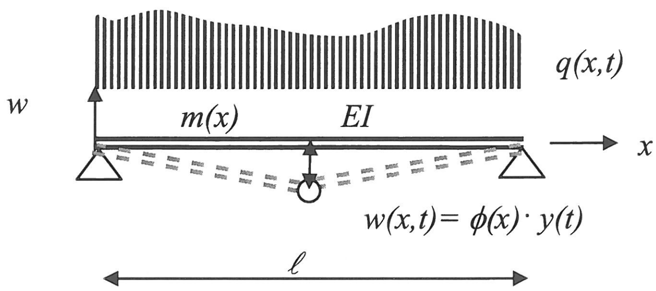

Figure 4.2 Beam in the mechanism mode

$$\varphi(x) = 1 - \frac{x}{\ell/2} \qquad x < \ell/2$$

with $\varphi(x) = 1 - \frac{x}{\ell/2}$ for the left hand side of the beam (similar on the right hand side). The dynamic equilibrium equation becomes

$$\frac{1}{3}m\ell\ddot{y} + 0 \cdot y = \frac{q\ell}{2} \tag{58}$$

or

$$0.66M\ddot{y} + 0 \cdot y = F \tag{59}$$

where the load-mass factor is 0.66.

For the fully clamped beam, hinges first form at the ends when the resistance

$$R_1 = q_1\ell = \frac{12M_p}{\ell} \tag{60}$$

is exceeded. Up to this level the stiffness is $K = \frac{384EI}{\ell^3}$. After hinges have been formed, the ends can no longer take more moment, and the beam behaves as if were simply supported. The stiffness in this phase is therefore $K = \frac{384EI}{\ell^3}$. Once the bending moments exceeds $R_{e\ell} = q_{e\ell}\ell = \frac{16M_p}{\ell}$, a hinge is also formed at midspan and a complete mechanism has been developed. Subsequently, the equilibrium is described by

$$0.66M\ddot{y} + 0 \cdot y = F$$

exactly as it is for the simply supported beam.

As seen I Figure 7 the resistance-deformation curve is bilinear up to the collapse resistance $R_{e\ell}$. This is inconvenient and hence, the concept of equivalent stiffness is introduced. It is defined such that the area under the resistance deformation curve is preserved when the bilinear curve is substituted by a single linear curve up to $R_{e\ell}$. The equivalent stiffness for the clamped beam becomes $K_{eq} = \frac{307EI}{\ell^3}$.

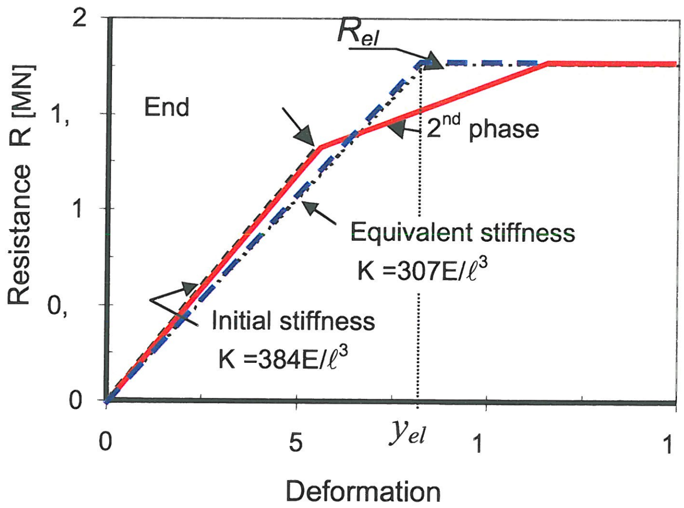

Figure 4.3 Equivalent stiffness for bi-linear resistance

4.3 Dynamic response charts

When the resistance curve is elastic-plastic or elastic plastic with large deformation "hardening" of "catenary" effect/membrane effect it becomes increasingly cumbersome to calculate the dynamic response analytically. An alternative to analytical analysis is numerical solution of equation (11) either by

i an incremental –iterative integration
ii solution of the Duhamel integral

$$y(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\lambda\omega_0(t-\tau)} \sin \omega_d d\tau$$

$$\omega_d = \omega_0 \sqrt{1-\lambda^2}$$

The response of such integration may be presented in dynamic response charts as shown in Figures 8 - 11.

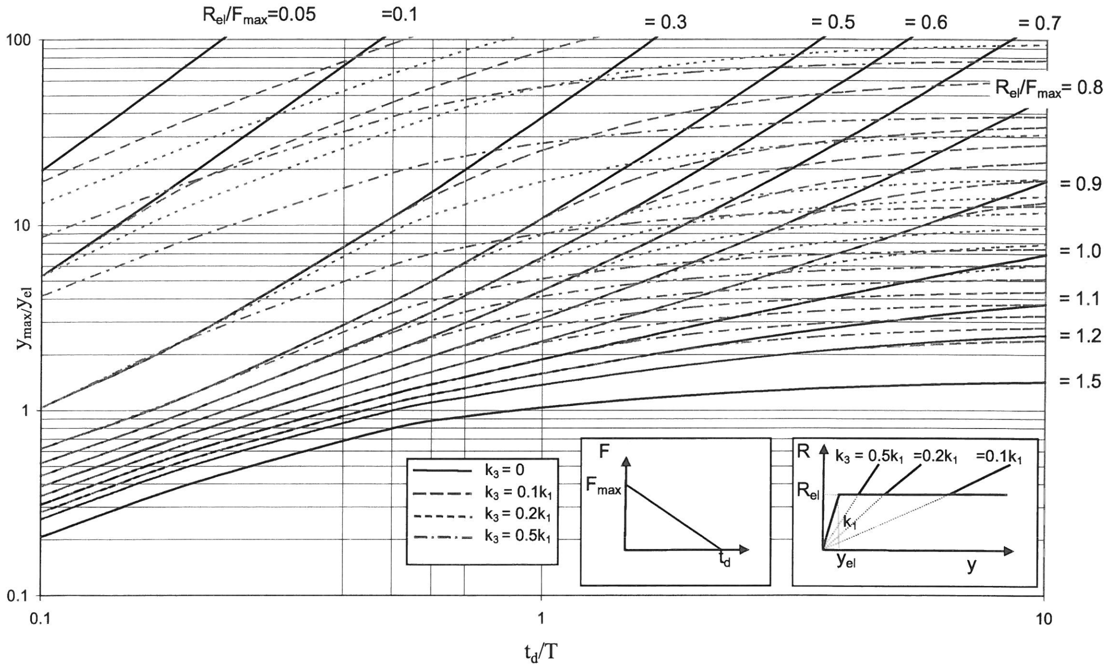

Figure 4.4 Response chart for triangular pressure pulse. Zero rise time

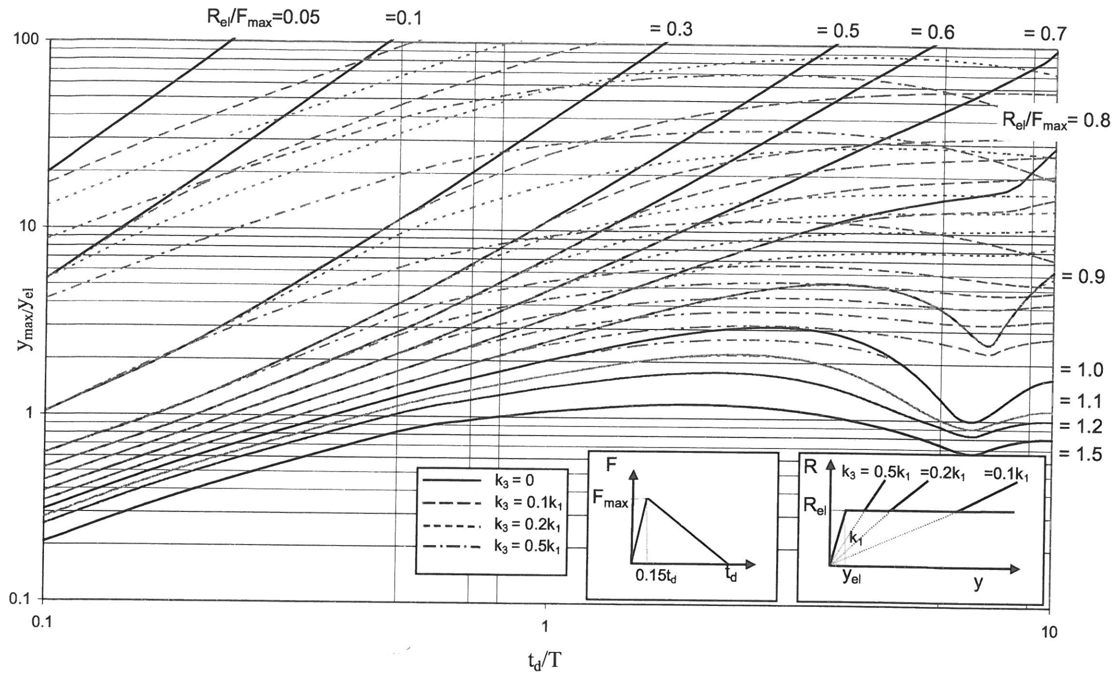

Figure 4.5 Response chart for triangular pressure pulse. Rise time $0.15t_d$

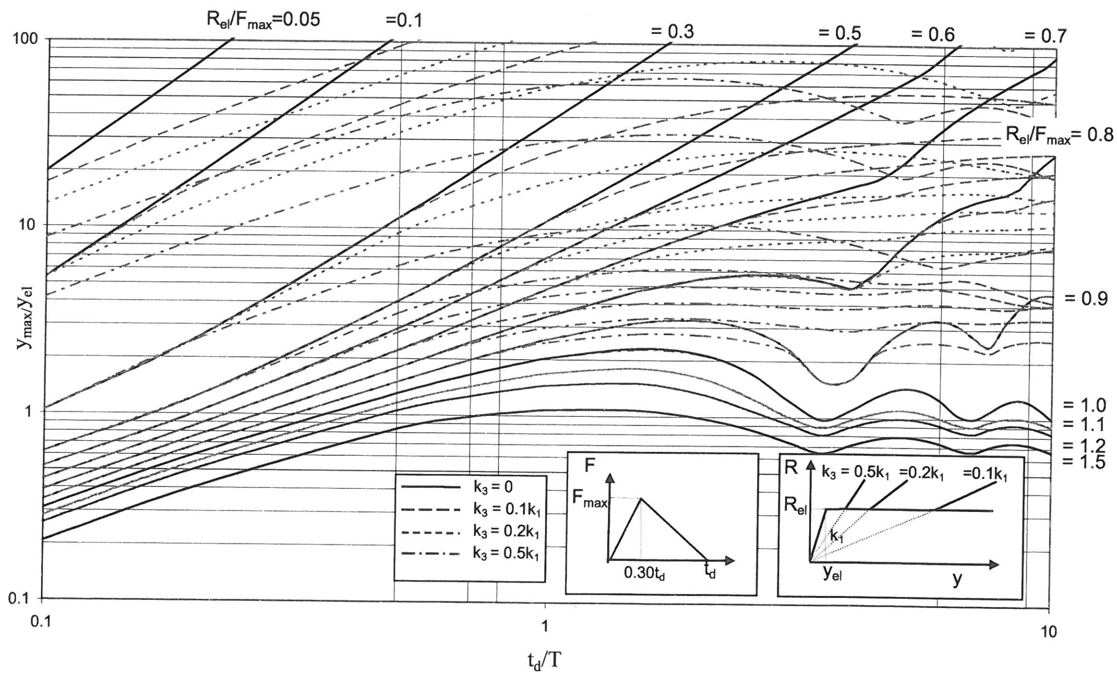

Figure 4.6 Response chart for triangular pressure pulse. Rise time $0.3t_d$

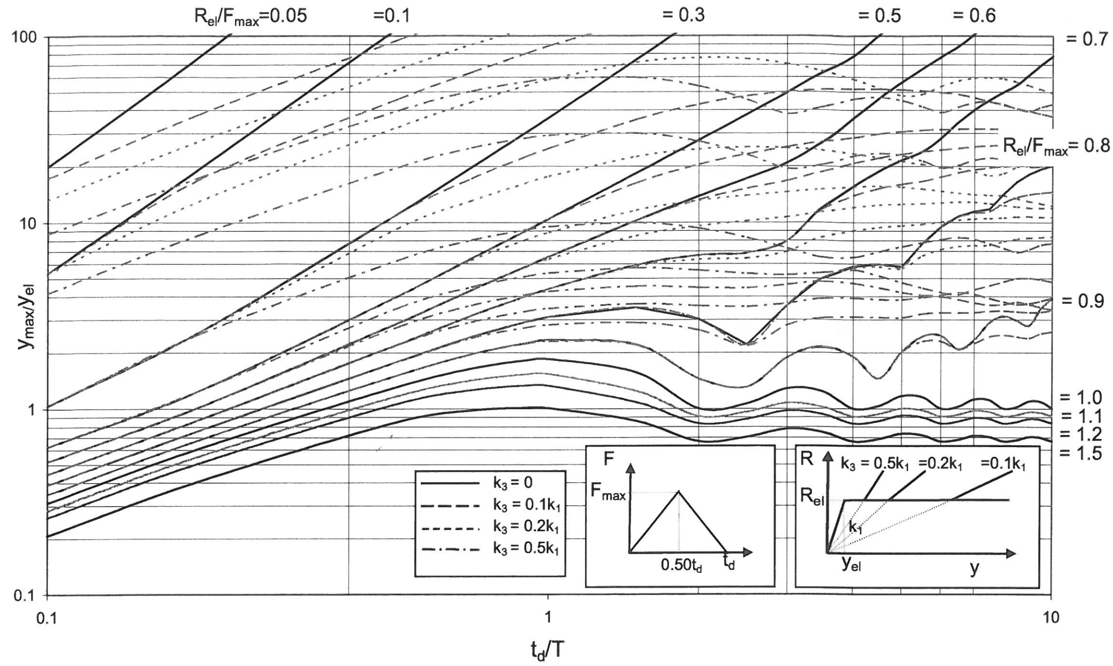

Figure 4.7 Response chart for triangular pressure pulse. Rise time $0.5t_d$

The maximum dynamic displacement $y_{\max}$ normalized versus the displacement at the end of the elastic region, $y_{e\ell}$, is plotted as a function of the pressure pulse, τ, duration normalized against the eigenperiod, T. There are different curves for each resistance/maximum pressure ratio $R_{e\ell} / F_{\max}$.

The curves plotted in Figures 15-18 relates to a triangular pressure pulse with a varying rise time-pulse duration ratio. It is observed that the curves are relatively smooth in the double logarithmic scale, except when $y_{\max} / y_{e\ell}$ is moderate for large t_d / T-ratios (e.g. for $R_{e\ell} / F_{\max} \gtrsim 0.7$): In other words for moderate or even negative "overpressures" and long duration pulses.

If the pulse duration is associated with some degree of uncertainty it is probably sound to use a regular curve and neglect the local troughs in the response.

4.4 P-I diagrams

From the dynamic response charts it is also possible to generate so-called pressure-impulse diagrams. Such diagrams may be used to identify which combinations of maximum pressures and impulses which are tolerable for a structure. They are also convenient because the starting point is an assessment of the maximum deformation the component can sustain before failure.

Say that a static analysis has shown that the component may deform up to a maximum value equal to 10 times the deformation at the end of the elastic region, i.e. $y_{max} / y_{el} = 10$. Failure could be due to excessive straining or severe local buckling on the compression side of a semi-compact cross-section. The line $y_{max} / y_{el} = 10$ is shown in Figure 4.8 where it is assumed that the pressure pulse is triangular with zero rise time. Each intersection with the response curve for a given R_{el} / F_{max} ratio represents combinations of maximum pressures and impulses creating a displacement of $y_{max} / y_{el} = 10$. The normalized pressure F_{max} / R_{el} is simply the inverse of R_{el} / F_{max} while the normalized impulse is

$$I^y = \frac{\frac{1}{2} F_{max} \cdot t_d}{R_{el} \cdot T} = \frac{1}{2} \frac{F_{max}}{R_{el}} \cdot \frac{t_d}{T} \tag{61}$$

where the normalized duration is the abscissa value at the intersection. The possible combinations of F_{max} / R_{el} versus $\frac{I}{R_{el} \cdot T}$ are plotted in Figure 4.9. The curve divides the area in two regions; below the curve all possible combinations of pressure and impulses are acceptable, above the curve they will produce excessive deformations. Such diagrams are very useful for screening of blast scenarios.

It is also observed that there are obtained an impulsive asymptote and pressure asymptote, below which all impulses and pressures are acceptable.

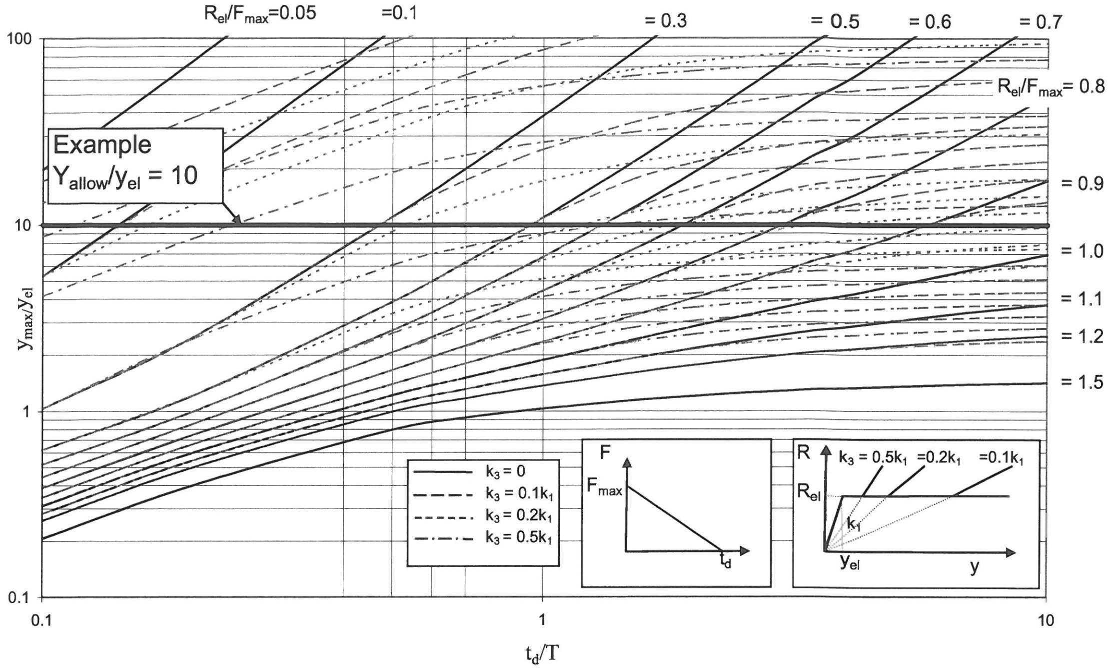

Figure 4.8 Determination of pressure and impulse for given deformation

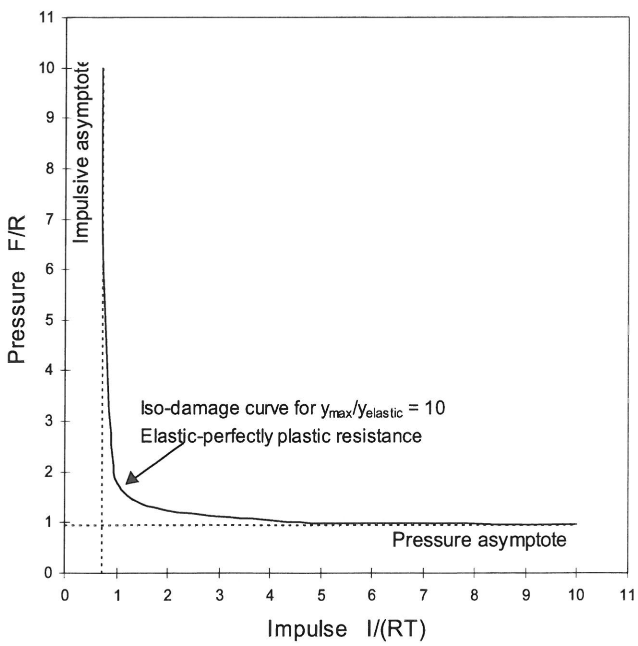

Figure 4.9 PI-diagram for given acceptable deformation

5 Simplified analysis

5.1 Maximum response

Impulsive asymptote

When the pulse is short, the speed at the end of the pulse for the SDOF model is

$$V = \int_0^{td} \frac{f(t)dt}{m} = \frac{I}{m} \tag{62}$$

The kinetic energy is given by

$$E_{kin} = \frac{1}{2} m V^2 = \frac{1}{2}\frac{I^2}{m} \tag{63}$$

Assuming that the response is elastic the kinetic strain energy is dissipated into strain energy, given by

$$E_{strain} = \frac{1}{2} k y_{\max}^2 = \frac{1}{2}\frac{1}{k} \cdot R_{el}^2 \left(\frac{y_{\max}}{y_{el}}\right)^2 \tag{64}$$

Equating strain energy and kinetic energy there is obtained

$$\frac{y_{\max}}{y_{el}} = \sqrt{\frac{k}{m}}\frac{I}{R_{el}} = \frac{2\pi}{T}\frac{I}{R_{el}} \tag{65}$$

For a triangular pulse

$$I = \frac{1}{2} F_{\max} \cdot t_d \tag{66}$$

and we get the *impulsive elastic solution*

$$\left(\frac{y_{\max}}{y_{el}}\right)_{imp,el} = \pi \frac{F_{\max}}{R_{el}} \cdot \frac{t_d}{T} \tag{67}$$

If the response becomes large, i.e., $y_{\max} / y_{el} > 1$, the structure yields. If we first neglect any hardening the strain energy becomes

$$E_{strain} = \frac{1}{2}ky_{el}^2 + R_{el}\left(y_{\max} - y_{el}\right) \tag{68}$$

If the structure experiences hardening, this must be taken into account. For large deformations we may disregard the phase with constant resistance and only take into account the "linear hardening" response

$$E_{strain} = \frac{1}{2}\alpha k y_{\max}^2 = \frac{1}{2}\alpha R_{el}\frac{y_{\max}^2}{y_{el}} \tag{69}$$

The resulting response may be considered as an interaction between the elastic-perfectly plastic response and the linear hardening response. Using elliptic interaction there is obtained

$$E_{strain} = \left\{\left(\frac{1}{2}ky_{el}^2 + R_{el}\left(y_{\max} - y_{el}\right)\right)^2 + \left(\frac{1}{2}\alpha R_{el}\frac{y_{\max}^2}{y_{el}}\right)^2\right\}^{\frac{1}{2}} \tag{70}$$

The kinetic energy is as before. Solving for the maximum displacement we get the *impulsive elastic-plastic solution*

$$\left(\frac{y_{\max}}{y_{el}}\right)_{imp,e-p} = \frac{1}{\alpha}\sqrt{\sqrt{4 + \alpha^2\pi^4\left(\frac{F_{\max}}{R_{el}}\right)^4\left(\frac{\tau}{T}\right)^4} - 2} \tag{71}$$

The results of the asymptotic impulsive analysis are plotted in Figure 5.1 When $R_{el}/F_{max} < 0.3$ the elastic-plastic with hardening response, Equation (71) is used. For $R_{el}/F_{max} > 0.5$ the displacement is based on the elastic response, Equation (67). It is observed that both equations give very good results when the duration is short ~ $\tau/T < 0.5$.

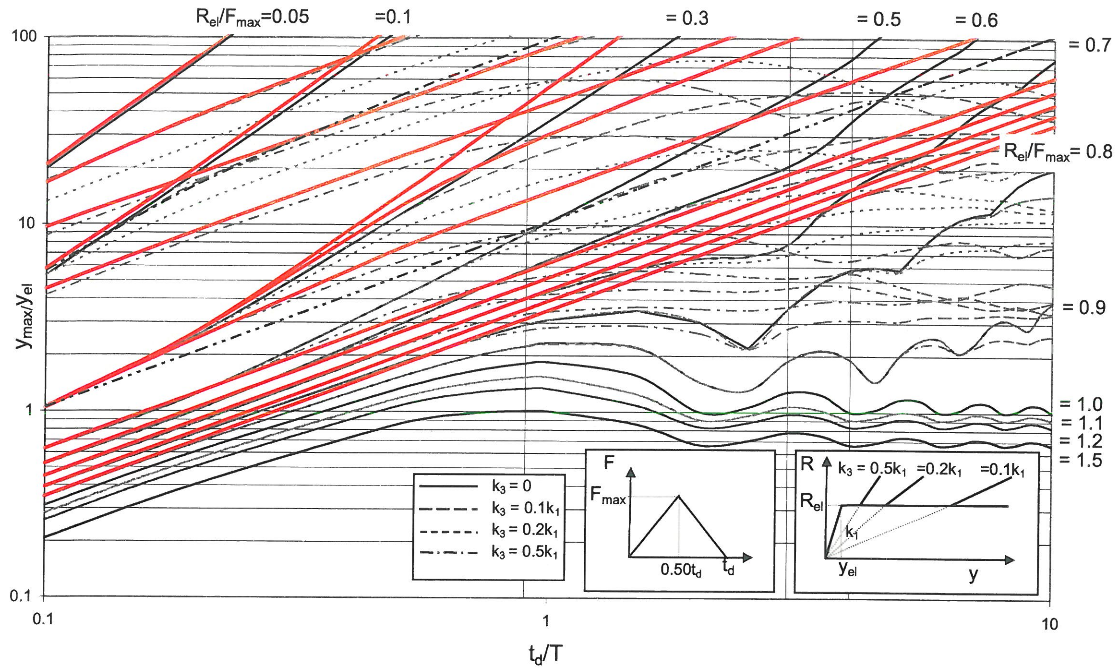

Figure 5.1 Maximum response according to asymptotic impulsive analysis

Long duration dynamic asymptote

Let us assume that the response curve is bilinear, i.e. an initial elastic phase followed by a mechanism phase with no "hardening". Assume that the response becomes much larger than the elastic region, i.e. $y_{max} / y_{e\ell} >> 1$. We then neglect the elastic-response and only take into account the plastic phase with constant resistance $R = R_{e\ell}$. This simplifies the solution. The resulting force is $F_{res} = F(t) - R_{e\ell}$. When $F_{res} > 0$ the mass is accelerated when $F_{res} < 0$ it is decelerated

$$a = \frac{F(t) - R_{e\ell}}{m} \tag{72}$$

The speed and displacement are now calculated by integration of equation (68) once and twice, respectively, i.e.:

$$V(t) = \int_0^t \frac{(F(t) - R_{e\ell})dt}{m} \tag{73}$$

$$S(t) = \int_0^t V(t)dt \tag{74}$$

Maximum displacement occurs when $\dot{V}(t = t_{\max}) = 0$. It is assumed that triangular pulse is given by

$$F(t) = F_{\max}\frac{t}{\beta\tau} \qquad \text{for } t \le \beta\tau$$
$$= F_{\max}\left(\frac{t-\beta\tau}{\tau-\beta\tau}\right) \qquad \text{for } \beta t \le t \le \tau \tag{75}$$

where $\beta\tau$ is the rise time, $0 \le \beta \le 1$. The following result is obtained for the *large displacement dynamic solution*

$$\left(\frac{y_{\max}}{y_{e\ell}}\right)_{\text{large,dyn}} = \frac{\pi^2}{6}\left(\frac{\tau}{T}\right)^2 \cdot \frac{1}{R/F}\left(1-\frac{R}{F}\right)^2$$
$$\left\{2+\frac{R}{F}(1-2\beta))4\beta+\frac{3}{R/F}\right\} \div \frac{\pi^2}{6}\left(\frac{\tau}{T}\right)^2 \cdot \left(\frac{R}{F}\right)^2 (1-\beta)^2 \tag{76}$$

If the displacement ceases before the force has become zero the following equation apply

$$\left(\frac{y_{\max}}{y_{e\ell}}\right)_{\text{large,}dyn} = \frac{2}{3}\pi^2 \cdot \frac{1}{R/F} \cdot \left(1-\frac{R}{F}\right)^3 \cdot \left(2\beta+2\sqrt{1-\beta}\right)\left(\frac{\tau}{T}\right)^2 \tag{77}$$

This holds true as long as

$$\frac{R}{F} \ge \frac{1}{1+\sqrt{1-\beta}} \tag{78}$$

so that the speed ends before pressure pulse has ceased.

The large displacement asymptote are shown in Figure 5.2. It is observed that for the asymptotic solution gives a quite good solution when the displacement becomes large.

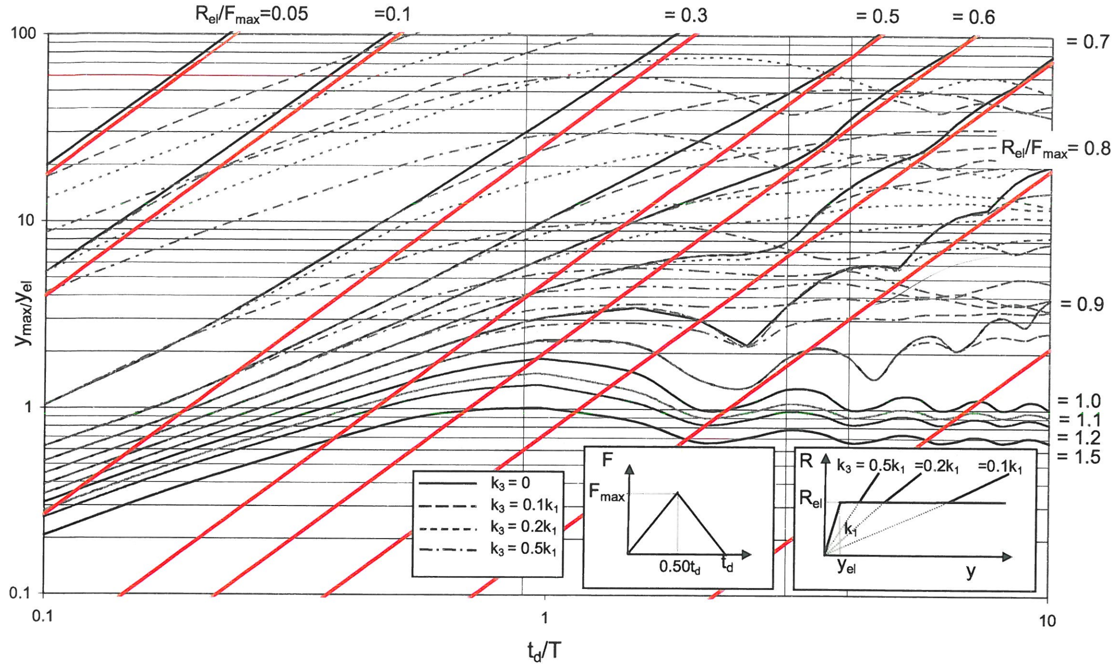

Figure 5.2 Maximum response according to large displacement asymptotic solution

Long duration static asymptote

When the pressure duration is long it may be assumed that the maximum displacement is reached when the force is maximum; it is actually presupposed that the force is constant during deformation. The external work is therefore

$$E_{ext} = F_{\max} \cdot y_{\max} \tag{79}$$

The strain energy contains three phases with stiffnesses K_1, $K_2 = 0$, $K_3 = \alpha K$. The total work can be expressed as, see Figure 13,

Figure 13

$$E_{strain} = \frac{1}{2}\alpha k \cdot y_{\max}^2 + \frac{1}{2}R_{e\ell}\left(\frac{y_{e\ell}}{\alpha} - y_{e\ell}\right) \tag{80}$$

Equating E_{ext} and E_{strain} we get *the large displacement static solution*

$$\left(\frac{y_{\max}}{y_{e\ell}}\right)_{\text{large,stat}} = \frac{F_{\max}}{R_{e\ell}} \cdot \frac{1}{\alpha}\left(1 + \sqrt{1 - \left(\frac{R_{e\ell}}{F_{\max}}\right)^2 (1-\alpha)}\right) \tag{81}$$

Sample values of the maximum displacement according to the static solution (Equation (81)) is plotted in Figure 5.3. Corresponding values according to complete solution and asymptotic solution are plotted with same indicators. It is observed that the static solution represents a conservative upper bound for the displacement.

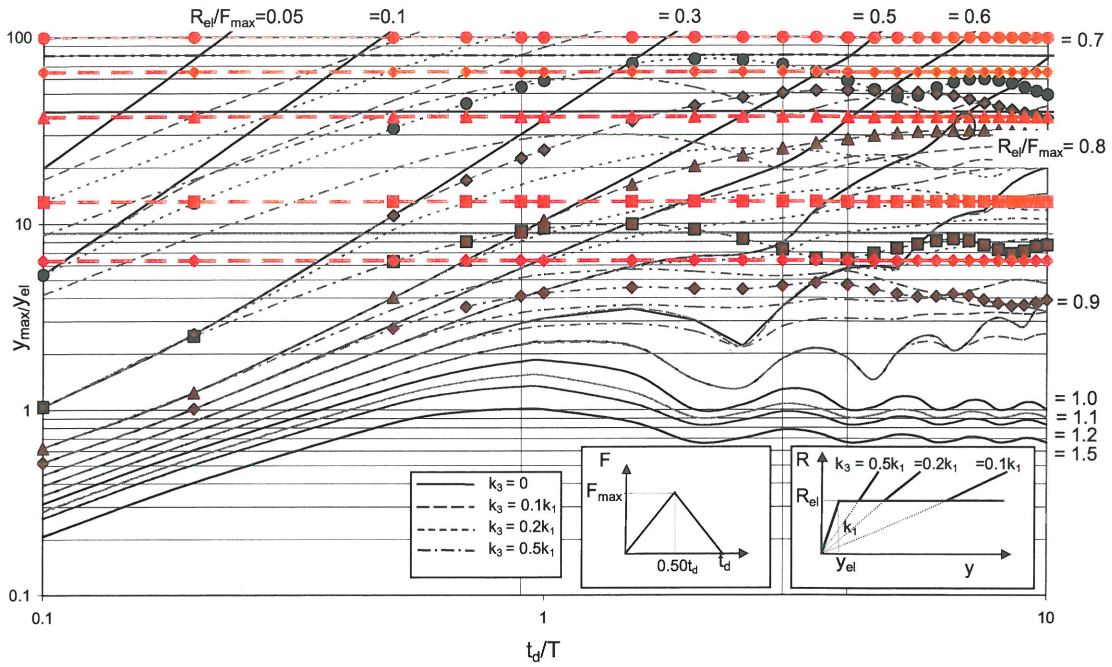

Figure 5.3 Maximum displacement according to static asymptotic analysis

Approximate calculation of maximum displacement

The maximum response may now be considered to be composed of an interaction between the various asymptotic solutions. This provides fairly good estimates of the response also in the dynamic range. When the pressure pulse is large, i.e. $R_{el} / F_{max} < 0.3$ a good solution is obtained using interaction between the impulsive and large displacement static, asymptote such that

$$\left(\frac{\left(\frac{y_{max}}{y_{e\ell}}\right)_{imp,e-p}}{\frac{y_{max}}{y_{e\ell}}}\right)^2 + \left(\frac{\left(\frac{y_{max}}{y_{e\ell}}\right)_{large,static}}{\frac{y_{max}}{y_{e\ell}}}\right)^2 = 1 \quad (82)$$

or

$$\frac{y_{max}}{y_{e\ell}} = \left(\left(\frac{y_{max}}{y_{el}}\right)^2_{imp,e-p} + \left(\frac{y_{max}}{y_{el}}\right)^2_{large,static}\right)^{1/2} \quad (83$$

When the pressure is small or moderate, $R_{el} / F_{max} < 0.3$ the elastic impulsive asymptote should be used for the impulsive range. In order to obtain good results the use of an "empirical" factor is also needed. The factor depends upon the pulse shape and R_{el} / F_{max} value. This constant is denoted $\left(\frac{y_{max}}{y_{el}}\right)_{emp}$. First, interaction between the elastic, impulsive asymptote and the empirical constant is assumed

$$\left(\frac{y_{max}}{y_{el}}\right)_1 = \left(\left(\frac{y_{max}}{y_{el}}\right)_{imp,el}^2 + \left(\frac{y_{max}}{y_{el}}\right)_{emp}^2\right)^{1/2} \tag{84}$$

Next, interaction between $\left(\frac{y_{max}}{y_{el}}\right)_1$ and the large displacement, dynamic solution is calculated according to the expression

$$\left(\frac{\left(\frac{y_{max}}{y_{el}}\right)_2}{\left(\frac{y_{max}}{y_{el}}\right)_1}\right)^2 + \left(\frac{\left(\frac{y_{max}}{y_{el}}\right)_2}{\left(\frac{y_{max}}{y_{el}}\right)_{large,dyn}}\right)^2 = 1 \tag{85}$$

which gives

$$\begin{aligned}\left(\frac{y_{max}}{y_{el}}\right)_2 &= \left\{\left(\frac{y_{max}}{y_{el}}\right)_1^{-2} + \left(\frac{y_{max}}{y_{el}}\right)_{large,dyn}^{-2}\right\}^{-\frac{1}{2}} \\ &= \left\{\left(\left(\frac{y_{max}}{y_{el}}\right)_{imp,el}^2 + \left(\frac{y_{max}}{y_{el}}\right)_{emp}^2\right)^{-1} + \left(\frac{y_{max}}{y_{el}}\right)_{large,dyn}^{-2}\right\}^{-\frac{1}{2}}\end{aligned} \tag{86}$$

Finally, interaction between $\left(\frac{y_{max}}{y_{el}}\right)_2$ and the large displacement, static solution is calculated:

$$\left(\frac{\left(\frac{y_{max}}{y_{el}}\right)}{\left(\frac{y_{max}}{y_{el}}\right)_2}\right)^2 + \left(\frac{\left(\frac{y_{max}}{y_{el}}\right)}{\left(\frac{y_{max}}{y_{el}}\right)_{large,stat}}\right)^2 = 1 \tag{87}$$

and the following relationship valid over the entire range is obtained

$$\left(\frac{y_{max}}{y_{e\ell}}\right)=\left\{\left(\frac{y_{max}}{y_{e\ell}}\right)_2^2+\left(\frac{y_{max}}{y_{e\ell}}\right)_{large,stat}^2\right\}^{\frac{1}{2}}$$

$$=\left[\left\{\left(\left(\frac{y_{max}}{y_{el}}\right)_{imp,el}^2+\left(\frac{y_{max}}{y_{el}}\right)_{emp}^2\right)^{-1}+\left(\frac{y_{max}}{y_{e\ell}}\right)_{large,dyn}^{-2}\right\}^{-1}+\left(\frac{y_{max}}{y_{e\ell}}\right)_{large,stat}^2\right]^{\frac{1}{2}} \quad (87)$$

The results of this simplified analysis are shown in Figure 5.4 for pressure pulse with equal rise time and decay time and in Figure 5.5 Maximum response for SDOF system-comparison between simplified analysis and exact solution for rise time $\tau/T = 0.15$ for rise time $\tau/T = 0.15$. A few response curves are plotted, only. It is observed that the exact curves are followed pretty well in some pulse duration ranges and with reasonable accuracy in other duration ranges. The "wave-type" response curves, especially apparent for $R_{el}/F_{max} = 0.8$ and 0.9, are not predicted. *On the other hand, unless the pulse duration is known with a very high degree of certainty, it is recommended to neglect the local "troughs" and use the more conservative "envelop" curve.*

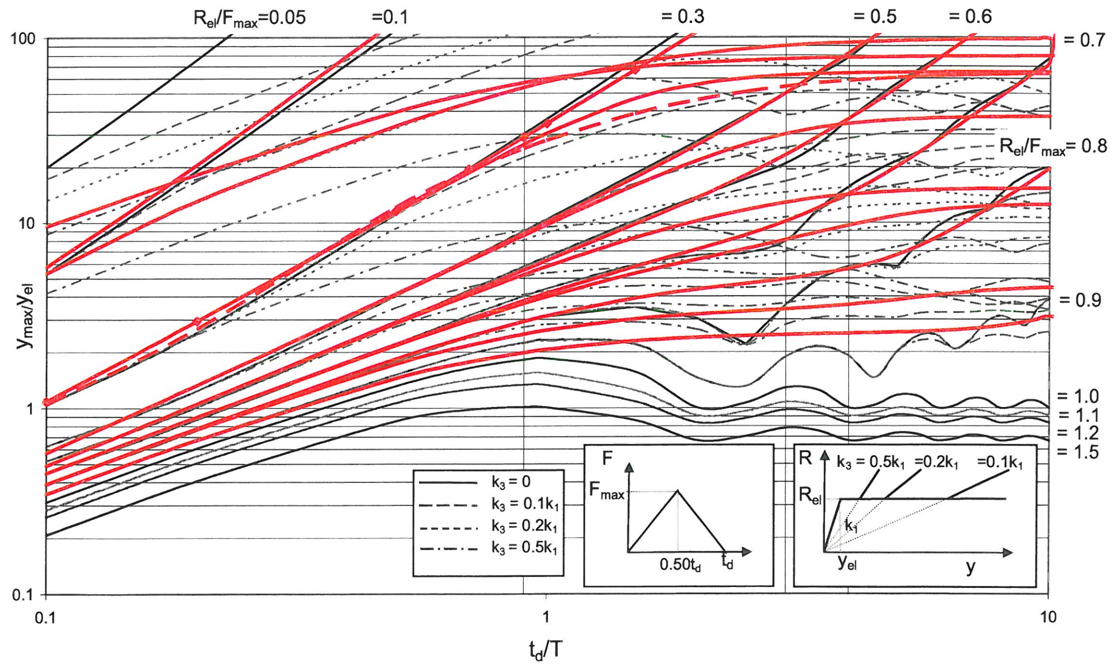

Figure 5.4 Maximum response for SDOF system-comparison between simplified analysis and exact solution for equal rise and decay time

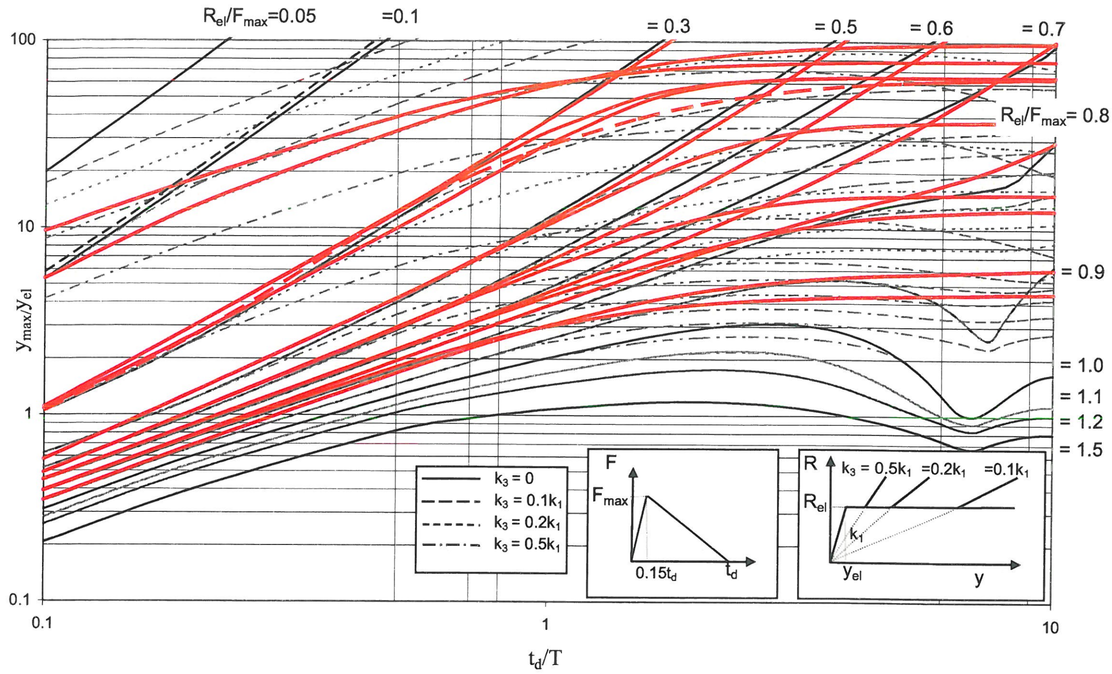

Figure 5.5 Maximum response for SDOF system-comparison between simplified analysis and exact solution for rise time τ/T = 0.15

5.2 PI-diagrams

The impulsive asymptote and the large displacement asymptote may be used to generate pressure – impulse relationships. From equation (67) we get when $I = \frac{1}{2} F_{\max} \tau$ is introduced

$$\left(\frac{I}{R_{e\ell}T}\right)_{imp} = \sqrt{\frac{1}{2}\left(\frac{y_{\max}}{y_{e\ell}}\right) - 1} \tag{88}$$

and from equation (70) the static asymptote:

$$\left(\frac{F_{\max}}{R_{e\ell}T}\right)_{stat} = \frac{1}{2}\left\{\alpha\frac{y_{\max}}{y_{e\ell}} + \left(\frac{y_{e\ell}}{\alpha y_{\max}} - \frac{y_{e\ell}}{y_{\max}}\right)\right\} \tag{89}$$

The total solution may be considered as an interaction between the impulsive asymptote and the static asymptote, i.e.

$$\left(\frac{\left(\frac{F_{\max}}{R_{e\ell}}\right)_{stat}}{\frac{F_{\max}}{R_{e\ell}}}\right)^{p}+\left(\frac{\left(\frac{I}{R_{e\ell}T}\right)_{imp}}{\frac{I}{R_{e\ell}T}}\right)^{p}=1 \tag{90}$$

or

$$\frac{F_{\max}}{R_{e\ell}}=\left(\frac{F_{\max}}{R_{e\ell}}\right)_{stat}\left\{1-\left(\frac{\left(\frac{I}{R_{e\ell}T}\right)_{imp}}{\left(\frac{I}{R_{e\ell}}T\right)}\right)^{p}\right\}^{-\frac{1}{p}} \tag{91}$$

where p is an exponent.