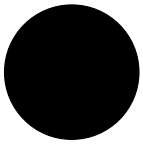
Classical Theory of Fields



Jan Myrheim
Department of Physics, NTNU

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Preface

This text has been written for an intermediate level, one semester course on the classical theory of fields and on general relativity, given during more than 40 years in Trondheim. The students entering the course should normally have some familiarity with electromagnetism and the calculus used there, with the Lagrange formulation of classical mechanics, and with special relativity.

The text consists, roughly speaking, of three main parts. The first part is mainly mathematical, introducing the differential geometry needed for field theory and especially for general relativity. The second part treats special relativity and the Lagrange formalism, with applications to the Klein–Gordon and the electromagnetic fields. The third part is an introduction to the general theory of relativity. The mathematics is collected at the beginning because it is a logical unit, and not necessarily because this is the natural order of teaching. An alternative approach may be to use the mathematical chapters not as a text book on differential geometry, but rather as a source of reference when the mathematics is needed in the field theory.

The selection of topics to be covered in a one semester course is necessarily somewhat arbitrary. For example, the theory of radiation, either electromagnetic or gravitational, is left out, although it would have found a natural place in the course. An important criterion for the selection of material has been that this course should fit in with other courses. Another guiding principle which may be visible, is the intention to teach principles and techniques. There is, unfortunately, not so much room for applications within the format of the course.

I want to thank especially Finn Bakke, who taught the course for many years and left his notes for me to use freely.

There exist of course already many good text books, and I can only hope that some students and teachers will find this one useful.

Trondheim, January 2011
Jan Myrheim

Front page illustration: A black hole with the mass of the Earth.

Some constants of nature

From "Review of Particle Physics", Physical Review D 54, 65 (1996).
See also the home page of the "Particle Data Group", `http://pdg.lbl.gov/`

Quantity	Symbol	Value
Hyperfine structure of ^{133}Cs		9 192 631 770 Hz
		(defines the second)
Speed of light in vacuum	c	299 792 458 m/s
		(defines the meter)
Planck's constant	h	$6.626\,076 \times 10^{-34}$ J s
Planck's constant, reduced	$\hbar = h/2\pi$	$1.054\,572\,7 \times 10^{-34}$ J s
		$= 6.582\,122 \times 10^{-22}$ MeV s
Elementary charge	e	$1.602\,177\,3 \times 10^{-19}$ C
Fine structure constant	$\alpha = e^2/4\pi\epsilon_0\hbar c$	1/137.035 990
Permeability of vacuum	μ_0	$4\pi \times 10^{-7}$ N/A^2
		(defines the unit of coulomb)
Permittivity of vacuum	$\epsilon_0 = 1/\mu_0 c^2$	$8.854\,187\,817\ldots \times 10^{-12}$ F/m
Newton's gravitational constant	G	6.6726×10^{-11} m^3 kg^{-1} s^{-2}
Planck mass	$m_P = \sqrt{\hbar c/G}$	2.1767×10^{-8} kg
Planck length	$L_P = \sqrt{\hbar G/c^3}$	1.6161×10^{-35} m
Planck time	$t_P = \sqrt{\hbar G/c^5}$	5.3906×10^{-44} s
Avogadro's constant	N_A	$6.022\,137 \times 10^{23}$ mol^{-1}
Boltzmann's constant	k_B	$1.380\,66 \times 10^{-23}$ J/K
		$= 8.617\,39 \times 10^{-5}$ eV/K
Electron mass	m_e	$9.109\,390 \times 10^{-31}$ kg
		$= 0.510\,999\,1$ MeV/c^2
Proton mass	m_p	$1.672\,623\,0 \times 10^{-27}$ kg
		$= 938.272\,3$ MeV/c^2
Neutron mass	m_n	$1.674\,928\,6 \times 10^{-27}$ kg
		$= 939.565\,6$ MeV/c^2
Bohr magneton	$\mu_B = e\hbar/2m_e$	$9.274\,015\,4 \times 10^{-24}$ J/T
		$= 5.788\,382\,6 \times 10^{-11}$ MeV/T
Nuclear magneton	$\mu_N = e\hbar/2m_p$	$5.050\,786\,6 \times 10^{-27}$ J/T
		$= 3.152\,451\,7 \times 10^{-14}$ MeV/T
Size of an electron:		
Classical electron radius	$r_e = e^2/4\pi\epsilon_0 m_e c^2$	$2.817\,940\,9 \times 10^{-15}$ m
Compton wavelength of electron	$\hbar/m_e c = r_e/\alpha$	$3.861\,593\,2 \times 10^{-13}$ m
Bohr atomic radius	$a = r_e/\alpha^2$	$0.529\,177\,25 \times 10^{-10}$ m
The astronomical unit	AU	$1.495\,978\,706\,6 \times 10^{11}$ m
Tropical year (equinox to equinox)		31 556 925.2 s
Siderial year (relative to fixed stars)		31 558 149.8 s
Mass of Sun	$M_\odot$	$1.988\,9 \times 10^{30}$ kg
Radius of Sun (at equator)	$R_\odot$	6.96×10^8 m
Mass of Earth	$M_\oplus$	$5.973\,70 \times 10^{24}$ kg
Radius of Earth (at equator)	$R_\oplus$	$6.378\,140 \times 10^6$ m

Contents

Chapter 1

A brief field guide

This first chapter is an assorted collection of notes on units, notation etc., which may be used as a field guide to be consulted whenever needed. The immediately following chapters give a brief introduction to differential geometry, which is the mathematics of classical field theory in general, and of the general theory of relativity in particular.

It is certainly possible to read the first chapters as a mathematics text book, before moving on to the physical applications. However, since the physics motivates the mathematics, some readers may prefer to start with Chapter 8, on the special theory of relativity, and to use all the mathematical chapters as a field guide.

1.1 Electromagnetic unit systems

We use here SI units, also called MKSA units: meter, kilogram, second and ampère. Unfortunately, this is only one out of at least three different electromagnetic unit systems in common use, the other two being Gaussian and Heaviside–Lorentz units. The differences show up in Coulomb's law for the potential energy V between two point charges q_1 and q_2 at a relative distance r,

$$V = k\,\frac{q_1 q_2}{r}\,. \tag{1.1}$$

The proportionality constant k is different in the three unit systems,

$$\begin{aligned} k &= \frac{1}{4\pi\epsilon_0} && \text{in SI units;} \\ k &= 1 && \text{in Gaussian units;} \\ k &= \frac{1}{4\pi} && \text{in Heaviside–Lorentz units.} \end{aligned}$$

The fine structure constant α, given by the elementary charge e, the reduced Planck's constant $\hbar$ and the speed of light c, has the same numerical value in the three unit systems,

$$\alpha = k\,\frac{e^2}{\hbar c} = \frac{1}{137.035\,990}\,. \tag{1.2}$$

The second place where differences show up is in the expression for the Lorentz force on an electric point charge q moving with a velocity $\boldsymbol{v}$ in an electromagnetic field in vacuum,

$$\begin{aligned} \boldsymbol{F} &= q\,(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) && \text{(SI)}; \\ \boldsymbol{F} &= q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) && \text{(Gauss or Heaviside–Lorentz).} \end{aligned} \tag{1.3}$$

Here $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ the magnetic flux density at the point where the charge is located. As can be seen, $\boldsymbol{E}$ and $\boldsymbol{B}$ have the same dimension in Gaussian units, and also in Heaviside–Lorentz units, whereas an expression like $|\boldsymbol{E}|/|\boldsymbol{B}|$ has the dimension of velocity in the SI system.

1.2 Index conventions

We use greek letters $\alpha, \beta, \mu, \nu, \ldots$ as indices to enumerate time and space coordinates. They run then from 0 to 3, and the time coordinate is usually labelled 0. Spatial coordinates alone we enumerate with latin indices $i, j, \ldots$, running from 1 to 3, or more generally from 1 to d, where d is the dimension. We use latin indices in most other cases, e.g. for enumerating generalized coordinates in Lagrangian mechanics, or field componentes.

Unless stated otherwise, we use the summation convention that an index occurring twice in a product is to be summed over. In tensor expressions, the summation index normally occurs as an upper and a lower index, otherwise the sum may depend on which coordinate system is used.

The Kronecker δ symbol

This has two indices that may be latin or greek, upper or lower, depending on the context. By definition,

$$\delta^i_j = \delta^{ij} = \delta_{ij} = \begin{cases} 1 & \text{when} \quad i = j, \\ 0 & \text{when} \quad i \neq j. \end{cases} \tag{1.4}$$

The Levi–Civita symbol

In an n dimensional space, the Levi–Civita symbol $\epsilon^{\mu\nu\ldots\sigma}$ has n indices, and is antisymmetric under an interchange of any two indices. The number of components is n^n. Any component with two indices taking the same value must vanish because of the antisymmetry. Of the $n!$ components whose n indices take n different values, half are $+1$ and the other half -1. We define here the Levi–Civita symbols with upper and lower indices to be equal, $\epsilon_{\mu\nu\ldots\sigma} = \epsilon^{\mu\nu\ldots\sigma}$.

The determinant $\det(A)$ of an $n \times n$ matrix A may be defined by the relation

$$\epsilon^{\kappa\lambda\ldots\nu} A^\alpha_\kappa A^\beta_\lambda \cdots A^\delta_\nu = \det(A)\, \epsilon^{\alpha\beta\ldots\delta} \,. \tag{1.5}$$

In fact, the left hand side of this equation must be proportional to $\epsilon^{\alpha\beta\ldots\delta}$, because it is totally antisymmetric in the indices $\alpha, \beta, \ldots, \delta$. The proportionality constant is the determinant.

The Levi–Civita symbol in a two dimensional space, as an example, has $2^2 = 4$ components,

$$\epsilon^{12} = -\epsilon^{21} = 1 \,, \qquad \epsilon^{11} = \epsilon^{22} = 0 \,, \tag{1.6}$$

and we see that

$$\epsilon^{kl} A^i_k A^j_l = A^i_1 A^j_2 - A^i_2 A^j_1 = (A^1_1 A^2_2 - A^1_2 A^2_1)\, \epsilon^{ij} = \det(A)\, \epsilon^{ij} \,. \tag{1.7}$$

In three dimensions the number of components is $3^3 = 27$, and we have

$$\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = -\epsilon^{213} = -\epsilon^{132} = -\epsilon^{321} = 1 \,, \qquad \text{all other } \epsilon^{ijk} = 0 \,. \tag{1.8}$$

In the four dimensional spacetime, the number of components is $4^4 = 256$, all determined by the antisymmetry and by the definition $\epsilon^{0123} = 1$.

Note that the convention used here, that $\epsilon_{\mu\nu\ldots\sigma} = \epsilon^{\mu\nu\ldots\sigma}$, differs from a convention frequently used in the special theory of relativity, that $\epsilon_{\mu\nu\rho\sigma} = -\epsilon^{\mu\nu\rho\sigma}$. The last convention is natural if $\epsilon^{\mu\nu\rho\sigma}$ is regarded as a tensor. Then the factor -1 appearing when the indices are lowered, is the determinant of the metric tensor. In the general theory of relativity, on the other hand, where the metric tensor is more freely variable and may have a determinant different from -1, it seems natural to define the Levi–Civita symbol independently of the metric tensor. Under a coordinate transformation which does not preserve volume, the Levi–Civita symbol, as defined here, transforms not as a tensor but as a *tensor density*, as defined in Chapter 3.

1.3 Three dimensional vector notation

In three dimensional Euclidean space we use a special vector notation, exemplified below. When indices are occasionally used, they are always written as lower latin indices. An index occurring twice in a product is to be summed over, but summing over two lower indices means that a formula is valid only in Euclidean coordinate systems, where the metric tensor equals the identity matrix, $g_{ij} = \delta_{ij}$. The examples will demonstrate the rules.

- Standard unit vectors: $\mathbf{i} = \mathbf{e}_x = \mathbf{e}_1 \,, \quad \mathbf{j} = \mathbf{e}_y = \mathbf{e}_2 \,, \quad \mathbf{k} = \mathbf{e}_z = \mathbf{e}_3 \,.$
- Position vector: $\boldsymbol{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3 = x_i\,\mathbf{e}_i \,,$
  differential: $\mathrm{d}\boldsymbol{r} = \mathrm{d}x\,\mathbf{i} + \mathrm{d}y\,\mathbf{j} + \mathrm{d}z\,\mathbf{k} \,.$
- General vector: $\boldsymbol{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k} = A_1\,\mathbf{e}_1 + A_2\,\mathbf{e}_2 + A_3\,\mathbf{e}_3 = A_i\,\mathbf{e}_i \,.$
- Scalar product between vectors: $\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z = A_i B_i \,,$
  and in particular: $|\boldsymbol{A}|^2 = \boldsymbol{A}^2 = \boldsymbol{A} \cdot \boldsymbol{A} = {A_x}^2 + {A_y}^2 + {A_z}^2 = A_i A_i \,.$
- Vector product: $\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\,\mathbf{i} + (A_z B_x - A_x B_z)\,\mathbf{j} + (A_x B_y - A_y B_x)\,\mathbf{k} \,,$
  in index notation: $(\boldsymbol{A} \times \boldsymbol{B})_i = \epsilon_{ijk} A_j B_k \,.$
- Metric: $\mathrm{d}s^2 = |\mathrm{d}\boldsymbol{r}|^2 = \mathrm{d}\boldsymbol{r}^2 = \mathrm{d}\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{r} = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \,.$
- Volume element: $dV = \mathrm{d}^3\boldsymbol{r} = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \,.$
- Nabla, the gradient operator:

$$\nabla = \frac{\partial}{\partial \boldsymbol{r}} = \mathbf{i}\,\frac{\partial}{\partial x} + \mathbf{j}\,\frac{\partial}{\partial y} + \mathbf{k}\,\frac{\partial}{\partial z} \,. \tag{1.9}$$

- Laplace operator:

$$\Delta = \nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \,. \tag{1.10}$$

Example: Polar coordinates

The polar coordinates (r, θ, φ) are related to the Euclidean coordinates (x, y, z) by the formulae

$$x = r\sin\theta\cos\varphi\ , \qquad y = r\sin\theta\sin\varphi\ , \qquad z = r\cos\theta\ . \tag{1.11}$$

The definition gives, by the chain rule for differentiation, that

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}r\,\frac{\partial \boldsymbol{r}}{\partial r} + \mathrm{d}\theta\,\frac{\partial \boldsymbol{r}}{\partial \theta} + \mathrm{d}\varphi\,\frac{\partial \boldsymbol{r}}{\partial \varphi} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\varphi\,\mathbf{e}_\varphi\ , \tag{1.12}$$

when we introduce the basis vectors

$$\begin{aligned}
\mathbf{e}_r &= \frac{\partial \boldsymbol{r}}{\partial r} = \sin\theta\cos\varphi\,\mathbf{i} + \sin\theta\sin\varphi\,\mathbf{j} + \cos\theta\,\mathbf{k}\ , \\
\mathbf{e}_\theta &= \frac{1}{r}\,\frac{\partial \boldsymbol{r}}{\partial \theta} = \cos\theta\cos\varphi\,\mathbf{i} + \cos\theta\sin\varphi\,\mathbf{j} - \sin\theta\,\mathbf{k}\ , \\
\mathbf{e}_\varphi &= \frac{1}{r\sin\theta}\,\frac{\partial \boldsymbol{r}}{\partial \varphi} = -\sin\varphi\,\mathbf{i} + \cos\varphi\,\mathbf{j}\ ,
\end{aligned} \tag{1.13}$$

which are orthonormal, i.e. they are orthogonal unit vectors,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}\ , \quad \text{with} \qquad i, j = r, \theta, \varphi\ . \tag{1.14}$$

It follows that

$$\mathrm{d}\boldsymbol{r}^2 = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\ . \tag{1.15}$$

Let $f = f(\boldsymbol{r})$. Then we have, according to the chain rule for differentiation, that

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial \boldsymbol{r}}{\partial r}\cdot\frac{\partial f}{\partial \boldsymbol{r}} = \mathbf{e}_r\cdot\nabla f\ , \\
\frac{\partial f}{\partial \theta} &= \frac{\partial \boldsymbol{r}}{\partial \theta}\cdot\frac{\partial f}{\partial \boldsymbol{r}} = r\,\mathbf{e}_\theta\cdot\nabla f\ , \\
\frac{\partial f}{\partial \varphi} &= \frac{\partial \boldsymbol{r}}{\partial \varphi}\cdot\frac{\partial f}{\partial \boldsymbol{r}} = r\sin\theta\,\mathbf{e}_\varphi\cdot\nabla f\ .
\end{aligned} \tag{1.16}$$

Consequently, we have

$$\begin{aligned}
\nabla f &= \mathbf{e}_r\,(\mathbf{e}_r\cdot\nabla f) + \mathbf{e}_\theta\,(\mathbf{e}_\theta\cdot\nabla f) + \mathbf{e}_\varphi\,(\mathbf{e}_\varphi\cdot\nabla f) \\
&= \mathbf{e}_r\,\frac{\partial f}{\partial r} + \mathbf{e}_\theta\,\frac{1}{r}\,\frac{\partial f}{\partial \theta} + \mathbf{e}_\varphi\,\frac{1}{r\sin\theta}\,\frac{\partial f}{\partial \varphi}\ ,
\end{aligned} \tag{1.17}$$

which may also be written as follows,

$$\nabla = \mathbf{e}_r\,\frac{\partial}{\partial r} + \mathbf{e}_\theta\,\frac{1}{r}\,\frac{\partial}{\partial \theta} + \mathbf{e}_\varphi\,\frac{1}{r\sin\theta}\,\frac{\partial}{\partial \varphi}\ . \tag{1.18}$$

1.4 The Dirac δ function

The Dirac δ function is the generalization of the Kronecker symbol δ_{ij} to the case when the indices i and j are continuous variables. The one dimensional δ function is defined by the relation

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x-y)\, f(x) = f(y)\ , \tag{1.19}$$

valid for an arbitrary continuous function $f = f(x)$. The δ function is symmetric, $\delta(x-y) = \delta(y-x)$, because the substitution $u = -x$ gives that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(y-x)\, f(x) = \int_{-\infty}^{\infty} \mathrm{d}u\, \delta(y+u)\, f(-u) = f(y)\ . \tag{1.20}$$

More generally we have that

$$\delta(ax) = \frac{1}{|a|}\, \delta(x) \tag{1.21}$$

when a is constant, $a \neq 0$, since the substitution $u = |a|\, x$ gives that

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(ax)\, f(x) = \frac{1}{|a|} \int_{-\infty}^{\infty} \mathrm{d}u\, \delta(\pm u)\, f\left(\frac{u}{|a|}\right) = \frac{1}{|a|}\, f(0)\ . \tag{1.22}$$

Even more generally, when g is a differentiable function with $g(0) = 0$, with $g(x) = 0$ only for $x = 0$, and with $g'(0) \neq 0$, we will have that

$$\delta(g(x)) = \frac{1}{|g'(0)|}\, \delta(x) \tag{1.23}$$

The derivatives of the δ function may be defined formally in the usual way,

$$\begin{aligned} \delta'(x) &= \lim_{h\to 0} \frac{\delta(x+h) - \delta(x-h)}{2h}\ , \\ \delta''(x) &= \lim_{h\to 0} \frac{\delta(x+h) + \delta(x-h) - 2\delta(x)}{h^2}\ . \end{aligned} \tag{1.24}$$

This then gives that

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}x\, \delta'(x-y)\, f(x) &= \lim_{h\to 0} \frac{f(y-h) - f(y+h)}{2h} = -f'(y)\ , \\ \int_{-\infty}^{\infty} \mathrm{d}x\, \delta''(x-y)\, f(x) &= \lim_{h\to 0} \frac{f(y-h) + f(y+h) - 2f(y)}{h^2} = f''(y)\ . \end{aligned} \tag{1.25}$$

These results can alternatively be derived by partial integrations, for example,

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \delta'(x-y)\, f(x) = -\int_{-\infty}^{\infty} \mathrm{d}x\, \delta(x-y)\, f'(x) = -f'(y)\ . \tag{1.26}$$

The δ functions in higher dimensions can be defined in a similar way, for example in three dimensions,

$$\int \mathrm{d}^3\boldsymbol{r}\, \delta^{(3)}(\boldsymbol{r}-\boldsymbol{s})\, f(\boldsymbol{r}) = f(\boldsymbol{s})\ . \tag{1.27}$$

The three dimensional δ function is written as $\delta^{(3)}(\boldsymbol{r})$, or simply as $\delta(\boldsymbol{r})$ if there is no danger of confusion. It is the product of one dimensional δ functions,

$$\delta(\boldsymbol{r}) = \delta(x)\, \delta(y)\, \delta(z)\ . \tag{1.28}$$

1.5 Green functions

Take an inhomogeneous differential equation, with boundary conditions that make the solution uniquely defined. For example the Poisson equation in three dimensions,

$$\nabla^2 f = g \ , \tag{1.29}$$

where $f = f(\boldsymbol{r})$ is an unknown function satisfying the boundary condition that $f(\boldsymbol{r}) \to 0$ as $|\boldsymbol{r}| \to \infty$. The right hand side of the equation, the known source $g = g(\boldsymbol{r})$, is assumed to be localized, so that it vanishes outside some region of finite extent.

Then the *Green function* of this equation with boundary conditions is the solution f in the case when the source g is the Dirac δ function. To be more precise, the Green function $G = G(\boldsymbol{r}; \boldsymbol{s})$ is a function of two points $\boldsymbol{r}$ and $\boldsymbol{s}$ such that $G(\boldsymbol{r}; \boldsymbol{s}) \to 0$ when $|\boldsymbol{r}| \to \infty$, and

$$\nabla^2 G(\boldsymbol{r}; \boldsymbol{s}) = \delta(\boldsymbol{r} - \boldsymbol{s}) \ , \tag{1.30}$$

where the Laplace operator ∇^2 differentiates with respect to $\boldsymbol{r}$. The Green function of the Poisson equation gives the solution for a general right hand side g as

$$f(\boldsymbol{r}) = \int \mathrm{d}^3\boldsymbol{s}\, G(\boldsymbol{r}; \boldsymbol{s})\, g(\boldsymbol{s}) \ , \tag{1.31}$$

as we can see directly by substitution into the equation. Thus, the operation $g \mapsto f$ defined by this integral, is the inverse of the operation $f \mapsto g = \nabla^2 f$.

In our example here we have $G(\boldsymbol{r}; \boldsymbol{s}) = G(\boldsymbol{r} - \boldsymbol{s})$, where $G = G(\boldsymbol{r})$ is the solution of the equation

$$\nabla^2 G = \delta \ . \tag{1.32}$$

This simplification is possible because of the translational symmetry of the Poisson equation. Rotational symmetry implies the further simplification that $G = G(\boldsymbol{r})$ is a function of the radius $r = |\boldsymbol{r}|$ alone, that is, $G = G(r)$.

One method for solving the equation $\nabla^2 G = \delta$ is to integrate over a volume V which is a sphere with centre at the origin and radius $R > 0$. The integral of the left hand side is, by the divergence theorem, equal to a surface integral over the surface S of the sphere,

$$\int_V \mathrm{d}V\, \nabla^2 G = \int_S \mathrm{d}S\, \mathbf{e}_r \cdot \nabla G = 4\pi R^2\, G'(R) \ . \tag{1.33}$$

Here $\mathbf{e}_r = \boldsymbol{r}/r$ is the unit vector in the radial direction. The integral of the right hand side is

$$\int_V \mathrm{d}V\, \delta(\boldsymbol{r}) = 1 \ . \tag{1.34}$$

Equating the two integrals gives the differential equation

$$G'(R) = \frac{1}{4\pi R^2} \ . \tag{1.35}$$

With the boundary condition $G(R) \to 0$ as $R \to \infty$ the solution is

$$G(R) = -\frac{1}{4\pi R} \ . \tag{1.36}$$

Thus we find the Green function

$$G(\boldsymbol{r};\boldsymbol{s}) = G(|\boldsymbol{r}-\boldsymbol{s}|) = -\frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{s}|}\,. \tag{1.37}$$

In this result we recognize the Coulomb potential. The Green function for the Poisson equation is the electrostatic potential of a unit point charge.

A second method for solving the same equation is the following Fourier transformation,

$$f(\boldsymbol{r}) = \int \mathrm{d}^3\boldsymbol{k}\,\widetilde{f}(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,, \qquad g(\boldsymbol{r}) = \int \mathrm{d}^3\boldsymbol{k}\,\widetilde{g}(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,. \tag{1.38}$$

It implies that

$$\nabla^2 f(\boldsymbol{r}) \;=\; -\int \mathrm{d}^3\boldsymbol{k}\,|\boldsymbol{k}|^2\,\widetilde{f}(\boldsymbol{k})\,\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,. \tag{1.39}$$

The inverse Fourier transformation gives that

$$\widetilde{g}(\boldsymbol{k}) = \frac{1}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{r}\,g(\boldsymbol{r})\,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}\,. \tag{1.40}$$

Thus, the Fourier transformed differential equation $\nabla^2 G = \delta$ is the simple algebraic equation

$$-|\boldsymbol{k}|^2\,\widetilde{G}(\boldsymbol{k}) = \frac{1}{(2\pi)^3}\,. \tag{1.41}$$

We see that the Fourier transform of the Green function, $\widetilde{G} = \widetilde{G}(\boldsymbol{k})$, becomes singular at $\boldsymbol{k} = 0$. A trick for avoiding this singularity is to solve instead the modified equation

$$\nabla^2 G - \kappa^2 G = \delta\,, \tag{1.42}$$

with $\kappa > 0$, and afterwards let $\kappa \to 0+$. The Green function of the modified equation is

$$\begin{aligned} G_\kappa(\boldsymbol{r}) \;&=\; -\frac{1}{(2\pi)^3}\int \mathrm{d}^3\boldsymbol{k}\,\frac{\mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}}}{|\boldsymbol{k}|^2+\kappa^2} \\ &=\; -\frac{1}{(2\pi)^3}\int_0^\infty k^2\,\mathrm{d}k\int_{-1}^1 \mathrm{d}(\cos\theta)\int_0^{2\pi}\mathrm{d}\varphi\,\frac{\mathrm{e}^{-\mathrm{i}kr\cos\theta}}{k^2+\kappa^2}\,. \end{aligned} \tag{1.43}$$

Here $k = |\boldsymbol{k}|$, and we define θ and φ as the polar angles of the vector $\boldsymbol{k}$ relative to an axis along $\boldsymbol{r}$, so that $\boldsymbol{k}\cdot\boldsymbol{r} = kr\cos\theta$. After performing the angular integrations, we get that

$$\begin{aligned} G_\kappa(\boldsymbol{r}) = G_\kappa(r) \;&=\; -\frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}k\,\frac{k^2}{k^2+\kappa^2}\left(-\frac{\mathrm{e}^{-\mathrm{i}kr}}{\mathrm{i}kr}+\frac{\mathrm{e}^{\mathrm{i}kr}}{\mathrm{i}kr}\right) \\ &=\; \frac{\mathrm{i}}{(2\pi)^2\,r}\int_{-\infty}^\infty \mathrm{d}k\,\frac{k\,\mathrm{e}^{\mathrm{i}kr}}{k^2+\kappa^2}\,. \end{aligned} \tag{1.44}$$

In the last integral, the integration contour along the real axis can be closed in the complex plane by a half circle of infinite radius in the upper half plane. The half circle does not contribute to the integral, because the integrand vanishes rapidly in the limit $k \to \infty$, when

k is complex with a positive imaginary part. The integrand has one single pole in the upper half plane, at $k = \mathrm{i}\kappa$, and the residue of the pole is

$$\lim_{k\to\mathrm{i}\kappa}\ (k-\mathrm{i}\kappa)\,\frac{k\,\mathrm{e}^{\mathrm{i}kr}}{k^2+\kappa^2} = \frac{1}{2}\,\mathrm{e}^{-\kappa r}\ . \tag{1.45}$$

The value of the integral is $2\pi\mathrm{i}$ times the residue. Hence,

$$G_\kappa(\boldsymbol{r}) = G_\kappa(r) = \frac{\mathrm{i}}{(2\pi)^2\,r}\,2\pi\mathrm{i}\,\frac{1}{2}\,\mathrm{e}^{-\kappa r} = -\frac{\mathrm{e}^{-\kappa r}}{4\pi r}\ . \tag{1.46}$$

This is the Yukawa potential, which is the static solution of the Klein–Gordon equation with a δ function source, and which has the Coulomb potential as its limit when $\kappa \to 0+$.

1.6 Active transformations

In order to be concrete we will talk here mostly about transformations of fields, even though much of what is said is more generally valid. We distinguish between *active* transformations, transforming the field, and *passive* transformations, transforming the coordinate system without transforming the field. A passive transformation is no more than a coordinate transformation, by which the same field is described relative to a new coordinate system. The mathematical description of active and passive transformations is exactly the same, and therefore we often do not bother to specify whether the transformation equations should be interpreted actively or passively.

An active transformation transforms one field configuration into another. We call it a *symmetry* of the system if it is invertible, so that it can be undone by some inverse transformation, and if in addition it preserves the field equation. Which means that the transformed field configuration is a solution of the field equation if and only if the original field configuration is a solution.

This definition implies that if a given transformation is a symmetry, then so is the inverse transformation. It implies furthermore that a composite transformation consisting of first one and then a second symmetry transformation, is again a symmetry. These two properties mean, in the language of mathematics, that the symmetries of a given physical system form a *group*.

Example: Translation

An example of an active transformation is a translation, or displacement, of a scalar function $\phi = \phi(x)$ of one variable x.

Displacing the whole function a constant distance d, means moving the function value $\phi(x)$ from a given point x to a new point $\widetilde{x} = x + d$. That is, we define a new function $\widetilde{\phi}$ such that

$$\widetilde{\phi}(\widetilde{x}) = \phi(x)\ . \tag{1.47}$$

See Figure 1.1. The transformation equation $\widetilde{\phi}(x+d) = \phi(x)$ means that the transformed function $\widetilde{\phi}$ is defined by the equation

$$\widetilde{\phi}(x) = \phi(x-d)\ . \tag{1.48}$$

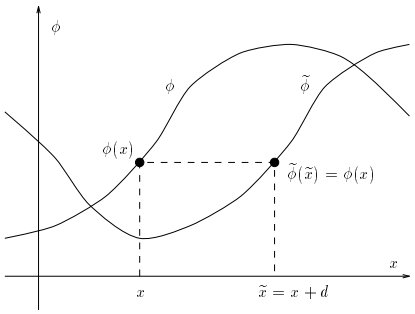

Figure 1.1: An active translation the distance d. Translation of the field configuration ϕ gives the field configuration $\widetilde{\phi}$.

Example: Rotation

We may take as a second example a rotation of a scalar function $\phi = \phi(x, y)$ of two variables (x, y). A rotation by an angle α transforms the point (x, y) into the point

$$(\widetilde{x}, \widetilde{y}) = (x\cos\alpha - y\sin\alpha\,,\; x\sin\alpha + y\cos\alpha)\;. \tag{1.49}$$

The inverse transformation is simply a rotation by the angle $-\alpha$,

$$(x, y) = (\widetilde{x}\cos\alpha + \widetilde{y}\sin\alpha\,,\; -\widetilde{x}\sin\alpha + \widetilde{y}\cos\alpha)\;. \tag{1.50}$$

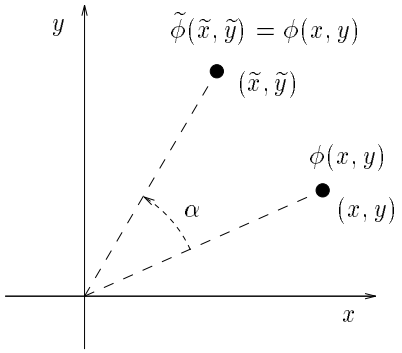

Figure 1.2: An active rotation by an angle α of a scalar field ϕ.

Rotating a scalar function $\phi = \phi(x, y)$ means transforming it into a new function $\widetilde{\phi}$, such that

$$\widetilde{\phi}(\widetilde{x}, \widetilde{y}) = \phi(x, y)\;, \tag{1.51}$$

in analogy with Equation (1.47). See Figure 1.2. From this definition, with Equation (1.50), we see that

$$\widetilde{\phi}(\widetilde{x}, \widetilde{y}) = \phi(\widetilde{x}\cos\alpha + \widetilde{y}\sin\alpha, -\widetilde{x}\sin\alpha + \widetilde{y}\cos\alpha)\;. \tag{1.52}$$

Or, if we write (x, y) instead of $(\widetilde{x}, \widetilde{y})$,

$$\widetilde{\phi}(x, y) = \phi(x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha)\;. \tag{1.53}$$

We see that if ϕ is a solution of the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \,, \tag{1.54}$$

then the rotated field $\widetilde{\phi}$ is a solution of exactly the same equation,

$$\frac{\partial^2\widetilde{\phi}}{\partial x^2} + \frac{\partial^2\widetilde{\phi}}{\partial y^2} = 0 \,. \tag{1.55}$$

This is what we mean when we say that Equation (1.54) is rotationally invariant.

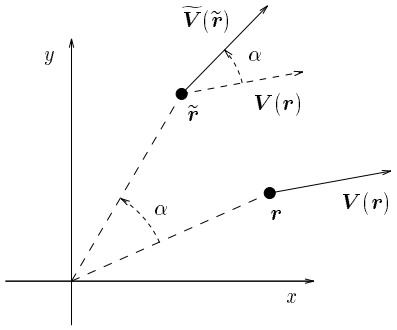

Figure 1.3: An active rotation by an angle α of a vector field $\boldsymbol{V}$.

Rotating a vector field $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{r})$ is slightly more complicated. A vector field in two dimensions has two components, $\boldsymbol{V} = (V_x(x,y), V_y(x,y))$, which are mixed together by the rotation, because the field coordinates (V_x, V_y) are transformed in the same way as the spatial coordinates (x, y). The transformed field $\widetilde{\boldsymbol{V}} = (\widetilde{V}_x(x,y), \widetilde{V}_y(x,y))$ is given by the transformation equations

$$\begin{aligned} \widetilde{V}_x(\widetilde{x},\widetilde{y}) &= \cos\alpha\, V_x(x,y) - \sin\alpha\, V_y(x,y) \,, \\ \widetilde{V}_y(\widetilde{x},\widetilde{y}) &= \sin\alpha\, V_x(x,y) + \cos\alpha\, V_y(x,y) \,. \end{aligned} \tag{1.56}$$

See figure 1.3. This means that

$$\begin{aligned} \widetilde{V}_x(x,y) = {} & \cos\alpha\, V_x(x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha) \\ & - \sin\alpha\, V_y(x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha) \,, \\ \widetilde{V}_y(x,y) = {} & \sin\alpha\, V_x(x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha) \\ & + \cos\alpha\, V_y(x\cos\alpha + y\sin\alpha, -x\sin\alpha + y\cos\alpha) \,. \end{aligned} \tag{1.57}$$

We see from these transformation equations that an equation such as $\nabla \cdot \boldsymbol{V} = \phi$ is rotationally invariant. In Cartesian coordinates it has the form

$$\frac{\partial}{\partial x} V_x(x,y) + \frac{\partial}{\partial y} V_y(x,y) = \phi(x,y) \,. \tag{1.58}$$

Assuming that the fields $\boldsymbol{V}$ and ϕ obey this equation, we see by direct insertion that the transformed fields $\widetilde{\boldsymbol{V}}$ and $\widetilde{\phi}$ obey the same equation,

$$\frac{\partial}{\partial x} \widetilde{V}_x(x,y) + \frac{\partial}{\partial y} \widetilde{V}_y(x,y) = \widetilde{\phi}(x,y) \,. \tag{1.59}$$

Example: Lorentz transformation

As a third example we consider a Lorentz transformation of charge density $\rho = \rho(x,t)$ and current density $j = j(x,t)$ in one spatial dimension.

An active Lorentz transformation transforms time t and space x as follows,

$$(t,x) \mapsto (\widetilde{t},\widetilde{x}) = \left(\gamma t + \frac{\gamma\beta}{c}\,x\,,\,\gamma x + \gamma\beta c t\right). \tag{1.60}$$

Here β is a dimensionless parameter, and $\gamma = 1/\sqrt{1-\beta^2}$. The inverse transformation is obtained by changing the sign of β,

$$(\widetilde{t},\widetilde{x}) \mapsto (t,x) = \left(\gamma\widetilde{t} - \frac{\gamma\beta}{c}\,\widetilde{x}\,,\,\gamma\widetilde{x} - \gamma\beta c\widetilde{t}\right). \tag{1.61}$$

The result of the transformation is e.g. that a particle at rest gets a velocity βc, since if x is constant, then

$$\widetilde{x} = \gamma x + \gamma\beta c t = \gamma x + \beta(c\widetilde{t} - \gamma\beta x) = \frac{x}{\gamma} + \beta c\widetilde{t} = \text{constant} + \beta c\widetilde{t}\,. \tag{1.62}$$

The density ρ and the current density j transform in the same way as t and x, that is,

$$\Big(\widetilde{\rho}(\widetilde{x},\widetilde{t}),\widetilde{j}(\widetilde{x},\widetilde{t})\Big) = \left(\gamma\rho(x,t) + \frac{\gamma\beta}{c}\,j(x,t)\,,\,\gamma j(x,t) + \gamma\beta c\rho(x,t)\right). \tag{1.63}$$

The explicit transformation equation is as follows,

$$\begin{aligned}
\widetilde{\rho}(x,t) &= \gamma\,\rho\Big(\gamma x - \gamma\beta c t, \gamma t - \frac{\gamma\beta}{c}\,x\Big) + \frac{\gamma\beta}{c}\,j\Big(\gamma x - \gamma\beta c t, \gamma t - \frac{\gamma\beta}{c}\,x\Big)\,,\\
\widetilde{j}(x,t) &= \gamma\,j\Big(\gamma x - \gamma\beta c t, \gamma t - \frac{\gamma\beta}{c}\,x\Big) + \gamma\beta c\,\rho\Big(\gamma x - \gamma\beta c t, \gamma t - \frac{\gamma\beta}{c}\,x\Big)\,.
\end{aligned} \tag{1.64}$$

We easily verify that if ρ and j satisfy the continuity equation,

$$\frac{\partial\rho}{\partial t} + \frac{\partial j}{\partial x} = 0\,, \tag{1.65}$$

then the transformed quantities satisfy the same equation,

$$\frac{\partial\widetilde{\rho}}{\partial t} + \frac{\partial\widetilde{j}}{\partial x} = 0\,. \tag{1.66}$$

Thus we see that the continuity equation is Lorentz invariant.

As a concrete example of the Lorentz transformation of density and current density, let us take a point charge q at rest at $x = a$. Then the charge density is a Dirac δ function,

$$\rho(x,t) = \rho(x) = q\,\delta(x-a)\,, \tag{1.67}$$

whereas the current density vanishes, $j(x,t) = 0$. The Lorentz transformed density and current density is

$$\begin{aligned}
\widetilde{\rho}(x,t) &= \gamma\,\rho(\gamma x - \gamma\beta c t) = q\gamma\,\delta(\gamma x - \gamma\beta c t - a) = q\,\delta\Big(x - \beta c t - \frac{a}{\gamma}\Big)\,,\\
\widetilde{j}(x,t) &= \gamma\beta c\,\rho(\gamma x - \gamma\beta c t) = \beta c\,\widetilde{\rho}(x,t)\,.
\end{aligned} \tag{1.68}$$

Note that the charge is invariant under the Lorentz transformation,

$$\widetilde{q} = \int_{-\infty}^{\infty}\mathrm{d}x\,\widetilde{\rho}(x,t) = q\gamma\int_{-\infty}^{\infty}\mathrm{d}x\,\delta(\gamma x - \gamma\beta c t - a) = q\,. \tag{1.69}$$

1.7 Passive transformations

As already stated, a passive transformation is just a coordinate transformation. It is a *symmetry* of the system if the transformed field equation has the same form as the original equation. However, this definition of a symmetry is vacuous until we define what we mean by the statement that two equations have the same form. We could always *define* that the transformed equation is the same as the original, only expressed in a different coordinate system. Clearly such an all embracing definition, that all field equations are invariant under all coordinate transformations, is not very interesting, and we are usually much more restrictive.

Example: Rotation

Figure 1.4 shows how a rotation by an angle α, as given in Equation (1.49), may be interpreted as a passive transformation, and by comparison with Figure 1.2 it illustrates the difference between active and passive transformations. A given point P in the plane has coordinates (x, y) in one rectangular coordinate system, and has coordinates $(\widetilde{x}, \widetilde{y})$ in another rectangular coordinate system rotated by the *opposite* angle $-\alpha$ relative to the first system. A scalar field ϕ is a function having the function value $\phi(P)$ at the point P, independent of which coordinate system we use. The passive transformation does not transform the field, but transforms our description of the field. The field is described mathematically as a function $\phi = \phi(x, y)$ of the coordinates (x, y) in the original coordinate system, while the same field is a different mathematical function $\widetilde{\phi} = \widetilde{\phi}(\widetilde{x}, \widetilde{y})$ of the coordinates $(\widetilde{x}, \widetilde{y})$ in the transformed coordinate system. The mathematical relation between ϕ and $\widetilde{\phi}$ is the same as in Equation (1.53).

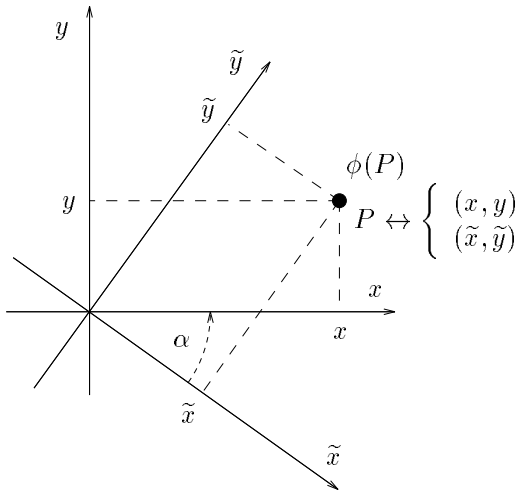

Figure 1.4: A passive rotation by an angle α.

A passive rotation of a vector field $\boldsymbol{V} = \boldsymbol{V}(\boldsymbol{r})$, as a second example, also does not change the field. But it changes the *components* of the field, because, like the spatial coordiantes, they always have to be measured relative to a specific coordinate system. In the two dimensional case we have

$$\boldsymbol{V} = (V_x(x, y), V_y(x, y)) \ , \tag{1.70}$$

using the original coordinate system, and $\boldsymbol{V} = (\widetilde{V}_x(\widetilde{x}, \widetilde{y}), \widetilde{V}_y(\widetilde{x}, \widetilde{y}))$ when we use instead the

transformed coordinate system. We write

$$\widetilde{\boldsymbol{V}} = (\widetilde{V}_x(\widetilde{x},\widetilde{y}), \widetilde{V}_y(\widetilde{x},\widetilde{y})) \ , \tag{1.71}$$

with the implicit understanding that $\boldsymbol{V}$ and $\widetilde{\boldsymbol{V}}$ represent the same field as described in two different coordinate systems. The mathematical transformation equation is still Equation (1.56), exactly as in the active case.

Example: Polar coordinates in the plane

As a closely related example we may consider the connection between Cartesian (Euclidean) coordinates (x, y) and polar coordinates (r, φ) in the plane. One and the same point P in the plane is given by the coordinates (x, y) in one system, or by the coordinates (r, φ) in the second system, and the relation between the two systems, as shown in Figure 1.5, is that

$$x = r\cos\varphi \ , \qquad y = r\sin\varphi \ . \tag{1.72}$$

A scalar field ϕ in the plane has a given value $\phi(P)$ at the point P, and we write

$$\phi = \phi(P) = \phi(x, y) = \phi(r, \varphi) \ . \tag{1.73}$$

One possible source of confusion is that the two functions $\phi(x, y)$ and $\phi(r, \varphi)$ are two very different mathematical functions, even though we denote them by the same letter ϕ.

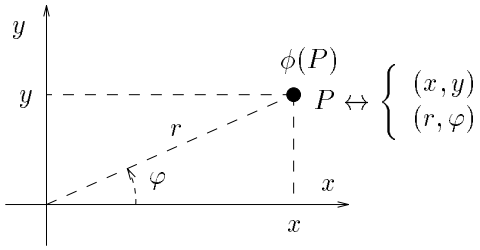

Figure 1.5: Cartesian coordinates (x, y) and polar coordinates (r, φ) in the plane.

Herrmann Weyl (in his book "Space—Time—Matter") points out this difference between physical and mathematical notation, that symbols are used in physics as the names of *quantities*, but in mathematics as the names of *functions*. In mathematical notation we would write

$$\phi(P) = f(x, y) = g(r, \varphi) \ . \tag{1.74}$$

This notation has the advantage of being unambiguous, but it is also less economical. It gives that

$$g(r, \varphi) = f(r\cos\varphi, r\sin\varphi) \ , \tag{1.75}$$

and hence, by the chain rule,

$$g_1(r, \varphi) = \cos\varphi\, f_1(r\cos\varphi, r\sin\varphi) + \sin\varphi\, f_2(r\cos\varphi, r\sin\varphi) \ . \tag{1.76}$$

Here $g_1 = \partial g/\partial r$ is the derivative of g with respect to its first argument, and similarly $f_1 = \partial f/\partial x$ and $f_2 = \partial f/\partial y$. In "physicist notation" we would write the same equation simply as

$$\frac{\partial\phi}{\partial r} = \frac{\partial x}{\partial r}\,\frac{\partial\phi}{\partial x} + \frac{\partial y}{\partial r}\,\frac{\partial\phi}{\partial y} = \cos\varphi\,\frac{\partial\phi}{\partial x} + \sin\varphi\,\frac{\partial\phi}{\partial y}\;. \tag{1.77}$$

This notation is usually sufficiently unambiguous, but in order to make it totally unambiguous one must specify explicitly what is constant during differentiation, in the following way,

$$\left.\frac{\partial\phi}{\partial r}\right|_{\varphi} = \left.\frac{\partial x}{\partial r}\right|_{\varphi}\,\left.\frac{\partial\phi}{\partial x}\right|_{y} + \left.\frac{\partial y}{\partial r}\right|_{\varphi}\,\left.\frac{\partial\phi}{\partial y}\right|_{x}\;. \tag{1.78}$$

Thus, at the one point $P = (x, y) = (r, \varphi)$, with two different coordinate systems we have four different partial derivatives, standing in the following relations,

$$\begin{aligned}
\frac{\partial}{\partial r} &= \frac{\partial x}{\partial r}\,\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\,\frac{\partial}{\partial y} = \quad\cos\varphi\,\frac{\partial}{\partial x} + \;\sin\varphi\,\frac{\partial}{\partial y}\;,\\
\frac{\partial}{\partial\varphi} &= \frac{\partial x}{\partial\varphi}\,\frac{\partial}{\partial x} + \frac{\partial y}{\partial\varphi}\,\frac{\partial}{\partial y} = -r\sin\varphi\,\frac{\partial}{\partial x} + r\cos\varphi\,\frac{\partial}{\partial y}\;,
\end{aligned} \tag{1.79}$$

and conversely,

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\,\frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial x}\,\frac{\partial}{\partial\varphi} = \cos\varphi\,\frac{\partial}{\partial r} - \frac{\sin\varphi}{r}\,\frac{\partial}{\partial\varphi}\;,\\
\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\,\frac{\partial}{\partial r} + \frac{\partial\varphi}{\partial y}\,\frac{\partial}{\partial\varphi} = \sin\varphi\,\frac{\partial}{\partial r} + \frac{\cos\varphi}{r}\,\frac{\partial}{\partial\varphi}\;.
\end{aligned} \tag{1.80}$$

We have the following expressions for the Laplace operator ∇^2 in the two coordinate systems,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2}{\partial\varphi^2}\;. \tag{1.81}$$

One would ordinarily tend to say that the two equations

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{1.82}$$

and

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\,\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\phi}{\partial\varphi^2} = 0 \tag{1.83}$$

are not of the same form. But this question is very much a matter of definition, since it is equally natural to see the two equations as just the same equation, the Laplace equation $\nabla^2\phi = 0$, expressed in two different coordinate systems.

Problems

1. Use Equation (1.18) to express the Laplace operator ∇^2 in polar coordinates. That is, compute $\nabla^2 f = \nabla \cdot (\nabla f)$ for an arbitrary function f. Remember to differentiate the unit vectors $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$.

2. We say that a scalar function $\phi = \phi(x)$ in one dimension is *invariant* under translation a distance d if $\phi(x) = \phi(x - d)$. This means that the translated function $\widetilde{\phi}$ as defined in Equation (1.48) is equal to the original function ϕ.
   Precisely which functions are invariant under translation a fixed distance d?
   Which functions are *translationally invariant*, in the sense that they are invariant under translation a distance d simultaneously for *all* values of d?

3. Similarly, we say that a scalar function (scalar field) $\phi = \phi(x, y)$ in two dimensions is invariant under rotation by a fixed angle α if $\widetilde{\phi} = \phi$ in Equation (1.53). We say simply that ϕ is invariant under rotations, or rotationally invariant, if it is invariant under rotation by *any* angle α.
   Precisely which scalar functions in two dimensions are invariant under rotation by a fixed angle α? (This question is actually somewhat tricky, and the answer depends on whether α is a rational or irrational multiple of π.)
   Which scalar functions in two dimensions are rotationally invariant?

4. Explain (for example by drawing a sketch) what the most general rotationally invariant vector field in the plane must look like. Again, "rotationally invariant" means "invariant under rotation by an arbitrary angle".
   What can you say about the value at the origin of a rotationally invariant vector field?

Chapter 2

Manifolds, vectors and tensors

Coordinates are arbitrarily assigned labels identifying times and places, and have no physical meaning in themselves. Therefore physical laws should be formulated in such a way that they do not depend on the use of one particular, or one particular type of, coordinate system. This philosophy is the basis of the general theory of relativity, and lies at the heart of much of classical mechanics and classical field theory, for example when the equations of motion and field equations are derived from variational principles. It finds an elegant expression in the branch of mathematics called differential geometry. The present and the following chapters summarize important concepts from differential geometry.

When we describe a physical system by means of one time coordinate and three space coordinates, it means that we treat spacetime as a four dimensional *manifold*, in the mathematical terminology. Differential geometry is the mathematical theory of manifolds.

We need coordinates to describe a manifold, but we imagine that the manifold has an existence independent of our coordinatization, and that it has something we may call geometrical structures existing independent of coordinates. Coordinate independence means that the description is invariant under general coordinate transformations, or in practical terms, that we write every formula in such a way that it is valid no matter what coordinate system we use.

In field theory we meet many examples of such geometrical structures: tensor algebra, differential forms, covariant differentiation and integration, curvature and metric. Geometry is a recurrent theme throughout physics, maybe more than any other theme, since Riemann, Maxwell and Einstein.

2.1 The surface of a sphere as an example

A simple, yet not entirely trivial example of a manifold is the surface of the Earth, if we idealize and assume that it is perfectly spherical, which is a good approximation as seen from the Moon. It is two dimensional, and we may specify an arbitrary point on it by specifying two coordinates, for example the polar angle $\theta \in [0, \pi]$ and the asimuthal angle $\varphi \in [0, 2\pi]$. $\theta = 0$ is the North Pole, and $\theta = \pi$ is the South Pole. The angles θ and φ are essentially what geographers call latitude and longitude.

This coordinate system covers the whole surface, but it is double valued along the null meridian, from pole to pole, where the longitude φ is either 0 or 2π. A more serious problem is that the longitude is discontinuous at both poles. In fact, at different points arbitrarily

close to either pole, φ takes any value from 0 to 2π. It is impossible to cover all the surface of the Earth by one single coordinate system which is everywhere free of discontinuities or other singularities. Of course, there is nothing intrinsically singular about the poles, or any other point on the surface of the Earth, the *coordinate singularities* at the poles are peculiarities of the polar coordinate system.

A coordinate system covering part of the surface of the Earth can be used for drawing a planar *map* of this region. A planar map of a curved surface must necessarily be distorted. It is possible to represent the surface of the Earth on a globe with all distances correctly scaled down by one common factor, but such a faithful representation is impossible on a planar map. A more modest demand is that the map should be *one to one*: there should correspond exactly one point on the map to every point in the region covered, and vice versa, exactly one point in the mapped region to every point on the map. The polar coordinates on the sphere are one to one if we exclude the null meridian and both poles, where the asimuthal angle is multivalued. This means that we restrict θ and φ to the open intervals $\langle 0, \pi \rangle$ and $\langle 0, 2\pi \rangle$, respectively.

A second reasonable demand when we draw a map is that we may use it for determining a compass course, a direction in which to move in order to get from one point to a nearby point. Intuitively, this means that the correspondence between the terrain and the map has to be both *continuous* and *differentiable*. It follows that every time two different maps overlap (remember our convention that one map is not allowed to overlap with itself), the correspondence between the two maps in the overlap region, or in other words, the *coordinate transformation* from one map to the other, must be continuous and differentiable.

A collection of maps covering together the whole surface of the Earth, is called an *atlas*. An atlas, including a complete list of all coordinate transformations between overlapping maps, describes the surface of the Earth completely, and in the mathematical sense this is all we need as a definition of a spherical surface.

An atlas of the surface of the Earth need not contain more than two maps. For example, the first map may cover the northern hemisphere, plus a little more, while the second map covers the southern hemisphere, plus a little more. The two maps then overlap around the equator. Let R be the radius of the Earth, and use e.g. the stereographic projection, mapping a point (X, Y, Z) on the surface, with $X^2 + Y^2 + Z^2 = R^2$, into the point (x, y) in the plane, such that

$$\begin{aligned} x = x_A = \frac{2RX}{R+Z}\,, \qquad y = y_A = \frac{2RY}{R+Z}\,, \qquad & \text{for map } A, \text{ and} \\ x = x_B = \frac{2RX}{R-Z}\,, \qquad y = y_B = \frac{2RY}{R-Z}\,, \qquad & \text{for map } B. \end{aligned} \tag{2.1}$$

Map A covers not only the northern hemisphere, with $Z > 0$, but actually the whole surface except the South Pole, where $Z = -R$. Similarly, map B covers the whole surface except the Nord Pole, where $Z = R$. The two maps overlap everywhere except at the South and North Poles. One point (X, Y, Z) on the surface of the Earth, with $|Z| < R$, is represented by the point (x_A, y_A) on map A and by (x_B, y_B) on map B. The coordinate transformations are:

$$\begin{aligned} &\text{from map } A \text{ to map } B: && x_B = \frac{4R^2 x_A}{{x_A}^2 + {y_A}^2} \,, && y_B = \frac{4R^2 y_A}{{x_A}^2 + {y_A}^2} \,, \\ &\text{from map } B \text{ to map } A: && x_A = \frac{4R^2 x_B}{{x_B}^2 + {y_B}^2} \,, && y_A = \frac{4R^2 y_B}{{x_B}^2 + {y_B}^2} \,. \end{aligned} \tag{2.2}$$

These transformations are analytic, thus we have an example of a C^ω manifold, as defined below.

Orientation

Both of these coordinate transformations have a negative Jacobi determinant, for example,

$$\det\left(\frac{\partial(x_B, y_B)}{\partial(x_A, y_A)}\right) = \begin{vmatrix} \frac{\partial x_B}{\partial x_A} & \frac{\partial x_B}{\partial y_A} \\ \frac{\partial y_B}{\partial x_A} & \frac{\partial y_B}{\partial y_A} \end{vmatrix} = -\left(\frac{4R^2}{{x_A}^2 + {y_A}^2}\right)^2 . \tag{2.3}$$

By definition, the minus sign means that the two maps have opposite *orientations* in their overlap region. One map defines a *right handed* coordinate system, and the other map defines a *left handed* system. Which coordinate system is right handed and which is left handed, is a matter of convention.

That the two maps define different orientations is easily amended. We may e.g. replace map B by a map C where the coordinates are $x_C = x_B$ and $y_C = -y_B$. Then the maps A and C have the same orientation, that is, the coordinate transformation from one to the other has a positive Jacobi determinant. Together they define a unique orientation of the whole surface.

A two dimensional spherical surface has an "outside" and an "inside", and a two dimensional plane has a "top side" and a "bottom side". Orienting a two dimensional surface, that is, defining an orientation by introducing an atlas of maps all having the same orientation, is intuitively the same as defining which side is the "outside" and which is the "inside", or the "top side" and the "bottom side", of the surface.

2.2 Manifolds in general

We may describe any d dimensional manifold $\mathcal{M}$ similarly by means of an atlas consisting of one or more d dimensional maps, together with a complete list of coordinate transformations between overlapping maps. This description *defines* the manifold, mathematically speaking.

Locally, in a small region of the manifold, we always have at least one coordinate system at our disposal, so that we may specify an arbitrary point in the region by specifying the coordinate system and the values of d coordinates. Whenever two different maps overlap, either in one connected region, or in two or more connected regions that are disconnected from one another, there must exist an invertible coordinate transformation between the two maps in each connected overlap region. We may use the coordinate transformation to change coordinates, going back and forth between the two maps.

Figure 2.1 illustrates the basic concepts. A map, or coordinate system, is a function f defined on a coordinate region O_1, and mapping every point $P \in O_1$ to a unique point

$x = f(P) \in \mathbf{R}^d$. By convention, we number the coordinates by means of upper indices, writing

$$x = (x^1, x^2, \ldots, x^d) \; . \tag{2.4}$$

Thus, $x^i = f^i(P)$ for $i = 1, 2, \ldots, d$, where each f^i is a real valued function defined on O_1. Since f is a one to one mapping from the region O_1 onto its image $f(O_1)$, there exists an inverse function f^{-1} which is a one to one mapping from $f(O_1)$ onto O_1. The subsets $O_1 \subset \mathcal{M}$ and $f(O_1) \subset \mathbf{R}^d$ should be open sets, as described below.

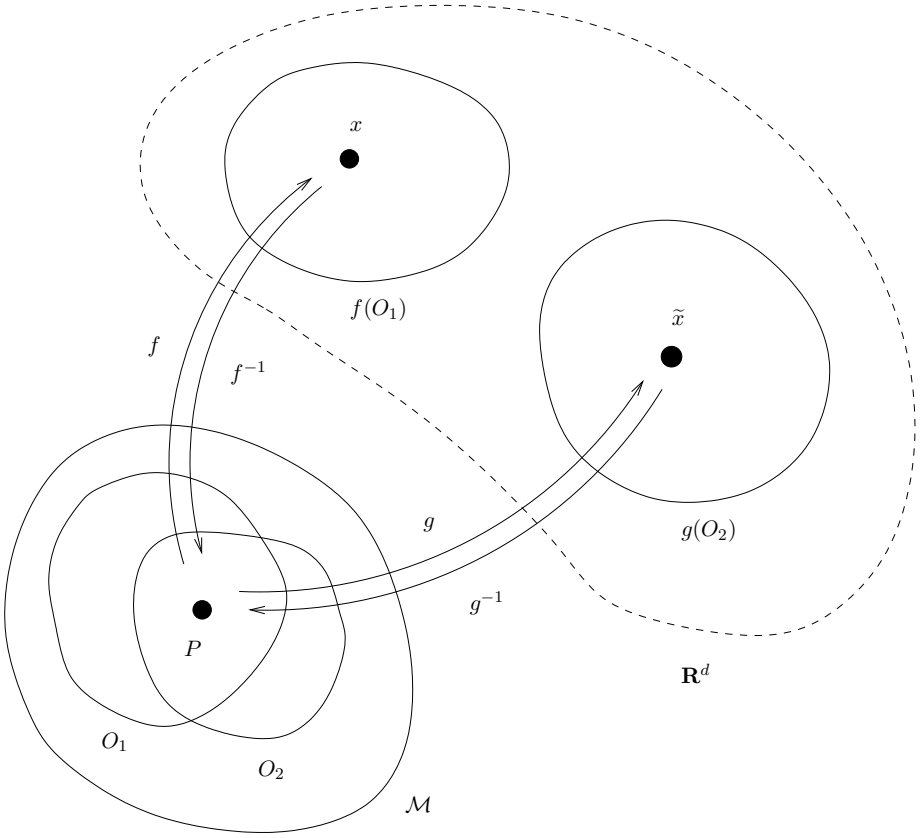

Figure 2.1: A d dimensional manifold $\mathcal{M}$, with two overlapping coordinate systems.

Assume now that $P \in O_1 \cap O_2$, where O_2 is another coordinate region with another coordinate function g. In this second coordinate system the point P has coordinates $\widetilde{x} = g(P) \in \mathbf{R}^d$. The relation between the two overlapping coordinate systems is the coordinate transformation gf^{-1},

$$x \mapsto \widetilde{x} = \widetilde{x}(x) = g(P) = g(f^{-1}(x)) \; , \tag{2.5}$$

and the inverse coordinate transformation $(gf^{-1})^{-1} = fg^{-1}$,

$$\widetilde{x} \mapsto x = x(\widetilde{x}) = f(P) = f(g^{-1}(\widetilde{x})) \; . \tag{2.6}$$

An atlas with its set of coordinate transformations defines an orientation of the manifold if all the coordinate transformations have positive Jacobi determinants. If a manifold is not already oriented, it may nevertheless be *orientable*, this means that we may make it oriented simply by switching the orientation of some of the maps. Not all manifolds are orientable, and the simplest counterexample is the Möbius strip. See Problem 1.

Topology

From the mathematical point of view, there are a number of technical points that we do not go into here in much detail.

For example, it is usually required that the region in the manifold $\mathcal{M}$ covered by a map should always be an *open* subset of $\mathcal{M}$. The list of all the open subsets of $\mathcal{M}$ is what mathematicians call the *topology* of $\mathcal{M}$. One of the defining characteristics of a manifold is that it has a topology, it is a *topological space*.

In the most general kind of topological space, only two simple properties are required for the list of open subsets. First, that the *union* of a *finite or infinite* number of open subsets must be an open subset, and second, that the *intersection* of a *finite* number of open subsets must be an open subset. By definition, a *closed* subset is the complement of an open subset, and hence the *intersection* of a *finite or infinite* number of closed subsets is always a closed subset, whereas the *union* of a *finite* number of closed subsets is always a closed subset. Again by definition, the manifold $\mathcal{M}$ is both an open and a closed subset of itself, and the empty subset is both open and closed.

One may ask which subsets of $\mathcal{M}$ are both open and closed, in addition to the two trivial cases just mentioned. The answer may be surprising when first encountered, and lies in the following definition. A topological space $\mathcal{M}$ is *connected* if and only if it has no subset which is both open and closed, apart from the two trivial ones. Any non-trivial subset which is simultaneously open and closed, and which itself has no non-trivial subset with the same two properties, is called a *connected component* of $\mathcal{M}$.

Metric topology

Most topological spaces encountered in physics are *metric spaces*, in which the topology is defined in terms of some measure of distances between points. For example, the Euclidean space $\mathbf{R}^n$, with $n = 1, 2, \ldots$, is a metric space where the distance between two points $x = (x^1, x^2, \ldots, x^n)$ and $y = (y^1, y^2, \ldots, y^n)$ is the Euclidean distance

$$s(x, y) = s(y, x) = \sqrt{(x^1 - y^1)^2 + (x^2 - y^2)^2 + \cdots + (x^n - y^n)^2} \, . \tag{2.7}$$

This metric is *positive definite* and *separates points*, that is, we have always $s(x, y) \geq 0$, and $s(x, y) = 0$ if and only if $x = y$. It also satisfies the *triangle inequality*,

$$s(x, y) \leq s(x, z) + s(z, y) \tag{2.8}$$

for any three points x, y, z. An open subset O of a metric space is defined by the property that for any given point $x \in O$ there exists a $\delta > 0$ such that $y \in O$ whenever $s(x, y) < \delta$.

The Euclidean distance $s(x, y)$ is the length of the shortest curve between x and y, which is a straight line. And the length of a curve in general is the integral along the curve of the *line element* $\mathrm{d}s = s(x, x + \mathrm{d}x)$. The square of $\mathrm{d}s$ is

$$\mathrm{d}s^2 = (\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2 + \cdots + (\mathrm{d}x^n)^2 \, . \tag{2.9}$$

The standard topology of a d dimensional manifold $\mathcal{M}$ is always metric, it is simply the standard Euclidean topology of $\mathbf{R}^d$, as defined by the Euclidean distance, transferred to the manifold by means of the coordinate mappings. In a manifold $\mathcal{M}$ in general we may not have a unique way of measuring distances, such as we have on the surface of the Earth. Fortunately, different measures of distance very often define the same open subsets, and hence it makes sense to speak of the topology of a manifold without reference to any particular metric defining that topology.

Note that the Euclidean line element $\mathrm{d}s^2$, Equation (2.9), is positive definit. Both in the special and in the general theory of relativity we are going to introduce a physical metric $g_{\mu\nu}$ on the four dimensional spacetime. This physical metric is not positive definit in the same way, and is not directly related to the Euclidean metric on $\mathbf{R}^4$ defining the spacetime topology.

Continuous functions

The concept of continuous functions is meaningful for general topological spaces. The standard "ϵ and δ" definition applies to metric spaces X and Y with metrics s_X and s_Y. It says that a function $f : X \mapsto Y$ is continuous at a point $x \in X$ if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $s_Y(f(x), f(x')) < \epsilon$ for every $x' \in X$ with $s_X(x, x') < \delta$. If f is continuous everywhere it is defined, then it is simply said to be continuous.

This definition is easily translated into the more general language of open subsets, and then amounts to the following. A function $f : X \mapsto Y$, where X and Y are topological spaces, is continuous if and only if the *inverse image* under f of an open subset $O \subset Y$ is always an open subset of X.

The inverse image of O is denoted by $f^{-1}(O)$, and consists of those points $x \in X$ such that $f(x) \in O$. Note that the notation $f^{-1}(O)$ does not imply that there has to exist a function f^{-1} which is inverse to the function f.

Differentiable functions

In the same way as in our mapping of the surface of the Earth, whenever we map an open subset of $\mathcal{M}$ onto an open subset of $\mathbf{R}^d$, we want the mapping to be both continuous and differentiable.

The topology on the manifold $\mathcal{M}$ defines which functions are continuous, but is not sufficient to define what is meant by differentiable functions on $\mathcal{M}$. Lacking any better definition, we simply select some maps and declare them to be continuous and differentiable. A necessary consistency condition is then that every time two maps overlap, the coordinate transformation from one map to the other must be continuous and differentiable. It is reasonable to impose even stronger conditions, for example that every coordinate transformation must belong to the function class C^n for some $n \geq 2$.

By definition, a function from $\mathbf{R}^k$ to $\mathbf{R}$, for $k = 1, 2, \ldots$, is of class C^n if all its n-th order partial derivatives exist and are continuous everywhere. We also say of a $\mathcal{C}^n$ function that it is n times *continuously differentiable*. The class of continuous functions is called C or C^0. A function is of class C^∞ if all its partial derivatives of all orders exist everywhere.

Analyticity is an even more restrictive condition: an analytic function is not only infinitely differentiable everywhere it is defined, but it can be represented by a power series in some

neighbourhood around any given point. The class of analytic functions is called C^ω, and it is different from $\mathcal{C}^\infty$. See Problem 2 for a famous counterexample.

The above definition of the function class $\mathcal{C}^n$ (or C^∞ or C^ω) applies to real valued functions on $\mathbf{R}^k$, i.e. functions $\mathbf{R}^k \mapsto \mathbf{R}$. But a function $f : \mathbf{R}^k \mapsto \mathbf{R}^m$ is no more nor less than m functions from $\mathbf{R}^k$ to $\mathbf{R}$, hence we use the obvious terminology and say that f is of class $\mathcal{C}^n$ if all these m functions are of class $\mathcal{C}^n$.

Definition of a manifold

To summarize, we may now define a little bit more precisely what we mean by a d dimensional manifold $\mathcal{M}$. It is a topological space, and it is covered by an atlas of one or more coordinate maps. Each coordinate map, or coordinate system, is a one to one contionuous mapping between an open subset of $\mathcal{M}$ and an open subset of $\mathbf{R}^d$.

Whenever two maps overlap, there exists an invertible coordinate transformation between them. The coordinate transformation is an invertible transformation between the two open regions of $\mathbf{R}^d$ corresponding to the single overlap region in $\mathcal{M}$.

A manifold where all coordinate transformations are of class C^n (or C^∞ or C^ω) is called a C^n (or C^∞ or C^ω) manifold.

2.3 Tensors and tensor fields

One way of writing equations so that they are valid in an arbitrary coordinate system is to write them as relations between *tensors*, or between *tensor fields*. To be concrete we will speak here about spacetime, which is a four dimensional "space", but most of what we say may be generalized directly to the case of a d dimensional manifold. A point in spacetime is given by the coordinates

$$x = x^\mu = (x^0, x^1, x^2, x^3) \, , \tag{2.10}$$

where, as a rule, x^0 is a time coordinate and the other three are spatial coordinates. We will stick to the usual convention that greek coordinate indices run from 0 til 3. More generally, in a d dimensional space, we would use latin indices running from 1 to d, thus we would write

$$x = x^i = (x^1, x^2, \ldots, x^d) \, . \tag{2.11}$$

A tensor is a quantity localized at one point in spacetime. Relative to a given coordinate system, the tensor has a number of components, and the components transform in certain well defined ways under coordinate transformations. There are many kinds of tensors, two special examples are scalars and vectors. If we have a tensor of a given kind at every point in spacetime, then we have a tensor field. Often we say "tensor" when we actually mean "tensor field".

A tensor at a given point is an element of a vector space, and we specify it by specifying a basis in the vector space, as well as the components of the tensor relative to this basis. We are free to choose whatever basis we want, but a standard basis is the *coordinate basis*, which is uniquely defined by the coordinate system we choose around the point where the tensor is located. The coupling between the tensor basis and the coordinate system means that the tensor components get transformed in a particular way when we change our coordinate

system. We will now see in more detail how the components of various types of tensors get transformed.

A *scalar* is a trivial tensor, because it has only one component, the value of which is independent of the coordinate system. A *scalar density* is slightly less trivial, its single component gets multiplied by a scale factor when we change coordinate system. We will have some more to say about densities in Chapter 3.

2.4 Contravariant vectors

A *contravariant vector* $\mathbf{A}$ at a point $x = (x^0, x^1, x^2, x^3)$ is the same as a *tangent vector* at x. It specifies a direction so that we may differentiate a scalar field $\phi = \phi(x) = \phi(x^0, x^1, x^2, x^3)$ in that direction.

To make the idea a little more concrete, consider a curve C described by a curve parameter u. Each point on the curve is identified by a corresponding value of u, and the coordinates of the point are given functions of u,

$$x = x(u) = (x^0(u), x^1(u), x^2(u), x^3(u)) \ . \tag{2.12}$$

Differentiation with respect to u gives the *tangent*, or *tangent vector*, of the curve C at the point $x = x(u)$,

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \left(\frac{\mathrm{d}x^0}{\mathrm{d}u}, \frac{\mathrm{d}x^1}{\mathrm{d}u}, \frac{\mathrm{d}x^2}{\mathrm{d}u}, \frac{\mathrm{d}x^3}{\mathrm{d}u} \right) . \tag{2.13}$$

Thus, the tangent vector has the components $\mathrm{d}x^\mu/\mathrm{d}u$ along the coordinate axes at the given point. Along the curve, the scalar field ϕ is a function of u,

$$\phi = \phi(u) = \phi(x(u)) = \phi(x^0(u), x^1(u), x^2(u), x^3(u)) \ , \tag{2.14}$$

which can be differentiated with respect to u,

$$\frac{\mathrm{d}\phi}{\mathrm{d}u} = \frac{\partial\phi}{\partial x^0}\frac{\mathrm{d}x^0}{\mathrm{d}u} + \frac{\partial\phi}{\partial x^1}\frac{\mathrm{d}x^1}{\mathrm{d}u} + \frac{\partial\phi}{\partial x^2}\frac{\mathrm{d}x^2}{\mathrm{d}u} + \frac{\partial\phi}{\partial x^3}\frac{\mathrm{d}x^3}{\mathrm{d}u} = \frac{\partial\phi}{\partial x^\mu}\frac{\mathrm{d}x^\mu}{\mathrm{d}u} \ . \tag{2.15}$$

We use the chain rule for differentiation, and the summation convention that the repeated index μ here is to be summed over from 0 to 3.

We use the notation $\mathrm{d}/\mathrm{d}u$ when differentiating a function of one single variable u, and the notation $\partial/\partial x^0$, $\partial/\partial x^1$, and so on, when differentiating a function of several variables x^0, x^1, x^2, x^3. When we write $\mathrm{d}\phi/\mathrm{d}u$ on the left hand side of the above equation, it means that we treat the scalar field ϕ as a function of the curve parameter u, and of no other variables, because we take into account only the values of ϕ along the curve. On the right hand side we take into account also the values of ϕ outside the curve, treating ϕ as a function of all the coordinates x^0, x^1, x^2, x^3.

Now we generalize from the last formula, introducing the concept of a tangent vector, or contravariant vector, $\mathbf{A}$, which need not be the tangent vector of a curve. The components of $\mathbf{A}$ relative to the coordinates x^0, x^1, x^2, x^3 we call $A^{x^0}, A^{x^1}, A^{x^2}, A^{x^3}$, or in a simpler notation, A^0, A^1, A^2, A^3. By convention, we number the components of a contravariant vector by an upper index. Given these components, we define the derivative of ϕ in the direction $\mathbf{A}$ as

$$\mathbf{A}(\phi) = \sum_{\mu=0}^{3} A^\mu \frac{\partial\phi}{\partial x^\mu} = A^\mu \frac{\partial\phi}{\partial x^\mu} = A^\mu \, \phi_{,\mu} \ . \tag{2.16}$$

We use again the summation convention for the repeated index μ, and we introduce a shorter notation for the partial derivative,

$$\phi_{,\mu} = \frac{\partial \phi}{\partial x^\mu} \ . \tag{2.17}$$

Note that the component A^ν is the derivative of the coordinate x^ν in the direction $\mathbf{A}$,

$$\mathbf{A}(x^\nu) = A^\mu \frac{\partial x^\nu}{\partial x^\mu} = A^\mu \, \delta^\nu_\mu = A^\nu \ . \tag{2.18}$$

We may write

$$\mathbf{A} = A^\mu \frac{\partial}{\partial x^\mu} = A^\mu \, \mathbf{e}_\mu \ , \tag{2.19}$$

expanding in the special basis vectors defined by the coordinate system,

$$\mathbf{e}_\mu = \frac{\partial}{\partial x^\mu} \ . \tag{2.20}$$

Instead of this coordinate basis for the space of contravariant vectors we could have chosen a more general basis

$$\mathbf{e}_A = e^\mu_A \, \mathbf{e}_\mu = e^\mu_A \frac{\partial}{\partial x^\mu} \ , \qquad A = 0, 1, 2, 3 \ , \tag{2.21}$$

where the $4 \times 4 = 16$ coefficients e^μ_A would be functions of $x = (x^0, x^1, x^2, x^3)$. However, we will stick to the coordinate basis, at least in the present chapter.

Equation (2.16) determines how the components of the vector get transformed when we change to a different set of coordinates $\widetilde{x} = (\widetilde{x}^0, \widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$. The chain rule for differentiation gives the following transformation formula for the differentiation operators,

$$\frac{\partial}{\partial \widetilde{x}^\mu} = \frac{\partial x^\kappa}{\partial \widetilde{x}^\mu} \frac{\partial}{\partial x^\kappa} \ . \tag{2.22}$$

Since the formula is completely general, we may interchange x and $\widetilde{x}$ to get that

$$\frac{\partial}{\partial x^\mu} = \frac{\partial \widetilde{x}^\kappa}{\partial x^\mu} \frac{\partial}{\partial \widetilde{x}^\kappa} \ . \tag{2.23}$$

The 4×4 matrix $\partial \widetilde{x}/\partial x$ is sometimes called the Jacobi matrix of the coordinate transformation, its determinant is the Jacobi determinant entering in the formula for changing variables in a multiple integral. The two matrices $\partial x/\partial \widetilde{x}$ and $\partial \widetilde{x}/\partial x$ are inverses of each other, a result which also follows directly from the chain rule,

$$\delta^\mu_\nu = \frac{\partial \widetilde{x}^\mu}{\partial \widetilde{x}^\nu} = \frac{\partial x^\kappa}{\partial \widetilde{x}^\nu} \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa} = \frac{\partial x^\mu}{\partial x^\nu} = \frac{\partial \widetilde{x}^\kappa}{\partial x^\nu} \frac{\partial x^\mu}{\partial \widetilde{x}^\kappa} \ . \tag{2.24}$$

Figure 2.2 illustrates the relation between the partial derivatives in two different coordinate systems in the plane. For example, $\partial/\partial x^1$ is a tangent vector of the curve $x^2 =$ constant, pointing in the direction in which x^1 increases.

Since the tangent vector $\mathbf{A}$ is the same, no matter which basis we expand it in, its components must get transformed so that

$$\mathbf{A} = A^\mu \frac{\partial}{\partial x^\mu} = \widetilde{A}^\mu \frac{\partial}{\partial \widetilde{x}^\mu} \ . \tag{2.25}$$

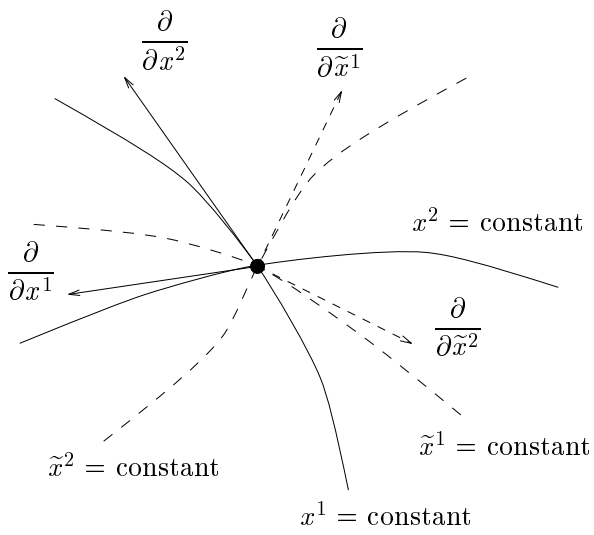

Figure 2.2: Two different coordinate systems in the plane, (x^1, x^2) and $(\widetilde{x}^1, \widetilde{x}^2)$, and the corresponding partial derivatives (directional derivatives) at one point.

Here $\widetilde{A}^0, \widetilde{A}^1, \widetilde{A}^2, \widetilde{A}^3$ is just a shorthand notation for $A^{\widetilde{x}^0}, A^{\widetilde{x}^1}, A^{\widetilde{x}^2}, A^{\widetilde{x}^3}$, the components of $\mathbf{A}$ relative to the coordinate system $\widetilde{x}^0, \widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3$.

The transformation formula for the components of the contravariant vector is easily found from the above relation, as follows,

$$\widetilde{A}^\mu = \mathbf{A}(\widetilde{x}^\mu) = \widetilde{A}^\kappa \frac{\partial \widetilde{x}^\mu}{\partial \widetilde{x}^\kappa} = A^\kappa \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa} \ . \tag{2.26}$$

The inverse transformation is

$$A^\mu = \widetilde{A}^\kappa \frac{\partial x^\mu}{\partial \widetilde{x}^\kappa} \ . \tag{2.27}$$

It is fully possible to regard this as a definition: a tangent vector is an object having components that get transformed according to these two formulae.

It is all important to understand how to read an equation such as the transformation formula

$$\widetilde{A}^\mu = A^\kappa \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa} \ . \tag{2.28}$$

The two greek indices μ and κ are both coordinate indices, but play fundamentally different roles. On each side of the equation μ is what we call a *free index*: it is free to take any one of the four possible values $0, 1, 2, 3$. In every meaningful tensor equation there must be exactly the same free indices on both sides of the equality sign. Since the one free index μ here may take four values, the single equation actually represents four equations: one equation for each value of μ. The other index κ occurs in the equation only on the right hand side. Since it occurs twice in a product, it is a *summation index*, and the right hand side is a sum over four terms in which κ takes successively the four values $0, 1, 2, 3$, whereas μ takes the same value in all four terms.

There is an easy way to remember the transformation formula from "old" coordinates x to "new" coordinates $\widetilde{x}$, Equation (2.28). Just observe that there is one index, κ, referring to the "old" coordinates, and one index, μ, referring to the "new" coordinates.

From contravariant vectors we generalize to contravariant tensors. That $\mathbf{A}$ is a *contravariant tensor of rank two* means that it can be written as

$$\mathbf{A} = A^{\mu\nu}\, \mathbf{e}_\mu \otimes \mathbf{e}_\nu \ , \tag{2.29}$$

where $\mathbf{e}_\mu \otimes \mathbf{e}_\nu$ is the *tensor product* of the basis vectors. This definition is circular, strictly speaking, since we have not defined the tensor product in an independent way. The important point is that the components $A^{\mu\nu}$ have two upper indices, and as long as we stick to the coordinate basis vectors, Equation (2.20), the following transformation formula is to be applied when we change our coordinate system,

$$\widetilde{A}^{\mu\nu} = \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\, \frac{\partial \widetilde{x}^\nu}{\partial x^\lambda}\, A^{\kappa\lambda} \ . \tag{2.30}$$

Each of the two indices gets transformed as a contravariant vector index. This transformation property of the tensor components may be taken as the definition of a contravariant tensor of rank two.

It should be obvious how to generalize further to tensors with any number of contravariant indices. Each contravariant index gets transformed under a coordinate transformation $x^\mu \mapsto \widetilde{x}^\mu$ by means of the matrix $\partial \widetilde{x}^\mu / \partial x^\nu$, consisting of all partial derivatives of the "new" coordinates with respct to the "old" ones.

Take as an example the connection between the Cartesian coordinates x, y in the plane and the polar coordinates r, φ, as given in the Equations (1.72), (1.79) and (1.80). Equation (2.26) gives for a contravariant vector $\mathbf{A}$ that

$$\begin{aligned} A^r &= \frac{\partial r}{\partial x}\, A^x + \frac{\partial r}{\partial y}\, A^y = \ \ \cos\varphi\, A^x + \sin\varphi\, A^y \ , \\ A^\varphi &= \frac{\partial \varphi}{\partial x}\, A^x + \frac{\partial \varphi}{\partial y}\, A^y = -\frac{\sin\varphi}{r}\, A^x + \frac{\cos\varphi}{r}\, A^y \ . \end{aligned} \tag{2.31}$$

Equation (2.30) gives for a contravariant tensor of rank two that, e.g.,

$$\begin{aligned} A^{r\varphi} &= \frac{\partial r}{\partial x}\, \frac{\partial \varphi}{\partial x}\, A^{xx} + \frac{\partial r}{\partial y}\, \frac{\partial \varphi}{\partial x}\, A^{yx} + \frac{\partial r}{\partial x}\, \frac{\partial \varphi}{\partial y}\, A^{xy} + \frac{\partial r}{\partial y}\, \frac{\partial \varphi}{\partial y}\, A^{yy} \\ &= \frac{\cos\varphi \sin\varphi}{r}\, (-A^{xx} + A^{yy}) - \frac{\sin^2\varphi}{r}\, A^{yx} + \frac{\cos^2\varphi}{r}\, A^{xy} \ . \end{aligned} \tag{2.32}$$

Commutation of contravariant vector fields

A vector field $\mathbf{A}$ acting on a scalar field ϕ gives a new scalar field $\mathbf{A}(\phi)$, by Equation (2.16). Hence, given two vector fields

$$\mathbf{A} = A^\mu\, \frac{\partial}{\partial x^\mu} \ , \qquad \mathbf{B} = B^\nu\, \frac{\partial}{\partial x^\nu} \ , \tag{2.33}$$

we may act on the scalar field ϕ with both of them in succession. For example, acting with $\mathbf{B}$ first and with $\mathbf{A}$ afterwards gives the following result,

$$\mathbf{A}(\mathbf{B}(\phi)) = A^\mu\, \frac{\partial}{\partial x^\mu} \left(B^\nu\, \frac{\partial \phi}{\partial x^\nu} \right) = A^\mu\, \frac{\partial B^\nu}{\partial x^\mu}\, \frac{\partial \phi}{\partial x^\nu} + A^\mu B^\nu\, \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} \ . \tag{2.34}$$

By definition, the operator product $\mathbf{AB}$ is the operator mapping ϕ into $\mathbf{A}(\mathbf{B}(\phi))$. It is not a vector field, because it involves second order derivatives of ϕ.

The partial derivative operators $\partial/\partial x^\mu$ and $\partial/\partial x^\nu$ commute, and therefore we have that

$$A^\mu B^\nu \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} = A^\mu B^\nu \frac{\partial^2 \phi}{\partial x^\nu \partial x^\mu} = A^\nu B^\mu \frac{\partial^2 \phi}{\partial x^\mu \partial x^\nu} \; . \tag{2.35}$$

The last equality sign is obtained by the frequently used trick of renaming the summation indices μ and ν: the value of a sum never depends on the name of a summation index. This relation implies that the second derivative terms cancel in the *commutator* between $\mathbf{A}$ and $\mathbf{B}$, which is the operator $\mathbf{C} = [\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$, explicitly defined by the relation

$$\mathbf{C}(\phi) = \mathbf{A}(\mathbf{B}(\phi)) - \mathbf{B}(\mathbf{A}(\phi)) = \left(A^\mu \frac{\partial B^\nu}{\partial x^\mu} - B^\mu \frac{\partial A^\nu}{\partial x^\mu} \right) \frac{\partial \phi}{\partial x^\nu} \; . \tag{2.36}$$

Thus, $\mathbf{C}$ is a vector field with the components

$$C^\nu = A^\mu \frac{\partial B^\nu}{\partial x^\mu} - B^\mu \frac{\partial A^\nu}{\partial x^\mu} \; . \tag{2.37}$$

It follows directly from the definition that the commutator is antisymmetric,

$$[\mathbf{B}, \mathbf{A}] = -[\mathbf{A}, \mathbf{B}] \; . \tag{2.38}$$

The Jacobi identity

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = 0 \tag{2.39}$$

follows directly from the definition $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$, together with the associative law $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

2.5 Covariant vectors, mixed tensors, and contraction

A *covariant vector* is the same as a *1-form*,

$$\mathbf{B} = B_\mu \, \mathbf{e}^\mu = B_\mu \, \mathrm{d}x^\mu \; , \tag{2.40}$$

where we have used the coordinate basis vectors

$$\mathbf{e}^\mu = \mathrm{d}x^\mu \; . \tag{2.41}$$

The components of $\mathbf{B}$ we call B_{x^μ}, or more simply B_μ, with a lower index.

The 1-form $\mathbf{B}$ is an object which can be integrated along the curve C defined by Equation (2.12). If the curve parameter u ranges from u_1 to u_2, then the integral is

$$\int_C \mathbf{B} = \int_C B_\mu \, \mathrm{d}x^\mu = \int_{u_1}^{u_2} \mathrm{d}u \; B_\mu(x(u)) \, \frac{\mathrm{d}x^\mu(u)}{\mathrm{d}u} \; . \tag{2.42}$$

The derivatives $\mathrm{d}x^\mu/\mathrm{d}u$ in the integral are the components of the tangent vector of the curve, as in Equation (2.13).

The covariant vector $\mathbf{B}$ is the same if we choose a different coordinate system, hence

$$\mathbf{B} = B_\mu \, \mathrm{d}x^\mu = \widetilde{B}_\mu \, \mathrm{d}\widetilde{x}^\mu \; , \tag{2.43}$$

and this determines the transformation formula for the components. Since

$$\mathrm{d}x^\kappa = \frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,\mathrm{d}\widetilde{x}^\mu\ , \tag{2.44}$$

by the chain rule, we must have that

$$\widetilde{B}_\mu = \frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,B_\kappa\ . \tag{2.45}$$

Again, $\widetilde{B}_\mu$ is a simpler notation for $B_{\widetilde{x}_\mu}$.

If $\mathbf{B}$ is a *covariant tensor of rank two*, this means that it can be written as

$$\mathbf{B} = B_{\mu\nu}\,\mathbf{e}^\mu \otimes \mathbf{e}^\nu = B_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu\ , \tag{2.46}$$

where $\mathbf{e}^\mu \otimes \mathbf{e}^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ is the tensor product of the basis vectors. The components $B_{\mu\nu}$ are written with two lower indices, each of which gets transformed in the same way as a covariant vector index,

$$\widetilde{B}_{\mu\nu} = \frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,B_{\kappa\lambda}\ . \tag{2.47}$$

Again it should be obvious how to generalize to tensors with an arbitrary number of covariant indices, and further to tensors with an arbitrary combination of contravariant and covariant indices. A tensor of rank n has n indices, whether they are contravariant or covariant. If we want to be a little more precise, we may say that a tensor has rank (m, n) if it has m contravariant and n covariant indices.

For a tensor A^μ_ν with one contravariant and one covariant index the following transformation formula holds,

$$\widetilde{A}^\mu_\nu = \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,A^\kappa_\lambda\ . \tag{2.48}$$

We may *contract* the upper and the lower index, that is, we put them equal and sum over all possible values. The result of this operation is independent of the coordinate system, in fact we have that

$$\widetilde{A}^\mu_\mu = \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\mu}\,A^\kappa_\lambda = \delta^\lambda_\kappa A^\kappa_\lambda = A^\kappa_\kappa\ . \tag{2.49}$$

The result of the contraction is a scalar, i.e. a quantity which is independent of the coordinate system. If we regard A^μ_ν as a matrix, then A^μ_μ is the sum of the diagonal matrix elements, the *trace* of the matrix.

We may always perform such a contraction between an upper and a lower index, no matter how many other indices a tensor has. The contraction produces a new tensor having one upper and one lower index less than the uncontracted tensor. We produce a tensor of rank $(m-1, n-1)$ from a tensor of rank (m, n). Tensor product and contraction are opposite operations in the sense that one increases and the other one decreases the number of indices.

Let us elaborate on our favourite example, the connection between Cartesian coordinates x, y and polar coordinates r, φ in the plane, as given in the Equations (1.72), (1.79) and (1.80). Equation (2.45) gives for a covariant vector $\mathbf{B}$ that

$$\begin{aligned} B_r &= \frac{\partial x}{\partial r}\,B_x + \frac{\partial y}{\partial r}\,B_y = \quad\ \cos\varphi\,B_x + \ \sin\varphi\,B_y\ , \\ B_\varphi &= \frac{\partial x}{\partial \varphi}\,B_x + \frac{\partial y}{\partial \varphi}\,B_y = -r\sin\varphi\,B_x + r\cos\varphi\,B_y\ . \end{aligned} \tag{2.50}$$

For a mixed contravariant and covariant tensor of rank 2, i.e. rank $(1,1)$, we get for the diagonal elements that

$$\begin{aligned} A^r_r &= \frac{\partial r}{\partial x}\frac{\partial x}{\partial r}A^x_x + \frac{\partial r}{\partial y}\frac{\partial x}{\partial r}A^y_x + \frac{\partial r}{\partial x}\frac{\partial y}{\partial r}A^x_y + \frac{\partial r}{\partial y}\frac{\partial y}{\partial r}A^y_y \\ &= \cos^2\varphi\, A^x_x + \cos\varphi\sin\varphi\,(A^y_x + A^x_y) + \sin^2\varphi\, A^y_y\,, \\ A^\varphi_\varphi &= \frac{\partial \varphi}{\partial x}\frac{\partial x}{\partial \varphi}A^x_x + \frac{\partial \varphi}{\partial y}\frac{\partial x}{\partial \varphi}A^y_x + \frac{\partial \varphi}{\partial x}\frac{\partial y}{\partial \varphi}A^x_y + \frac{\partial \varphi}{\partial y}\frac{\partial y}{\partial \varphi}A^y_y \\ &= \sin^2\varphi\, A^x_x - \cos\varphi\sin\varphi\,(A^y_x + A^x_y) + \cos^2\varphi\, A^y_y\,. \end{aligned} \tag{2.51}$$

This shows explicitly that contraction gives the same result in Cartesian and polar coordinates,

$$A^r_r + A^\varphi_\varphi = A^x_x + A^y_y\,. \tag{2.52}$$

2.6 Symmetry and antisymmetry of tensors

An important consequence of the transformation formula in Equation (2.30) is that if a contravariant tensor of rank two has either symmetric or antisymmetric components in one coordinate system, that is, if

$$A^{\mu\nu} = \epsilon A^{\nu\mu}\,, \tag{2.53}$$

with $\epsilon = 1$ or $\epsilon = -1$, then it has components with the same symmetry in any coordinate system,

$$\widetilde{A}^{\mu\nu} = \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\frac{\partial \widetilde{x}^\nu}{\partial x^\lambda}A^{\kappa\lambda} = \epsilon\,\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\frac{\partial \widetilde{x}^\nu}{\partial x^\lambda}A^{\lambda\kappa} = \epsilon\widetilde{A}^{\nu\mu}\,. \tag{2.54}$$

Thus, the property of such a tensor that it has symmetric components ($\epsilon = +1$), or antisymmetric components ($\epsilon = -1$), is independent of the coordinate system, it is a property of the tensor itself. This observation is quite generally valid, for contravariant or covariant tensors, for tensors with any number of indices, and for general symmetry classes.

Totally symmetric and totally antisymmetric tensors of arbitrary rank are of particular interest. A contravariant tensor, as an example, is symmetric, or totally symmetric, if the order of the indices is irrelevant for the value of a component $A^{\mu\nu\cdots\rho}$. It is antisymmetric, or totally antisymmetric, if the order of the indices is relevant only for the sign of the component $A^{\mu\nu\cdots\rho}$, so that the sign changes for every interchange of two indices. A couple of examples should illustrate the concepts.

An arbitrary contravariant tensor of rank two, $A^{\mu\nu}$, may be written as a sum

$$A^{\mu\nu} = A^{(\mu\nu)} + A^{[\mu\nu]} \tag{2.55}$$

of a symmetric part

$$A^{(\mu\nu)} = \frac{1}{2}\,(A^{\mu\nu} + A^{\nu\mu}) \tag{2.56}$$

and an antisymmetric part

$$A^{[\mu\nu]} = \frac{1}{2}\left(A^{\mu\nu} - A^{\nu\mu}\right) . \tag{2.57}$$

This splitting is unique and, as already stated, invariant under coordinate transformations.

A contravariant tensor of rank three, $A^{\lambda\mu\nu}$, has one totally symmetric part,

$$A^{(\lambda\mu\nu)} = \frac{1}{6}\left(A^{\lambda\mu\nu} + A^{\nu\lambda\mu} + A^{\mu\nu\lambda} + A^{\mu\lambda\nu} + A^{\lambda\nu\mu} + A^{\nu\mu\lambda}\right) , \tag{2.58}$$

one totally antisymmetric part,

$$A^{[\lambda\mu\nu]} = \frac{1}{6}\left(A^{\lambda\mu\nu} + A^{\nu\lambda\mu} + A^{\mu\nu\lambda} - A^{\mu\lambda\nu} - A^{\lambda\nu\mu} - A^{\nu\mu\lambda}\right) , \tag{2.59}$$

and in addition a third part of mixed symmetry,

$$A^{\lambda\mu\nu} - A^{(\lambda\mu\nu)} - A^{[\lambda\mu\nu]} = \frac{1}{3}\left(2A^{\lambda\mu\nu} - A^{\nu\lambda\mu} - A^{\mu\nu\lambda}\right) . \tag{2.60}$$

This splitting into different symmetry components is still invariant under coordinate transformations.

In the same way, we may split covariant tensors with two or three indices into symmetry components. Tensors with more than three indices, contravariant or covariant, may also be split into symmetry components, but the number of different components increases rapidly with the number of indices. Partial symmetrization or antisymmetrization is possible, we may define for example

$$A^{\lambda[\mu\nu]} = \frac{1}{2}\left(A^{\lambda\mu\nu} - A^{\lambda\nu\mu}\right) . \tag{2.61}$$

Contraction, symmetry and antisymmetry

One result which is trivial enough, but is used so often that it merits special attention, is that contraction of two symmetric indices against two antisymmetric indices must always give zero. For example, $A^{\mu\nu}B_{\mu\nu} = 0$ if $A^{\mu\nu} = A^{\nu\mu}$ and $B_{\mu\nu} = -B_{\nu\mu}$. The proof is as follows,

$$A^{\mu\nu}B_{\mu\nu} = A^{\nu\mu}B_{\mu\nu} = -A^{\nu\mu}B_{\nu\mu} = -A^{\mu\nu}B_{\mu\nu} . \tag{2.62}$$

The last equality follows from the fact that μ and ν are summation indices, so that we are free to rename them as we like, for example to interchange their names. When $A^{\mu\nu}B_{\mu\nu} = -A^{\mu\nu}B_{\mu\nu}$, it means that $A^{\mu\nu}B_{\mu\nu} = 0$.

2.7 The metric tensor

The Kronecker symbol with one upper and one lower index,

$$\delta^{\mu}_{\nu} = \begin{cases} 1 & \text{when} \quad \mu = \nu, \\ 0 & \text{when} \quad \mu \neq \nu, \end{cases} \tag{2.63}$$

actually defines a tensor. In fact, if we define A^μ_ν to be a tensor with components $A^\mu_\nu = \delta^\mu_\nu$ in one coordinate system, then when we transform to another coordinate system the transformed components are

$$\widetilde{A}^\mu_\nu = \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,A^\kappa_\lambda = \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\delta^\kappa_\lambda = \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\frac{\partial x^\kappa}{\partial \widetilde{x}^\nu} = \delta^\mu_\nu\ . \tag{2.64}$$

This is a rather special tensor: it is invariant in the sense that it has the same components δ^μ_ν in any coordinate system.

The metric tensor $g_{\mu\nu}$, the metric for short, is used for lowering indices. It is a covariant tensor of rank two. For example, we convert a contravariant vector A^μ into a covariant vector A_μ, which we call by the same name A, by defining

$$A_\mu = g_{\mu\kappa}A^\kappa\ . \tag{2.65}$$

We define the scalar product between two contravariant vectors $\mathbf{A}$ and $\mathbf{B}$ as

$$\mathbf{A}\cdot\mathbf{B} = A^\mu B_\mu = g_{\mu\nu}A^\mu B^\nu\ . \tag{2.66}$$

Thus, the metric tensor enables us to contract two upper indices against each other. The scalar product $\mathbf{A}\cdot\mathbf{B}$ is a scalar, as the name says. If the scalar product of a vector $\mathbf{A}$ with itself, $\mathbf{A}^2 = \mathbf{A}\cdot\mathbf{A}$, is non-negative, then we may define the length of $\mathbf{A}$ as

$$|\mathbf{A}| = \sqrt{\mathbf{A}^2}\ . \tag{2.67}$$

If $\mathbf{A}^2 > 0$ and $\mathbf{B}^2 > 0$, then we may define the angle θ between the vectors $\mathbf{A}$ and $\mathbf{B}$ by the relation

$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta\ . \tag{2.68}$$

We assume that the matrix $g_{\mu\nu}$ is always non-singular, i.e. invertible. The inverse matrix we write as $g^{\mu\nu}$, and we use it for raising indices,

$$A^\mu = g^{\mu\kappa}A_\kappa\ . \tag{2.69}$$

That $g_{\mu\nu}$ and $g^{\mu\nu}$ are each others inverses, means that

$$g^{\mu\kappa}g_{\kappa\nu} = g_{\nu\kappa}g^{\kappa\mu} = \delta^\mu_\nu\ . \tag{2.70}$$

This definition is independent of the coordinate system, since contraction is a coordinate independent operation, and since the Kronecker tensor δ^μ_ν has the same components in all coordinate systems. By definition, the result of raising one index of the metrix tensor is δ^μ_ν, thus the Kronecker tensor is simply the metric tensor with one upper and one lower index.

We will always assume here that the metric tensor is symmetric, i.e. that $g_{\mu\nu} = g_{\nu\mu}$. It follows that $g^{\mu\nu} = g^{\nu\mu}$, since the inverse of a symmetric matrix is symmetric.

The symmetry has another less obvious consequence. By a suitable coordinate transformation,

$$\widetilde{g}_{\mu\nu} = \frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,g_{\kappa\lambda}\ , \tag{2.71}$$

we may always diagonalize the metric at one arbitrary isolated point, that is, we may obtain that $\widetilde{g}_{\mu\nu} = 0$ for $\mu \neq \nu$, at this one point. The number of positive and negative diagonal elements of the metric in such a coordinate system is a characteristic property which is called the *signature* of the metric. The metric in spacetime has everywhere signature $(1,3)$, that is, when diagonalized it has always one positive diagonal element, corresponding to the time direction, and three negative diagonal elements, corresponding to the spatial directions. The overall sign has no very deep significance, so that a metric of signature $(3,1)$ may serve just as well as a metric of signature $(1,3)$. In fact, these opposite sign conventions are both in common use, but we have to fix a convention and we decide on the $(1,3)$ convention.

An important quantity is the determinant of the metric tensor,

$$g = \det(g_{\mu\nu}) \; . \tag{2.72}$$

It may be positive or negative, dependent on the signature of the metric, and for the signature $(1,3)$ we have $g < 0$.

Variation of the metric

An infinitesimal variation $\Delta g_{\mu\nu}$ of $g_{\mu\nu}$ will lead to infinitesimal variations $\Delta g^{\mu\nu}$ of $g^{\mu\nu}$ and Δg of g. The definition of $g^{\mu\nu}$, Equation (2.70), gives immediately that

$$\Delta g^{\mu\kappa} g_{\kappa\lambda} + g^{\mu\kappa} \Delta g_{\kappa\lambda} = 0 \; . \tag{2.73}$$

Multiplying with $g^{\nu\lambda}$ and using the relation $g^{\nu\lambda} g_{\kappa\lambda} = \delta^{\nu}_{\kappa}$, we deduce that

$$\Delta g^{\mu\nu} = -g^{\mu\kappa} g^{\nu\lambda} \Delta g_{\kappa\lambda} \; . \tag{2.74}$$

The relation between Δg and $\Delta g_{\mu\nu}$ is slightly more involved. In order to derive it we may use the connection between the determinant and the Levi–Civita symbol, Equation (1.5). In four dimensions we have that

$$g_{\alpha\kappa}\, g_{\beta\lambda}\, g_{\gamma\mu}\, g_{\delta\nu}\, \epsilon^{\kappa\lambda\mu\nu} = g\, \epsilon_{\alpha\beta\gamma\delta} \; . \tag{2.75}$$

The Levi–Civita symbol with upper or lower indices, $\epsilon^{\kappa\lambda\mu\nu}$ or $\epsilon_{\kappa\lambda\mu\nu}$, is totally antisymmetric, and $\epsilon^{0123} = \epsilon_{0123} = 1$. The convention that the metric tensor is used for lowering and raising indices does not hold for the Levi–Civita symbol, as we define it here. The above relation means e.g. that

$$g_{\beta\lambda}\, g_{\gamma\mu}\, g_{\delta\nu}\, \epsilon^{\kappa\lambda\mu\nu} = g\, g^{\kappa\rho}\, \epsilon_{\rho\beta\gamma\delta} \; . \tag{2.76}$$

By varying $g_{\mu\nu}$, we get for example that

$$\begin{aligned} \Delta g\, \epsilon_{\alpha\beta\gamma\delta} \;=\; & (\Delta g_{\alpha\kappa}\, g_{\beta\lambda}\, g_{\gamma\mu}\, g_{\delta\nu} + g_{\alpha\kappa}\, \Delta g_{\beta\lambda}\, g_{\gamma\mu}\, g_{\delta\nu} \\ & + g_{\alpha\kappa}\, g_{\beta\lambda}\, \Delta g_{\gamma\mu}\, g_{\delta\nu} + g_{\alpha\kappa}\, g_{\beta\lambda}\, g_{\gamma\mu}\, \Delta g_{\delta\nu})\, \epsilon^{\kappa\lambda\mu\nu} \\ =\; & \Delta g_{\alpha\kappa}\, g^{\kappa\rho}\, g\, \epsilon_{\rho\beta\gamma\delta} + \Delta g_{\beta\lambda}\, g^{\lambda\rho}\, g\, \epsilon_{\alpha\rho\gamma\delta} \\ & + \Delta g_{\gamma\mu}\, g^{\mu\rho}\, g\, \epsilon_{\alpha\beta\rho\delta} + \Delta g_{\delta\nu}\, g^{\nu\rho}\, g\, \epsilon_{\alpha\beta\gamma\rho} \; . \end{aligned} \tag{2.77}$$

We multiply this equation by $\epsilon^{\alpha\beta\gamma\delta}$ and use that, e.g.,

$$\epsilon^{\alpha\beta\gamma\delta} \epsilon_{\rho\beta\gamma\delta} = 6\delta^{\alpha}_{\rho} \; , \qquad \epsilon^{\alpha\beta\gamma\delta} \epsilon_{\alpha\beta\gamma\delta} = 24 \; . \tag{2.78}$$

Then we arrive at the following relation, which can be proved by similar methods in an arbitrary dimension,

$$\Delta g = g\, g^{\nu\mu}\, \Delta g_{\mu\nu} = -g\, g_{\nu\mu}\, \Delta g^{\mu\nu} \ . \tag{2.79}$$

Rewriting this formula in the form

$$g^{\nu\mu} = \frac{1}{g}\, \frac{\partial g}{\partial g_{\mu\nu}} \ , \tag{2.80}$$

we may recognize it as Kramer's rule for inverting a matrix. Here we differentiate with respect to one component, say g_{01}, as if g_{01} and g_{10} were independent, in spite of the symmetry condition $g_{\mu\nu} = g_{\nu\mu}$.

Later on, we will need a formula for the variation of $\sqrt{|g|}$, which is

$$\Delta\sqrt{|g|} = \frac{1}{2\sqrt{|g|}}\, \frac{|g|}{g}\, \Delta g = \frac{1}{2}\, \sqrt{|g|}\, g^{\nu\mu}\, \Delta g_{\mu\nu} = -\frac{1}{2}\, \sqrt{|g|}\, g_{\nu\mu}\, \Delta g^{\mu\nu} \ . \tag{2.81}$$

The special theory of relativity

In the special theory of relativity we normally choose time and space coordinates such that the metric gets the simple form

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} , \tag{2.82}$$

first introduced by Minkowski in 1908, although with the opposite sign. Then we have

$$g^{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{2.83}$$

It is not always possible to choose the coordinate system so that the metric gets this simple form. The necessary and sufficient condition is that the curvature tensor defined by the metric vanishes, and of course that the signature is $(1, 3)$. See the Chapters 5 and 6. Thus, the specialty of the special theory of relativity is the assumption that spacetime is flat.

Problems

1. A Möbius strip is made e.g. from a rectangle in the (x, y) plane with its corners at $(0, 0)$, $(5, 0)$, $(0, 1)$ and $(5, 1)$. We glue the two edges $x = 0$ and $x = 5$ against each other, with the two opposite corners $(0, 0)$ and $(5, 1)$ together, and the corners $(0, 1)$ and $(5, 0)$ also together.
   Prove that the Möbius strip is not orientable.

2. Define $f(0) = 0$, and
$$f(x) = \mathrm{e}^{-\frac{1}{x^2}} \quad \text{for} \quad x \neq 0 \, .$$
   Show that this function $f : \mathbf{R} \mapsto \mathbf{R}$ is C^∞ (infinitely differentiable) everywhere.
   Find its Taylor series expansion about $x = 0$.
   Is it C^ω (analytic) at $x = 0$? At $x \neq 0$?

3. Express the Euclidean metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$ in the (x, y) plane in the polar coordinates (r, φ). By definition, $r \in [0, \infty\rangle$, $\varphi \in [0, 2\pi]$, and $x = r\cos\varphi$, $y = r\sin\varphi$.

4. Express the three dimensional Euclidean metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ in the polar coordinates (r, θ, φ), defined by the relations $r \in [0, \infty\rangle$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$, and
$$x = r\sin\theta\cos\varphi \, , \qquad y = r\sin\theta\sin\varphi \, , \qquad z = r\cos\theta \, .$$
   Compare with Equation (1.15).

5. Put $r = a =$ constant in Problem 4. This defines the surface of a sphere of radius a, such that a point on the surface is given by the polar coordinates (θ, φ).
   Write in polar coordinates the metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ on the surface of the sphere.

Chapter 3

Tensor algebra

3.1 Tensor product

As an example, let $\mathbf{A}$ be a contravariant vector with components A^μ and $\mathbf{B}$ a covariant vector with components B_ν. Then the tensor product of $\mathbf{A}$ and $\mathbf{B}$ is a tensor $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ having the components

$$C^\mu{}_\nu = A^\mu B_\nu \, . \tag{3.1}$$

If $\mathbf{A}$ and $\mathbf{B}$ are tensor fields, then $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ is a tensor field. The tensor product of fields is a pointwise product: the value of the field $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ at the point x is $\mathbf{C}(x) = \mathbf{A}(x) \otimes \mathbf{B}(x)$.

The tensor product is identical to the usual product in the special case where at least one of the factors is a scalar, or a scalar field. In general, the tensor product of two tensors of ranks m and n is a tensor of rank $m + n$. The fact that the tensor product is usually a tensor of higher rank than each of the factors, distinguishes it from other types of product, such as multiplication of numbers or matrices, scalar product between vectors, or the three dimensional vector product.

The tensor product is non-commutative,

$$\mathbf{A} \otimes \mathbf{B} \neq \mathbf{B} \otimes \mathbf{A} \quad \text{when} \quad \mathbf{A} \neq \mathbf{B} \, , \tag{3.2}$$

except when one of the factors is a scalar. In the above example, the tensor product $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ of a contravariant vector $\mathbf{A}$ and a covariant vector $\mathbf{B}$, the components of $\mathbf{D} = \mathbf{B} \otimes \mathbf{A}$ are

$$D_\nu{}^\mu = B_\nu A^\mu = C^\mu{}_\nu \, . \tag{3.3}$$

As another example, the tensor products $\mathbf{C} = \mathbf{A} \otimes \mathbf{B}$ and $\mathbf{D} = \mathbf{B} \otimes \mathbf{A}$ of two contravariant vectors $\mathbf{A} \neq \mathbf{B}$ are also different, because

$$C^{\mu\nu} = A^\mu B^\nu = D^{\nu\mu} \neq D^{\mu\nu} \, . \tag{3.4}$$

The tensor product is assosiative,

$$(\mathbf{A} \otimes \mathbf{B}) \otimes \mathbf{C} = \mathbf{A} \otimes (\mathbf{B} \otimes \mathbf{C}) \, , \tag{3.5}$$

and therefore we write $\mathbf{A} \otimes \mathbf{B} \otimes \mathbf{C}$ without parentheses. It is bilinear: when a and b are scalars, then

$$\begin{aligned} \mathbf{A} \otimes (a\,\mathbf{B} + b\,\mathbf{C}) &= a\,\mathbf{A} \otimes \mathbf{B} + b\,\mathbf{A} \otimes \mathbf{C} \, , \\ (a\,\mathbf{A} + b\,\mathbf{B}) \otimes \mathbf{C} &= a\,\mathbf{A} \otimes \mathbf{C} + b\,\mathbf{B} \otimes \mathbf{C} \, . \end{aligned} \tag{3.6}$$

3.2 Forms and exterior product

Antisymmetric covariant tensors play a major role in the theory of differentiation and integration. Therefore they have got a formalism of their own with its own terminology. An antisymmetric covariant tensor of rank n is called an *n-form*. In particular, a 0-form is the same as a scalar, and a 1-form is the same as a covariant vector.

The simplest example where the antisymmetry enters is a 2-form $\mathbf{A}$. Using the coordinate basis, Equation (2.41), and exploiting the antisymmetry we get that

$$\mathbf{A} = A_{\mu\nu}\,\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = A_{\mu\nu}\,\frac{1}{2}\left(\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu\right) \ . \tag{3.7}$$

At this point it is natural to introduce the antisymmetric tensor product "$\wedge$" ("wedge"), also called the *exterior product*. For two basis vectors $\mathbf{e}^\mu = \mathrm{d}x^\mu$ and $\mathbf{e}^\nu = \mathrm{d}x^\nu$ we define

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu \ . \tag{3.8}$$

This definition gives that

$$\mathbf{A} = \frac{1}{2}\,A_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \sum_{\mu<\nu} A_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ . \tag{3.9}$$

In the first sum here we sum freely over all μ and ν, but since $A_{\mu\nu} = -A_{\nu\mu}$ and $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, two and two terms in the sum are equal. The factor of $1/2$ compensates for the double counting. In the last sum we sum over terms that are all independent, thus avoiding the extra factor of $1/2$. The following notation is sometimes used,

$$\mathbf{A} = \sum_{\mu<\nu} A_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = A_{|\mu\nu|}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \ , \tag{3.10}$$

with the absolute value sign in $|\mu\nu|$ meaning that $\mu < \nu$. The explicit sum is

$$\begin{aligned}\mathbf{A} = \quad & A_{01}\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 + A_{02}\,\mathrm{d}x^0 \wedge \mathrm{d}x^2 + A_{03}\,\mathrm{d}x^0 \wedge \mathrm{d}x^3 \\ & + A_{12}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 + A_{13}\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 + A_{23}\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \ .\end{aligned} \tag{3.11}$$

For three basis vectors we define

$$\begin{aligned}\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \quad & \mathrm{d}x^\lambda \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu + \mathrm{d}x^\nu \otimes \mathrm{d}x^\lambda \otimes \mathrm{d}x^\mu + \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\lambda \\ & - \mathrm{d}x^\mu \otimes \mathrm{d}x^\lambda \otimes \mathrm{d}x^\nu - \mathrm{d}x^\lambda \otimes \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu \otimes \mathrm{d}x^\lambda \ .\end{aligned} \tag{3.12}$$

And so on. The exterior product $\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \ldots \wedge \mathrm{d}x^\rho$ of n arbitrary basis vectors is by definition antisymmetric under an interchange of any two indices, hence it is zero if two indices are equal. The antisymmetry may be used in order to interchange the indices, e.g. so that they always come in an increasing order, $\lambda < \mu < \ldots < \rho$.

Unfortunately (but not surprisingly!) two different normalization conventions exist for the exterior product. The alternative convention prescribes an extra factor of $1/n!$ in the definition of the exterior product of n basis vectors. Whatever convention is adopted, factors like n or $n!$ must turn up somewhere.

In general we define the exterior product of an m-form

$$\mathbf{A} = \frac{1}{m!}\,A_{\alpha\beta\ldots\eta}\,\mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \ldots \wedge \mathrm{d}x^\eta \tag{3.13}$$

and an n-form

$$\mathbf{B} = \frac{1}{n!}\, B_{\mu\nu\ldots\rho}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \ldots \wedge \mathrm{d}x^\rho \tag{3.14}$$

as

$$\mathbf{A} \wedge \mathbf{B} = \frac{1}{m!\, n!}\, A_{\alpha\beta\ldots\eta}\, B_{\mu\nu\ldots\rho}\, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \ldots \wedge \mathrm{d}x^\eta \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \ldots \wedge \mathrm{d}x^\rho \,. \tag{3.15}$$

Thus, $\mathbf{C} = \mathbf{A} \wedge \mathbf{B}$ is an $(m+n)$-form,

$$\mathbf{C} = \frac{1}{(m+n)!}\, C_{\alpha\beta\ldots\eta\mu\nu\ldots\rho}\, \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta \wedge \ldots \wedge \mathrm{d}x^\eta \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \ldots \wedge \mathrm{d}x^\rho \,, \tag{3.16}$$

with components

$$C_{\alpha\beta\ldots\eta\mu\nu\ldots\rho} = \frac{(m+n)!}{m!\, n!}\, A_{[\alpha\beta\ldots\eta}\, B_{\mu\nu\ldots\rho]} \,. \tag{3.17}$$

The square parentheses "[ ]" imply an antisymmetrization over all of the $m+n$ indices.

The exterior product is associative: for an arbitrary m-form $\mathbf{A}$, n-form $\mathbf{B}$ and p-form $\mathbf{C}$ we have that

$$(\mathbf{A} \wedge \mathbf{B}) \wedge \mathbf{C} = \mathbf{A} \wedge (\mathbf{B} \wedge \mathbf{C}) \,, \tag{3.18}$$

and therefore we write $\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}$ without parentheses. Note that

$$\mathbf{A} \wedge \mathbf{B} = (-1)^{mn}\, \mathbf{B} \wedge \mathbf{A} \,. \tag{3.19}$$

The exterior product is antisymmetric if both factors are of odd rank, otherwise it is symmetric, i.e. commutative.

Let us take as example three dimensional space with Euclidean coordinates (x, y, z). The exterior product of two 1-forms

$$\mathbf{A} = A_x \mathrm{d}x + A_y \mathrm{d}y + A_z \mathrm{d}z \,, \qquad \mathbf{B} = B_x \mathrm{d}x + B_y \mathrm{d}y + B_z \mathrm{d}z \,, \tag{3.20}$$

is a 2-form

$$\begin{aligned} \mathbf{A} \wedge \mathbf{B} = \quad & (A_y B_z - A_z B_y)\, \mathrm{d}y \wedge \mathrm{d}z + (A_z B_x - A_x B_z)\, \mathrm{d}z \wedge \mathrm{d}x \\ & + (A_x B_y - A_y B_x)\, \mathrm{d}x \wedge \mathrm{d}y \,. \end{aligned} \tag{3.21}$$

In three dimensions there exist three linearly independent 1-forms, and just as many linearly independent 2-forms. It is therefore possible to define a *duality* between 1-forms and 2-forms, e.g. by postulating that

$$\mathrm{d}x \leftrightarrow \mathrm{d}y \wedge \mathrm{d}z \,, \qquad \mathrm{d}y \leftrightarrow \mathrm{d}z \wedge \mathrm{d}x \,, \qquad \mathrm{d}z \leftrightarrow \mathrm{d}x \wedge \mathrm{d}y \,. \tag{3.22}$$

In this way, the exterior product becomes identical to the well known vector product,

$$\mathbf{A} \wedge \mathbf{B} \leftrightarrow \boldsymbol{A} \times \boldsymbol{B} \,. \tag{3.23}$$

The exterior product between three 1-forms corresponds to the triple product between vectors,

$$\begin{aligned} \mathbf{A}\wedge\mathbf{B}\wedge\mathbf{C} \;&=\; ((A_yB_z - A_zB_y)C_x + (A_zB_x - A_xB_z)C_y \\ &\qquad +(A_xB_y - A_yB_x)C_z)\,\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z \\ &\leftrightarrow\; (\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{C} = \boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C})\;. \end{aligned} \tag{3.24}$$

At this point a warning may be useful. The exterior product $\mathbf{A}\wedge\mathbf{B}\wedge\mathbf{C}$ of three covariant vectors in three dimensions is not a vector, it is a 3-form, which is the dual of a scalar. Thus, it is fundamentally different from either of the two vector products $(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}$ and $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})$, which are both vectors. Unlike most products we are familiar with, the vector product is not associative, in general we have $(\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}\neq\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})$.

As we will see later on, it is possible to define quite generally a duality between n-forms and $(d-n)$-forms in d dimensions, for arbitrary d and n, as soon as we have given a metric tensor. The present example with $d=3$ and $n=1$ (or $n=2$) is just a special case.

Symmetric tensor product

We might want to define a symmetric tensor product similar to the antisymmetric product, writing for example

$$\mathrm{d}x^\mu\vee\mathrm{d}x^\nu = \mathrm{d}x^\mu\otimes\mathrm{d}x^\nu + \mathrm{d}x^\nu\otimes\mathrm{d}x^\mu\;. \tag{3.25}$$

The factors in the symmetric tensor product must be tensors of the same rank, and the rank of the product is twice the rank of each factor. In other respects, the symmetric tensor product is much like ordinary multiplication, since it is both associative and commutative.

For example, when we define the metric tensor $g_{\mu\nu}$ by writing the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\;, \tag{3.26}$$

we might interpret this as an expansion of the covariant tensor $\mathbf{g}$, of rank two, in terms of the basis $\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu$ in the space of all rank two covariant tensors,

$$\mathbf{g} = g_{\mu\nu}\,\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu = \frac{1}{2}\,g_{\mu\nu}\,(\mathrm{d}x^\mu\otimes\mathrm{d}x^\nu + \mathrm{d}x^\nu\otimes\mathrm{d}x^\mu)\;. \tag{3.27}$$

The last equality follows because the metric is symmetric, $g_{\mu\nu}=g_{\nu\mu}$. We will make no use of the symmetric tensor product here.

3.3 Tensor densities

We say that a quantity T^κ_λ, for example, is a *tensor density*, as opposed to a tensor, if it is subject to the following transformation formula,

$$\widetilde{T}^\kappa_\lambda = \left|\det\left(\frac{\partial x}{\partial\widetilde{x}}\right)\right|\,\frac{\partial\widetilde{x}^\kappa}{\partial x^\alpha}\,\frac{\partial x^\beta}{\partial\widetilde{x}^\lambda}\,T^\alpha_\beta\;, \tag{3.28}$$

containing the absolute value of the Jacobi determinant for *the inverse* coordinate transformation. Scalar densities are important in field theory, because a sufficient condition for the invariance of a theory under general coordinate transformations is that the Lagrange density transforms as a scalar density.

More generally, we may have tensor densities with an arbitrary number of contravariant and covariant indices, and of arbitrary *weight* ν, meaning that the transformation involves the ν-th power of the absolute value of the Jacobi determinant. The most important class of tensor densities are the antisymmetric contravariant densities of weight 1, which we call simply *densities*. A density of rank n, having n antisymmetric contravariant indices, may be called an n-density.

A tensor density of weight 0 is the same as a tensor. The difference between tensors and tensor densities is of course invisible if we restrict ourselves to volume preserving coordinate transformations, which have determinants of ± 1. Examples are space and time translations and Lorentz transformations.

A *pseudo tensor density* transforms exactly like a tensor density under a coordinate transformation with a positive Jacobi determinant, but with the opposite sign when the Jacobi determinant is negative. Thus, the above formula would be modified as follows,

$$\widetilde{T}^{\kappa}_{\lambda} = \det\left(\frac{\partial x}{\partial \widetilde{x}}\right) \frac{\partial \widetilde{x}^{\kappa}}{\partial x^{\alpha}} \frac{\partial x^{\beta}}{\partial \widetilde{x}^{\lambda}} T^{\alpha}_{\beta} \ . \tag{3.29}$$

The distinction between tensor densities and pseudo tensor densities is often not essential, and sometimes we refer to both of them as tensor densities.

The prototype of a scalar density is $\sqrt{|g|}$, since

$$\sqrt{|\widetilde{g}|} = \sqrt{|\det(\widetilde{g}_{\mu\nu})|} = \sqrt{\left|\det\left(\frac{\partial x^{\rho}}{\partial \widetilde{x}^{\mu}} \, g_{\rho\sigma} \, \frac{\partial x^{\sigma}}{\partial \widetilde{x}^{\nu}}\right)\right|} = \left|\det\left(\frac{\partial x}{\partial \widetilde{x}}\right)\right| \sqrt{|g|} \ . \tag{3.30}$$

This proves, by the way, that $\sqrt{|\det A_{\mu\nu}|}$ is a scalar density whenever $A_{\mu\nu}$ is a covariant tensor of rank two.

The product of a tensor density and a tensor is again a tensor density. For example, if A^{κ}_{λ} is a tensor, then $T^{\kappa}_{\lambda} = \sqrt{|g|}\, A^{\kappa}_{\lambda}$ is a tensor density.

The prototype of a pseudo density, in contrast to a density, is the Levi–Civita symbol with upper indices, $\epsilon^{\kappa\lambda\ldots\rho}$, i.e. $\epsilon^{\kappa\lambda\mu\nu}$ in the four dimensional case. In fact, if we define a quantity $T^{\kappa\lambda\mu\nu}$ having the components

$$T^{\kappa\lambda\mu\nu} = \epsilon^{\kappa\lambda\mu\nu} \tag{3.31}$$

in every coordinate system, then the transformation formula

$$\widetilde{T}^{\kappa\lambda\mu\nu} = \det\left(\frac{\partial x}{\partial \widetilde{x}}\right) \frac{\partial \widetilde{x}^{\kappa}}{\partial x^{\alpha}} \frac{\partial \widetilde{x}^{\lambda}}{\partial x^{\beta}} \frac{\partial \widetilde{x}^{\mu}}{\partial x^{\gamma}} \frac{\partial \widetilde{x}^{\nu}}{\partial x^{\delta}} T^{\alpha\beta\gamma\delta} \tag{3.32}$$

holds, because

$$\frac{\partial \widetilde{x}^{\kappa}}{\partial x^{\alpha}} \frac{\partial \widetilde{x}^{\lambda}}{\partial x^{\beta}} \frac{\partial \widetilde{x}^{\mu}}{\partial x^{\gamma}} \frac{\partial \widetilde{x}^{\nu}}{\partial x^{\delta}} \epsilon^{\alpha\beta\gamma\delta} = \det\left(\frac{\partial \widetilde{x}}{\partial x}\right) \epsilon^{\kappa\lambda\mu\nu} = \left(\det\left(\frac{\partial x}{\partial \widetilde{x}}\right)\right)^{-1} \epsilon^{\kappa\lambda\mu\nu} \ . \tag{3.33}$$

Thus, if we multiply a tensor by $\epsilon^{\kappa\lambda\mu\nu}$, and maybe contract one or more upper indices against the same number of lower indices, the result will be a (pseudo) tensor density.

Note that the scalar density $\sqrt{|g|}$ simply gives the "size" of the tensor density $\epsilon^{\kappa\lambda\mu\nu}$, as measured by the metric tensor. In fact, we have that

$$g_{\alpha\kappa}\, g_{\beta\lambda}\, g_{\gamma\mu}\, g_{\delta\nu}\, \epsilon^{\alpha\beta\gamma\delta}\, \epsilon^{\kappa\lambda\mu\nu} = 24g \ . \tag{3.34}$$

3.4 Duality

The Levi–Civita symbol may be used in a similar way as the metric for raising and lowering indices. In particular, if A, B_ν, $C_{\mu\nu}$, $D_{\lambda\mu\nu}$ and $E_{\kappa\lambda\mu\nu}$ are antisymmetric covariant tensors of ranks zero, one, two, three and four, respectively, then we define the *dual* quantities

$$\begin{aligned}
{}^\circ A^{\kappa\lambda\mu\nu} &= \epsilon^{\kappa\lambda\mu\nu} A \,, \\
{}^\circ B^{\kappa\lambda\mu} &= \epsilon^{\kappa\lambda\mu\nu} B_\nu \,, \\
{}^\circ C^{\kappa\lambda} &= \frac{1}{2}\,\epsilon^{\kappa\lambda\mu\nu} C_{\mu\nu} \,, \\
{}^\circ D^{\kappa} &= \frac{1}{6}\,\epsilon^{\kappa\lambda\mu\nu} D_{\lambda\mu\nu} \,, \\
{}^\circ E &= \frac{1}{24}\,\epsilon^{\kappa\lambda\mu\nu} E_{\kappa\lambda\mu\nu} \,.
\end{aligned} \tag{3.35}$$

They are then antisymmetric contravariant tensor densities of ranks four, three, two, one and zero, respectively.

We see that the dual of an n-form is a $(4-n)$-density, or more precisely a $(4-n)$ pseudo density, where the number 4 is the dimension of spacetime. The particular duality transformation defined here has the effect of raising indices, and we denote it by the symbol "$\circ$" placed to the *upper* left. The inverse duality transformation lowers the indices again, and it is effected by means of the Levi–Civita symbol with lower indices, $\epsilon_{\kappa\lambda\mu\nu}$. The dual of an n-density is a $(4-n)$-form, and we denote it here by the symbol "$_\circ$" placed to the *lower* left. In particular, if we take the duals of ${}^\circ A$, ${}^\circ B$, ${}^\circ C$, ${}^\circ D$ and ${}^\circ E$ above, we get that

$$\begin{aligned}
{}_\circ({}^\circ A) &= \frac{1}{24}\,\epsilon_{\kappa\lambda\mu\nu}\,{}^\circ A^{\kappa\lambda\mu\nu} = A \,, \\
{}_\circ({}^\circ B)_\alpha &= \frac{1}{6}\,\epsilon_{\kappa\lambda\mu\alpha}\,{}^\circ B^{\kappa\lambda\mu} = B_\alpha \,, \\
{}_\circ({}^\circ C)_{\alpha\beta} &= \frac{1}{2}\,\epsilon_{\kappa\lambda\alpha\beta}\,{}^\circ C^{\kappa\lambda} = C_{\alpha\beta} \,, \\
{}_\circ({}^\circ D)_{\alpha\beta\gamma} &= \epsilon_{\kappa\alpha\beta\gamma}\,{}^\circ D^{\kappa} = D_{\alpha\beta\gamma} \,, \\
{}_\circ({}^\circ E)_{\alpha\beta\gamma\delta} &= \epsilon_{\alpha\beta\gamma\delta}\,{}^\circ E = E_{\alpha\beta\gamma\delta} \,.
\end{aligned} \tag{3.36}$$

Recall our definition of the Levi–Civita symbol: it is totally antisymmetric in its indices, and $\epsilon^{0123} = \epsilon_{0123} = 1$.

Duality dependent on the metric

The duality transformation as defined above is independent of the metric. Like the metric, it raises lower indices and lowers upper indices, but unlike the metric it changes the number of indices, turning n indices into $4-n$. By invoking the metric, we may define a duality transformation that changes the number of indices without moving indices up and down. This means in 4 dimensions that an n-form is transformed into a dual $(4-n)$-form. This duality transformation between forms is called *Hodge duality*.

As an example, if $C_{\mu\nu}$ is a 2-form, then

$$ {}^{\circ *}C^{\kappa\lambda} = \sqrt{|g|}\, g^{\kappa\mu} g^{\lambda\nu} C_{\mu\nu} \tag{3.37} $$

is a 2-density, whereas

$$ {}^{*}C_{\alpha\beta} = {}_{\circ}({}^{\circ *}C)_{\alpha\beta} = \frac{1}{2}\, \epsilon_{\kappa\lambda\alpha\beta} \sqrt{|g|}\, g^{\kappa\mu} g^{\lambda\nu} C_{\mu\nu} \tag{3.38} $$

is a new 2-form, the dual of $C_{\mu\nu}$. We also have that

$$ {}^{*}C_{\alpha\beta} = \frac{\sqrt{|g|}}{2g}\, g_{\alpha\gamma} g_{\beta\delta}\, \epsilon^{\gamma\delta\mu\nu} C_{\mu\nu} \ , \tag{3.39} $$

because

$$ g_{\alpha\gamma} g_{\beta\delta}\, \epsilon^{\gamma\delta\mu\nu} = g_{\alpha\gamma} g_{\beta\delta}\, \epsilon^{\gamma\delta\rho\sigma} g_{\rho\kappa} g_{\sigma\lambda} g^{\kappa\mu} g^{\lambda\nu} = g\, \epsilon_{\alpha\beta\kappa\lambda} g^{\kappa\mu} g^{\lambda\nu} = g\, \epsilon_{\kappa\lambda\alpha\beta} g^{\kappa\mu} g^{\lambda\nu} \ . \tag{3.40} $$

Thus, making the same transformation twice gives that

$$ {}^{**}C_{\alpha\beta} = \frac{|g|}{4g}\, g_{\alpha\gamma} g_{\beta\delta} \epsilon^{\gamma\delta\rho\sigma} \epsilon_{\kappa\lambda\rho\sigma}\, g^{\kappa\mu} g^{\lambda\nu} C_{\mu\nu} = \frac{|g|}{g}\, C_{\alpha\beta} = \pm C_{\alpha\beta} \ . \tag{3.41} $$

The sign here depends only on the signature of the metric, in particular, the sign is minus when the signature is $(1, 3)$.

We have seen here only one special example, but it is generally true that the Hodge duality transformation of forms is its own inverse, up to a sign.

Problems

1. How many linearly independent 0-forms, 1-forms, 2-forms, and so on, exist at one point in a space of dimension d (e.g. $d = 1, 2, 3, 4$)?
   Show that the total number of n-forms in dimension d, summed over $n = 0, 1, 2, \ldots, d$, is 2^d.

2. Express the 2-form $\mathrm{d}x \wedge \mathrm{d}y$, in the (x, y) plane, in terms of the two dimensional polar coordinates (r, φ).

3. Express the 3-form $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$, in the (x, y, z) space, in terms of the three dimensional polar coordinates (r, θ, φ).

Chapter 4

Transformations and Lie algebras

We have seen in Chapter 2 how tensor components get transformed under general coordinate transformations. The transformations that we are interested in often belong to some continuous group of transformations, which means that they can be built up continuously from infinite successions of infinitesimal transformations. We can learn much of what is worth knowing about continuous transformations by studying only the infinitesimal transformations, and this restriction is often an important simplification. The mathematical theory of continuous transformation groups was developed by Sophus Lie.

4.1 Infinitesimal coordinate transformations

An infinitesimal coordinate transformation has the form

$$\widetilde{x}^\mu = x^\mu + \xi^\mu \ , \tag{4.1}$$

where $\xi^\mu = \xi^\mu(x)$ is infinitesimal. That is,

$$\xi^\mu = \epsilon A^\mu \ , \tag{4.2}$$

where ϵ is an infinitesimal parameter, and $A^\mu = A^\mu(x)$ is a finite contravariant vector field. The inverse transformation is

$$x = \widetilde{x} - \xi(x) = \widetilde{x} - \xi(\widetilde{x}) \ . \tag{4.3}$$

In these relations it does not matter whether we use $\xi(x)$ or $\xi(\widetilde{x})$. Since the difference between $A^\mu(x)$ and $A^\mu(\widetilde{x})$ is of order ϵ, the difference between $\xi^\mu(x)$ and $\xi^\mu(\widetilde{x})$ is of order ϵ^2 and hence negligible.

Fully written out, we have the relations

$$\begin{aligned} \widetilde{x}^\mu &= x^\mu + \epsilon A^\mu(x^0, x^1, x^2, x^3) \ , \\ x^\mu &= \widetilde{x}^\mu - \epsilon A^\mu(\widetilde{x}^0, \widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) \ . \end{aligned} \tag{4.4}$$

From the infinitesimal transformation we obtain a finite transformation

$$x \mapsto \widetilde{x} = \widetilde{x}(u) = f_u(x) \ , \tag{4.5}$$

dependent on a parameter u, by integrating the first order ordinary differential equations

$$\frac{\mathrm{d}}{\mathrm{d}u}\,\widetilde{x}^\mu(u) = A^\mu(\widetilde{x}(u))\ , \tag{4.6}$$

with the initial conditions $\widetilde{x}^\mu(0) = x^\mu$. These coordinate transformations are well defined for a range of positive and negative values of the parameter u, although it may happen that the integration of Equation (4.6) diverges at a finite value of u. If there is no divergence, so that the coordinate transformations are well defined for all positive and negative values of u, then they form a *one parameter group* of transformations, which means that

$$f_v(f_u(x)) = f_{v+u}(x) \tag{4.7}$$

for any values of u and v. By definition, f_0 is the identity transformation, $f_0(x) = x$.

Example: Rotation

In three dimensions, with coordinates $x^1 = x$, $x^2 = y$, and $x^3 = z$, a counterclockwise rotation about the z axis by an infinitesimal angle ϵ has the form

$$\widetilde{x} = x - \epsilon y\ , \qquad \widetilde{y} = y + \epsilon x\ , \qquad \widetilde{z} = z\ . \tag{4.8}$$

Thus, it is generated by the vector field

$$\mathbf{A} = (A^x, A^y, A^z) = (-y, x, 0)\ . \tag{4.9}$$

Now let the transformation

$$(x, y, z) \mapsto (\widetilde{x}, \widetilde{y}, \widetilde{z}) = (\widetilde{x}(\alpha), \widetilde{y}(\alpha), \widetilde{z}(\alpha)) \tag{4.10}$$

be the rotation about the z axis by the finite angle α. It is the solution of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\,(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (A^x(\widetilde{x}, \widetilde{y}, \widetilde{z}), A^y(\widetilde{x}, \widetilde{y}, \widetilde{z}), A^z(\widetilde{x}, \widetilde{y}, \widetilde{z})) = (-\widetilde{y}, \widetilde{x}, 0)\ , \tag{4.11}$$

with the initial condition $(\widetilde{x}(0), \widetilde{y}(0), \widetilde{z}(0)) = (x, y, z)$. It is easy to verify that the solution is

$$(\widetilde{x}, \widetilde{y}, \widetilde{z}) = (x\cos\alpha - y\sin\alpha\ ,\ x\sin\alpha + y\cos\alpha\ ,\ z)\ . \tag{4.12}$$

Example: Linear vector fields

The above rotation provides an example of a linear vector field, of the general form

$$A^\mu(x) = A^\mu_\nu\, x^\nu\ . \tag{4.13}$$

Here the coefficients

$$A^\mu_\nu = A^\mu_{,\nu} = \frac{\partial A^\mu}{\partial x^\nu} \tag{4.14}$$

are assumed to be constant, they are identical to the partial derivatives of the vector field components A^μ, which we usually denote by $A^\mu_{,\nu}$.

It is natural in this case to introduce matrix notation, writing, for example in the four dimensional case,

$$x = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} . \tag{4.15}$$

In matrix notation the linear equation

$$\frac{\mathrm{d}}{\mathrm{d}u}\,\widetilde{x}^\mu(u) = A^\mu(\widetilde{x}(u)) = A^\mu_\nu\,\widetilde{x}^\nu(u) \tag{4.16}$$

takes the form

$$\frac{\mathrm{d}}{\mathrm{d}u}\,\widetilde{x}(u) = A\widetilde{x}(u) . \tag{4.17}$$

The solution is, with the initial condition $\widetilde{x}(0) = x$, and with a constant matrix A,

$$\widetilde{x}(u) = \mathrm{e}^{uA}\,x , \tag{4.18}$$

where we define the exponential function of matrices by the standard power series expansion,

$$\mathrm{e}^{uA} = I + uA + \frac{u^2}{2}\,A^2 + \cdots + \frac{u^n}{n!}\,A^n + \cdots . \tag{4.19}$$

4.2 Transformations of tensor fields, Lie derivatives

Transformation of a scalar field

Consider first the transformation of a scalar field $\phi = \phi(x)$. A coordinate transformation is sometimes called a passive transformation, as discussed in Section 1.7. We write $\phi(P) = \phi(x)$ for the value of the field ϕ at a point P having the coordinates $x = (x^0, x^1, x^2, x^3)$ in one coordinate system. The coordinate transformation does not change the field value $\phi(P)$, but it changes the values of the coordinates of the point P from $x = (x^0, x^1, x^2, x^3)$ to $\widetilde{x} = (\widetilde{x}^0, \widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$. Hence, in the new coordinate system, the field is a new mathematical function $\widetilde{\phi}$ of the coordinates, and the relation between the two functions is that

$$\widetilde{\phi}(\widetilde{x}) = \phi(x) = \phi(P) . \tag{4.20}$$

We now use the relation $\widetilde{x} = x + \xi(x)$, with ξ infinitesimal, or the inverse relation $x = \widetilde{x} - \xi(\widetilde{x})$. Thus we get that

$$\widetilde{\phi}(\widetilde{x}) = \phi(\widetilde{x} - \xi(\widetilde{x})) . \tag{4.21}$$

This equation defines the function $\widetilde{\phi}$, and $\widetilde{x}$ is here a dummy variable which we may just as well call x. Thus,

$$\widetilde{\phi}(x) = \phi(x - \xi(x)) = \phi(x) - \xi^\nu(x)\,\phi_{,\nu}(x) = \phi(x) - \epsilon A^\nu(x)\,\phi_{,\nu}(x) , \tag{4.22}$$

where we use the notation for the partial derivative that

$$\phi_{,\nu} = \frac{\partial\phi}{\partial x^\nu} . \tag{4.23}$$

Suppressing the argument x, we write the transformation of a scalar field ϕ under the infinitesimal coordinate transformation $\widetilde{x}^\mu = x^\mu + \xi^\mu = x^\mu + \epsilon A^\mu$ as

$$\widetilde{\phi} = \phi - \xi^\nu \phi_{,\nu} = \phi - \epsilon A^\nu \phi_{,\nu} \ . \tag{4.24}$$

Note that the transformation involves $A^\nu \phi_{,\nu}$, which is simply the derivative of the scalar field ϕ along the vector field A^ν.

Example: Rotation

Consider again rotations in three dimensions about the z axis, generated by the vector field $\mathbf{A} = (-y, x, 0)$. The rotation by an infinitesimal angle ϵ transforms a scalar field $\phi = \phi(x, y, z)$ into

$$\widetilde{\phi} = \phi - \epsilon \left(A^x \frac{\partial \phi}{\partial x} + A^y \frac{\partial \phi}{\partial y} + A^z \frac{\partial \phi}{\partial z} \right) = \phi - \epsilon \left(-y \frac{\partial \phi}{\partial x} + x \frac{\partial \phi}{\partial y} \right) . \tag{4.25}$$

Consider, as a special case, the three linear functions $\phi_1 = \phi_1(x, y, z) = x$, $\phi_2 = y$, and $\phi_3 = z$. We have for example that

$$\widetilde{\phi}_1 = \phi_1 - \epsilon \left(-y \frac{\partial \phi_1}{\partial x} + x \frac{\partial \phi_1}{\partial y} \right) = x + \epsilon y = \phi_1 + \epsilon \phi_2 \ . \tag{4.26}$$

The three functions are rotated into linear combinations of each other, as follows,

$$\widetilde{\phi}_1 = \phi_1 + \epsilon \phi_2 \ , \qquad \widetilde{\phi}_2 = \phi_2 - \epsilon \phi_1 \ , \qquad \widetilde{\phi}_3 = \phi_3 \ . \tag{4.27}$$

Rotation by a finite angle α gives that

$$\widetilde{\phi}_1 = \phi_1 \cos\alpha + \phi_2 \sin\alpha \ , \qquad \widetilde{\phi}_2 = \phi_2 \cos\alpha - \phi_1 \sin\alpha \ , \qquad \widetilde{\phi}_3 = \phi_3 \ . \tag{4.28}$$

Thus, a general linear combination $\phi = c^1\phi_1 + c^2\phi_2 + c^3\phi_3$ with constant coefficients c^1, c^2, c^3 is transformed into

$$\widetilde{\phi} = c^1 \widetilde{\phi}_1 + c^2 \widetilde{\phi}_2 + c^3 \widetilde{\phi}_3 = \widetilde{c}^1 \phi_1 + \widetilde{c}^2 \phi_2 + \widetilde{c}^3 \phi_3 \ . \tag{4.29}$$

The coefficients c^1, c^2, c^3 are transformed in exactly the same way as the coordinate values $x^1 = x$, $x^2 = y$, $x^3 = z$ of a given point in space,

$$(\widetilde{c}^1, \widetilde{c}^2, \widetilde{c}^3) = (c^1 \cos\alpha - c^2 \sin\alpha \, , \, c^2 \cos\alpha + c^1 \sin\alpha \, , \, c^3) \ . \tag{4.30}$$

The Lie derivative of a contravariant vector field

Knowing how to transform a scalar field ϕ we consider next a contravariant vector field B^μ. The relation corresponding to Equation (4.20) is

$$\widetilde{B}^\mu(\widetilde{x}) = \frac{\partial \widetilde{x}^\mu}{\partial x^\nu} B^\nu(x) \ . \tag{4.31}$$

The extra complication compared with the scalar case is that the Jacobi matrix

$$\frac{\partial \widetilde{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu + \xi^\mu_{,\nu} = \delta^\mu_\nu + \epsilon A^\mu_{,\nu} \tag{4.32}$$

introduces an extra term in the transformation formula. To first order in ϵ we get that

$$\widetilde{B}^\mu = B^\mu - \xi^\nu B^\mu_{,\nu} + \xi^\mu_{,\nu} B^\nu = B^\mu - \epsilon(A^\nu B^\mu_{,\nu} - A^\mu_{,\nu} B^\nu) \; . \tag{4.33}$$

In the same way as the infinitesimal coordinate transformation of the scalar field ϕ involves $A^\nu \phi_{,\nu}$, the derivative of ϕ along the vector field A^ν, the transformation of the contravariant vector field B^μ involves a quantity $A^\nu B^\mu_{,\nu} - A^\mu_{,\nu} B^\nu$. We may interpret this as a kind of generalized derivative, which is called the *Lie derivative* of the contravariant vector field $\mathbf{B}$ along the contravariant vector field $\mathbf{A}$. Comparing with Equation (2.37), we see that the Lie derivative of $\mathbf{B}$ along $\mathbf{A}$ is equal to the commutator $[\mathbf{A}, \mathbf{B}]$. We write

$$\mathcal{L}_{\mathbf{A}} \mathbf{B} = [\mathbf{A}, \mathbf{B}] \; . \tag{4.34}$$

We have seen already that the commutator of two contravariant vector fields is itself a contravariant vector field.

The Lie derivative of a covariant vector field

Deducing the transformation formula for a covariant vector field C_μ is no more difficult. The basic relation is that

$$\widetilde{C}_\mu(\widetilde{x}) = \frac{\partial x^\nu}{\partial \widetilde{x}^\mu} \, C_\nu(x) \; . \tag{4.35}$$

As compared with the contravariant case we now need the inverse Jacobi matrix, which is

$$\frac{\partial x^\nu}{\partial \widetilde{x}^\mu} = \delta^\nu_\mu - \xi^\nu_{,\mu} = \delta^\nu_\mu - \epsilon A^\nu_{,\mu} \; . \tag{4.36}$$

In fact, it is easy to check that this matrix is the inverse of the matrix in Equation (4.32), to first order in ϵ, which is all we need. To first order in ϵ we get now that

$$\widetilde{C}_\mu = C_\mu - \xi^\nu C_{\mu,\nu} - \xi^\nu_{,\mu} C_\nu = C_\mu - \epsilon(A^\nu C_{\mu,\nu} + A^\nu_{,\mu} C_\nu) \; . \tag{4.37}$$

Thus, the Lie derivative of the covariant vector field $\mathbf{C}$ along the contravariant vector field $\mathbf{A}$ is a quantity $\mathbf{D} = \mathcal{L}_{\mathbf{A}} \mathbf{C}$ having the components

$$D_\mu = A^\nu C_{\mu,\nu} + A^\nu_{,\mu} C_\nu \; . \tag{4.38}$$

We ought to check that $\mathbf{D}$ is actually a covariant vector field. The proof that the Lie derivative of a general tensor is a tensor of the same rank, will be given in the next section.

The Lie derivative of a scalar density

The infinitesimal coordinate transformation $x^\mu \mapsto \widetilde{x}^\mu = x^\mu + \xi^\mu$ gives that

$$\left| \det\left(\frac{\partial x}{\partial \widetilde{x}} \right) \right| = \left| \det\left(\frac{\partial \widetilde{x}^\mu}{\partial x^\nu} \right) \right|^{-1} = \left| \det\left(\delta^\mu_\nu + \xi^\mu_{,\nu} \right) \right|^{-1} = \left| 1 + \xi^\mu_{,\mu} \right|^{-1} = 1 - \xi^\mu_{,\mu} \; . \tag{4.39}$$

A scalar density $D = D(x)$ transforms in the following way,

$$\widetilde{D}(\widetilde{x}) = \det\left(\frac{\partial x}{\partial \widetilde{x}} \right) D(x) \; , \tag{4.40}$$

that is,

$$\widetilde{D}(x) = (1 - \xi^\mu_{,\mu})\, D(x - \xi) = (1 - \xi^\mu_{,\mu})(D(x) - \xi^\mu D_{,\mu}(x)) \; . \tag{4.41}$$

We write this as $\widetilde{D}(x) = D(x) + \Delta D(x)$, with

$$\Delta D = -\xi^\mu D_{,\mu} - \xi^\mu_{,\mu} D = -\frac{\partial(\xi^\mu D)}{\partial x^\mu} \; . \tag{4.42}$$

Thus, the Lie derivative of a scalar density D along a vector field A^μ is

$$\mathcal{L}_{\mathbf{A}} D = \frac{\partial(A^\mu D)}{\partial x^\mu} \; . \tag{4.43}$$

4.3 The Lie derivative of a general tensor

The generalization from contravariant and covariant vectors to tensors of arbitrary rank should be obvious. For example, the Lie derivative along $\mathbf{A}$ of a tensor B^μ_ν is

$$\mathcal{L}_{\mathbf{A}} B^\mu_\nu = A^\rho B^\mu_{\nu,\rho} - A^\mu_{,\rho} B^\rho_\nu + A^\rho_{,\nu} B^\mu_\rho \; . \tag{4.44}$$

Note that the Lie derivative along $\mathbf{A}$ is identical to the ordinary derivative along $\mathbf{A}$ if the components of $\mathbf{A}$ are constant. We now want to verify that $\mathcal{L}_{\mathbf{A}} B^\mu_\nu$ is a tensor of the same rank as B^μ_ν. The proof for general tensors is a direct generalization.

Thus, let $x^\mu \mapsto \widetilde{x}^\mu$ be an arbitrary coordinate transformation, infinitesimal or finite, not related to the transformation $x^\mu \mapsto x^\mu + \epsilon A^\mu$ used for defining the Lie derivative. To see how the Lie derivative $D^\mu_\nu = \mathcal{L}_{\mathbf{A}} B^\mu_\nu$ is transformed, we compute it using the new coordinate system,

$$\begin{aligned}
\widetilde{D}^\mu_\nu \;&=\; \widetilde{A}^\rho \widetilde{B}^\mu_{\nu,\rho} - \widetilde{A}^\mu_{,\rho} \widetilde{B}^\rho_\nu + \widetilde{A}^\rho_{,\nu} \widetilde{B}^\mu_\rho \\
&=\; \frac{\partial \widetilde{x}^\rho}{\partial x^\alpha} A^\alpha \frac{\partial}{\partial \widetilde{x}^\rho} \left(\frac{\partial \widetilde{x}^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\nu} B^\beta_\gamma \right) - \left(\frac{\partial}{\partial \widetilde{x}^\rho} \left(\frac{\partial \widetilde{x}^\mu}{\partial x^\alpha} A^\alpha \right) \right) \frac{\partial \widetilde{x}^\rho}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\nu} B^\beta_\gamma \\
&\quad + \left(\frac{\partial}{\partial \widetilde{x}^\nu} \left(\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha} A^\alpha \right) \right) \frac{\partial \widetilde{x}^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\rho} B^\beta_\gamma \; .
\end{aligned} \tag{4.45}$$

This is to be compared to the transformation formula for a tensor,

$$\widetilde{D}^\mu_\nu = \frac{\partial \widetilde{x}^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\nu} D^\beta_\gamma = \frac{\partial \widetilde{x}^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\nu} \left(A^\rho \frac{\partial B^\beta_\gamma}{\partial x^\rho} - \frac{\partial A^\beta}{\partial x^\rho} B^\rho_\gamma + \frac{\partial A^\rho}{\partial x^\gamma} B^\beta_\rho \right) . \tag{4.46}$$

The difference between the two expressions consists of the terms containing second order derivatives of the coordinate transformation,

$$\begin{aligned}
\Delta^\mu_\nu \;&=\; \left[\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha} \frac{\partial}{\partial \widetilde{x}^\rho} \left(\frac{\partial \widetilde{x}^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\nu} \right) - \left(\frac{\partial}{\partial \widetilde{x}^\rho} \left(\frac{\partial \widetilde{x}^\mu}{\partial x^\alpha} \right) \right) \frac{\partial \widetilde{x}^\rho}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\nu} \right. \\
&\quad \left. + \left(\frac{\partial}{\partial \widetilde{x}^\nu} \left(\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha} \right) \right) \frac{\partial \widetilde{x}^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\rho} \right] A^\alpha B^\beta_\gamma \\
&=\; \left[\frac{\partial}{\partial x^\alpha} \left(\frac{\partial \widetilde{x}^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\nu} \right) - \frac{\partial^2 \widetilde{x}^\mu}{\partial x^\beta \, \partial x^\alpha} \frac{\partial x^\gamma}{\partial \widetilde{x}^\nu} + \left(\frac{\partial}{\partial \widetilde{x}^\nu} \left(\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha} \right) \right) \frac{\partial \widetilde{x}^\mu}{\partial x^\beta} \frac{\partial x^\gamma}{\partial \widetilde{x}^\rho} \right] A^\alpha B^\beta_\gamma \; .
\end{aligned} \tag{4.47}$$

Since the partial derivatives $\partial/\partial x^\alpha$ and $\partial/\partial x^\beta$ commute, we have that

$$\Delta^\mu_\nu = \left[\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\gamma}{\partial \widetilde{x}^\nu}\right) + \left(\frac{\partial}{\partial \widetilde{x}^\nu}\left(\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha}\right)\right)\frac{\partial x^\gamma}{\partial \widetilde{x}^\rho}\right]\frac{\partial \widetilde{x}^\mu}{\partial x^\beta}\, A^\alpha B^\beta_\gamma \,. \tag{4.48}$$

Next, by differentiating the identity

$$\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha}\,\frac{\partial x^\gamma}{\partial \widetilde{x}^\rho} = \delta^\gamma_\alpha \tag{4.49}$$

with respect to $\widetilde{x}^\nu$, and using that the partial derivatives $\partial/\partial\widetilde{x}^\nu$ and $\partial/\partial\widetilde{x}^\rho$ commute, we get that

$$\left(\frac{\partial}{\partial \widetilde{x}^\nu}\left(\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha}\right)\right)\frac{\partial x^\gamma}{\partial \widetilde{x}^\rho} = -\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha}\,\frac{\partial^2 x^\gamma}{\partial \widetilde{x}^\nu\,\partial \widetilde{x}^\rho} = -\frac{\partial \widetilde{x}^\rho}{\partial x^\alpha}\,\frac{\partial}{\partial \widetilde{x}^\rho}\left(\frac{\partial x^\gamma}{\partial \widetilde{x}^\nu}\right) = -\frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^\gamma}{\partial \widetilde{x}^\nu}\right). \tag{4.50}$$

We conclude that $\Delta^\mu_\nu = 0$, and hence D^μ_ν transforms as a tensor.

4.4 Commutation of Lie derivatives

Again we consider the typical example of a tensor D^μ_ν of rank (1,1). Its Lie derivative along the vector field $\mathbf{B}$ is

$$\mathcal{L}_{\mathbf{B}} D^\mu_\nu = B^\sigma D^\mu_{\nu,\sigma} - B^\mu_{,\sigma} D^\sigma_\nu + B^\sigma_{,\nu} D^\mu_\sigma \,. \tag{4.51}$$

Computing the Lie derivative of this new tensor along the vector field $\mathbf{A}$ gives

$$\begin{aligned}\mathcal{L}_{\mathbf{A}}(\mathcal{L}_{\mathbf{B}} D^\mu_\nu) \;=\; & A^\rho(B^\sigma_{,\rho} D^\mu_{\nu,\sigma} + B^\sigma D^\mu_{\nu,\sigma\rho} - B^\mu_{,\sigma\rho} D^\sigma_\nu - B^\mu_{,\sigma} D^\sigma_{\nu,\rho} + B^\sigma_{,\nu\rho} D^\mu_\sigma + B^\sigma_{,\nu} D^\mu_{\sigma,\rho}) \\ & - A^\mu_{,\rho}(B^\sigma D^\rho_{\nu,\sigma} - B^\rho_{,\sigma} D^\sigma_\nu + B^\sigma_{,\nu} D^\rho_\sigma) + A^\rho_{,\nu}(B^\sigma D^\mu_{\rho,\sigma} - B^\mu_{,\sigma} D^\sigma_\rho + B^\sigma_{,\rho} D^\mu_\sigma) \,.\end{aligned} \tag{4.52}$$

By definition, the commutator $[\mathcal{L}_{\mathbf{A}}, \mathcal{L}_{\mathbf{B}}]$ of the Lie derivatives in the two directions $\mathbf{A}$ and $\mathbf{B}$ acts on D^μ_ν to give

$$\begin{aligned}[\mathcal{L}_{\mathbf{A}}, \mathcal{L}_{\mathbf{B}}]\, D^\mu_\nu \;=\; & \mathcal{L}_{\mathbf{A}}(\mathcal{L}_{\mathbf{B}} D^\mu_\nu) - \mathcal{L}_{\mathbf{B}}(\mathcal{L}_{\mathbf{A}} D^\mu_\nu) \\ =\; & A^\rho(B^\sigma_{,\rho} D^\mu_{\nu,\sigma} - B^\mu_{,\sigma\rho} D^\sigma_\nu + B^\sigma_{,\nu\rho} D^\mu_\sigma) + A^\mu_{,\rho} B^\rho_{,\sigma} D^\sigma_\nu + A^\rho_{,\nu} B^\sigma_{,\rho} D^\mu_\sigma \\ & - B^\rho(A^\sigma_{,\rho} D^\mu_{\nu,\sigma} - A^\mu_{,\sigma\rho} D^\sigma_\nu + A^\sigma_{,\nu\rho} D^\mu_\sigma) - B^\mu_{,\rho} A^\rho_{,\sigma} D^\sigma_\nu - B^\rho_{,\nu} A^\sigma_{,\rho} D^\mu_\sigma \,.\end{aligned} \tag{4.53}$$

A number of terms cancel to give this result. Reordering terms, we get that

$$\begin{aligned}[\mathcal{L}_{\mathbf{A}}, \mathcal{L}_{\mathbf{B}}]\, D^\mu_\nu \;=\; & (A^\rho B^\sigma_{,\rho} - B^\rho A^\sigma_{,\rho}) D^\mu_{\nu,\sigma} - (A^\rho B^\mu_{,\rho} - B^\rho A^\mu_{,\rho})_{,\sigma} D^\sigma_\nu \\ & + (A^\rho B^\sigma_{,\rho} - B^\rho A^\sigma_{,\rho})_{,\nu} D^\mu_\sigma \,.\end{aligned} \tag{4.54}$$

This proves the nice result that $[\mathcal{L}_{\mathbf{A}}, \mathcal{L}_{\mathbf{B}}] = \mathcal{L}_{\mathbf{C}}$, where the vector field $\mathbf{C} = [\mathbf{A}, \mathbf{B}]$ is the commutator of the vector fields $\mathbf{A}$ and $\mathbf{B}$, with components

$$C^\mu = A^\rho B^\mu_{,\rho} - B^\rho A^\mu_{,\rho} \,. \tag{4.55}$$

4.5 Isometries and Killing vector fields

Whenever a manifold is equipped with a metric tensor field $g_{\mu\nu}$, this is definitely one of the most important geometrical structures of the manifold. It is then always interesting to ask for *symmetries* of the metric. By definition, a coordinate transformation is a symmetry of the metric, or more briefly an *isometry*, if it leaves the metric invariant, in the sense that the components of the metric are the same mathematical functions of the coordinates in both coordinate systems.

In particular, an infinitesimal coordinate transformation $\widetilde{x}^\mu = x^\mu + \epsilon A^\mu$ is a symmetry of the metric if and only if the Lie derivative of the metric along the vector field vanishes, or in the mathematical language,

$$\mathcal{L}_{\mathbf{A}} g_{\mu\nu} = A^\rho g_{\mu\nu,\rho} + A^\rho_{,\mu} g_{\rho\nu} + A^\rho_{,\nu} g_{\mu\rho} = 0 \; . \tag{4.56}$$

A contravariant vector field $\mathbf{A}$ having this property is called a *Killing vector field*, or more briefly a Killing vector, named after the Austrian mathematician Killing.

It follows directly from the definition that if $\mathbf{A}$ and $\mathbf{B}$ are Killing vector fields, then so is any linear combination $a\mathbf{A} + b\mathbf{B}$, with a and b constant. Furthermore, the commutator $\mathbf{C} = [\mathbf{A}, \mathbf{B}]$ is a Killing vector field, as we may easily verify. In fact, if $\mathcal{L}_{\mathbf{A}} g_{\mu\nu} = 0$ and $\mathcal{L}_{\mathbf{B}} g_{\mu\nu} = 0$, then we have that

$$\mathcal{L}_{\mathbf{C}} g_{\mu\nu} = \mathcal{L}_{\mathbf{A}}(\mathcal{L}_{\mathbf{B}} g_{\mu\nu}) - \mathcal{L}_{\mathbf{B}}(\mathcal{L}_{\mathbf{A}} g_{\mu\nu}) = 0 \; . \tag{4.57}$$

Thus, the Killing vector fields of a given metric tensor form a Lie algebra, which is the Lie algebra of the group of continuous transformations that are isometries.

A very brief summary of group theory is given in Appendix B.

Problems

1. Check explicitly that the Lie derivative of a covariant vector field, as defined in Equation (4.38), transforms as a covariant vector field.

Chapter 5

Differentiation

In the classical theory of fields we encounter many kinds of derivatives, all cousins of the basic derivative $f' = \mathrm{d}f/\mathrm{d}x$ where f is a function of the single variable x. There are time derivatives, gradients, partial derivatives, absolute (or total) derivatives, covariant derivatives, exterior derivatives, divergences, functional derivatives, as well as the Lie derivative already encountered in Chapter 4. We need a variety of notations to keep track of the various kinds of derivatives, and it is not always easy to avoid conflicts in the notation.

5.1 Partial and total derivatives

Time derivatives

The time derivative of a quantity $x = x(t)$ which is a function of time t, is written as

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} \,. \tag{5.1}$$

If $x = x(t)$ and $f = f(x,t) = f(x(t),t)$, then f has both an explicit time dependence in its second argument t, and an implicit time dependence in its first argument x. We distinguish then between $\partial/\partial t$, partial differentiation, which we apply to the explicit time dependence only, and $\mathrm{d}/\mathrm{d}t$, called total or absolute differentiation, which we apply to both the explicit and the implicit time dependence. Thus,

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\,\dot{x} + \frac{\partial f}{\partial t} \,. \tag{5.2}$$

Partial derivatives

We use at least three different notations for the standard partial differentiation of a scalar function $f = f(x) = f(x^0, x^1, x^2, x^3)$ with respect to one coordinate x^μ,

$$f_{,\mu} = \partial_\mu f = \frac{\partial f}{\partial x^\mu} \,. \tag{5.3}$$

We use occasionally even a fourth notation,

$$\frac{\mathrm{d}f}{\mathrm{d}x^\mu} \,, \tag{5.4}$$

in the case where f is both an explicit function of $x = (x^0, x^1, x^2, x^3)$ and an implicit function of x through some field $\phi = \phi(x)$,

$$f = f(\phi(x), x) = f(\phi(x^0, x^1, x^2, x^3), x^0, x^1, x^2, x^3) \; . \tag{5.5}$$

Then we may use $\mathrm{d}/\mathrm{d}x^\mu$ to denote "absolute partial differentiation", acting on both the explicit and the implicit dependence on x^μ, whereas we take $\partial/\partial x^\mu$ to act only on the explicit dependence. Thus,

$$\frac{\mathrm{d}f}{\mathrm{d}x^\mu} = \frac{\partial f}{\partial \phi}\,\frac{\partial \phi}{\partial x^\mu} + \frac{\partial f}{\partial x^\mu} \; . \tag{5.6}$$

5.2 Covariant derivatives of tensor fields

Partial differentiation of a scalar (i.e. a scalar field) gives a covariant vector (i.e. a covariant vector field), since the following formula holds for a coordinate transformation,

$$\widetilde{\phi}_{,\mu} = \frac{\partial \phi}{\partial \widetilde{x}^\mu} = \frac{\partial x^\rho}{\partial \widetilde{x}^\mu}\,\frac{\partial \phi}{\partial x^\rho} = \frac{\partial x^\rho}{\partial \widetilde{x}^\mu}\,\phi_{,\rho} \; . \tag{5.7}$$

Partial differentiation of a vector, on the other hand, does *not* in general produce a tensor. Thus, for the partial derivatives of a contravariant vector A^μ we have the transformation formula

$$\widetilde{A}^\mu{}_{,\nu} = \frac{\partial \widetilde{A}^\mu}{\partial \widetilde{x}^\nu} = \frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\frac{\partial}{\partial x^\lambda}\left(\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,A^\kappa\right) = \frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,A^\kappa{}_{,\lambda} + \frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\frac{\partial^2 \widetilde{x}^\mu}{\partial x^\lambda \partial x^\kappa}\,A^\kappa \; . \tag{5.8}$$

The term with second derivatives of the coordinate transformation disqualifies the derivative $A^\mu{}_{,\nu}$ from being called a tensor. One way to avoid the offending term is to admit only linear coordinate transformations, and that is precisely what we do in the special theory of relativity. Then $A^\mu{}_{,\nu}$ becomes a tensor.

A more general method for producing a tensor is to replace the partial differentiation by a *covariant* differentiation, which we denote by a semicolon ";" instead of a comma ",". We define

$$A^\mu{}_{;\nu} = A^\mu{}_{,\nu} + \Gamma^\mu_{\kappa\nu} A^\kappa \; . \tag{5.9}$$

The quantities $\Gamma^\lambda_{\mu\nu}$ are called *connection coefficients*, and by a coordinate transformation they are transformed into $\widetilde{\Gamma}^\lambda_{\mu\nu}$, so that $A^\mu{}_{;\nu}$ is transformed as a tensor,

$$\widetilde{A}^\mu{}_{;\nu} = \widetilde{A}^\mu{}_{,\nu} + \widetilde{\Gamma}^\mu_{\rho\nu}\widetilde{A}^\rho = \frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,A^\kappa{}_{;\lambda} = \frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,(A^\kappa{}_{,\lambda} + \Gamma^\kappa_{\sigma\lambda} A^\sigma) \, . \tag{5.10}$$

We see that this equation will hold if

$$\begin{aligned} \widetilde{\Gamma}^\mu_{\rho\nu}\widetilde{A}^\rho &= \frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\Gamma^\kappa_{\sigma\lambda} A^\sigma - \frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\frac{\partial^2 \widetilde{x}^\mu}{\partial x^\lambda \partial x^\kappa}\,A^\kappa \\ &= \frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\left(\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\Gamma^\kappa_{\sigma\lambda} - \frac{\partial^2 \widetilde{x}^\mu}{\partial x^\lambda \partial x^\sigma}\right)\frac{\partial x^\sigma}{\partial \widetilde{x}^\rho}\,\widetilde{A}^\rho \; . \end{aligned} \tag{5.11}$$

Since $\widetilde{A}^\rho$ here is arbitrary, this means that we must define

$$\widetilde{\Gamma}^\mu_{\rho\nu} = \frac{\partial x^\sigma}{\partial \widetilde{x}^\rho}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\left(\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\Gamma^\kappa_{\sigma\lambda} - \frac{\partial^2 \widetilde{x}^\mu}{\partial x^\lambda \partial x^\sigma}\right). \tag{5.12}$$

For the partial derivatives of a covariant vector field B_μ the following transformation formula holds,

$$\begin{aligned}\widetilde{B}_{\mu,\nu} &= \frac{\partial \widetilde{B}_\mu}{\partial \widetilde{x}^\nu} = \frac{\partial}{\partial \widetilde{x}^\nu}\left(\frac{\partial x^\rho}{\partial \widetilde{x}^\mu}\,B_\rho\right) = \frac{\partial x^\rho}{\partial \widetilde{x}^\mu}\,\frac{\partial B_\rho}{\partial \widetilde{x}^\nu} + \frac{\partial^2 x^\rho}{\partial \widetilde{x}^\nu \partial \widetilde{x}^\mu}\,B_\rho \\ &= \frac{\partial x^\rho}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\sigma}{\partial \widetilde{x}^\nu}\,B_{\rho,\sigma} + \frac{\partial^2 x^\rho}{\partial \widetilde{x}^\nu \partial \widetilde{x}^\mu}\,B_\rho\,.\end{aligned} \tag{5.13}$$

Again we define a covariant derivative,

$$B_{\mu;\nu} = B_{\mu,\nu} - \Delta^\rho_{\mu\nu} B_\rho\,, \tag{5.14}$$

and require $\Delta^\rho_{\mu\nu}$ to transform in such a way that $B_{\mu;\nu}$ transforms as a tensor,

$$\widetilde{B}_{\mu;\nu} = \widetilde{B}_{\mu,\nu} - \widetilde{\Delta}^\rho_{\mu\nu}\widetilde{B}_\rho = \frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,B_{\kappa;\lambda} = \frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,(B_{\kappa,\lambda} - \Delta^\sigma_{\kappa\lambda}B_\sigma)\,. \tag{5.15}$$

We see that this equation holds if

$$\begin{aligned}\widetilde{\Delta}^\rho_{\mu\nu}\widetilde{B}_\rho &= \frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\Delta^\sigma_{\kappa\lambda}B_\sigma + \frac{\partial^2 x^\rho}{\partial \widetilde{x}^\nu \partial \widetilde{x}^\mu}\,B_\rho \\ &= \left(\frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\Delta^\sigma_{\kappa\lambda} + \frac{\partial^2 x^\sigma}{\partial \widetilde{x}^\nu \partial \widetilde{x}^\mu}\right)\frac{\partial \widetilde{x}^\rho}{\partial x^\sigma}\,\widetilde{B}_\rho\,.\end{aligned} \tag{5.16}$$

This condition holds for an arbitrary $\widetilde{B}_\rho$ provided that

$$\widetilde{\Delta}^\rho_{\mu\nu} = \frac{\partial \widetilde{x}^\rho}{\partial x^\sigma}\left(\frac{\partial x^\kappa}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\Delta^\sigma_{\kappa\lambda} + \frac{\partial^2 x^\sigma}{\partial \widetilde{x}^\nu \partial \widetilde{x}^\mu}\right). \tag{5.17}$$

The transformation formulae for Γ and Δ look different, but are in fact identical, because

$$\frac{\partial \widetilde{x}^\rho}{\partial x^\sigma}\,\frac{\partial^2 x^\sigma}{\partial \widetilde{x}^\nu \partial \widetilde{x}^\mu} = \frac{\partial}{\partial \widetilde{x}^\nu}\left(\frac{\partial \widetilde{x}^\rho}{\partial x^\sigma}\,\frac{\partial x^\sigma}{\partial \widetilde{x}^\mu}\right) - \left(\frac{\partial}{\partial \widetilde{x}^\nu}\,\frac{\partial \widetilde{x}^\rho}{\partial x^\sigma}\right)\frac{\partial x^\sigma}{\partial \widetilde{x}^\mu} = -\frac{\partial x^\sigma}{\partial \widetilde{x}^\mu}\,\frac{\partial x^\lambda}{\partial \widetilde{x}^\nu}\,\frac{\partial^2 \widetilde{x}^\rho}{\partial x^\lambda \partial x^\sigma}\,. \tag{5.18}$$

It is therefore very reasonable to choose $\Delta^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}$. This choice is not only the simplest possible, but is necessary if we want covariant differentiation to commute with contraction. See the remark below.

For a tensor with several indices, e.g. $C^{\kappa\lambda}_{\mu\nu}$, we find by similar computations that the covariant derivative $C^{\kappa\lambda}_{\mu\nu;\rho}$ transforms as a tensor if we define

$$C^{\kappa\lambda}_{\mu\nu;\rho} = C^{\kappa\lambda}_{\mu\nu,\rho} + \Gamma^\kappa_{\sigma\rho}C^{\sigma\lambda}_{\mu\nu} + \Gamma^\lambda_{\sigma\rho}C^{\kappa\sigma}_{\mu\nu} - \Gamma^\sigma_{\mu\rho}C^{\kappa\lambda}_{\sigma\nu} - \Gamma^\sigma_{\nu\rho}C^{\kappa\lambda}_{\mu\sigma}\,. \tag{5.19}$$

In particular, the covariant derivative of a scalar field ϕ is the same as the partial derivative,

$$\phi_{;\rho} = \phi_{,\rho}\,. \tag{5.20}$$

This general definition of covariant differentiation implies e.g. that the Leibniz rule holds for differentiation of a product of tensors. For example,

$$(A^\mu B_\nu)_{;\rho} = A^\mu{}_{;\rho} B_\nu + A^\mu B_{\nu;\rho} \, . \tag{5.21}$$

It implies furthermore that covariant differentiation commutes with contraction. We have for example that

$$A^\mu_{\nu;\rho} = A^\mu_{\nu,\rho} + \Gamma^\mu_{\sigma\rho} A^\sigma_\nu - \Gamma^\sigma_{\nu\rho} A^\mu_\sigma \, , \tag{5.22}$$

and contraction of the indices μ and ν gives that

$$A^\mu_{\mu;\rho} = A^\mu_{\mu,\rho} + \Gamma^\mu_{\sigma\rho} A^\sigma_\mu - \Gamma^\sigma_{\mu\rho} A^\mu_\sigma = A^\mu_{\mu,\rho} = (A^\mu_\mu)_{;\rho} \, . \tag{5.23}$$

The result is the same whether we differentiate first and then contract: $A^\mu_\nu \to A^\mu_{\nu;\rho} \to A^\mu_{\mu;\rho}$, or contract first and then differentiate: $A^\mu_\nu \to A^\mu_\mu \to (A^\mu_\mu)_{;\rho}$.

Covariant derivative and Lie derivative

An alternative notation for the covariant derivative of a tensor field $\mathbf{C}$, as in Equation (5.19), is $\nabla_\rho \mathbf{C}$. Given a contravariant vector field $\mathbf{A}$ we define the covariant derivative along $\mathbf{A}$ as

$$\nabla_{\mathbf{A}} \mathbf{C} = A^\rho \, \nabla_\rho \mathbf{C} \, . \tag{5.24}$$

It may be interesting to compare the covariant derivative

$$\nabla_{\mathbf{A}} C^{\kappa\lambda}_{\mu\nu} = A^\rho C^{\kappa\lambda}_{\mu\nu;\rho} = A^\rho (C^{\kappa\lambda}_{\mu\nu,\rho} + \Gamma^\kappa_{\sigma\rho} C^{\sigma\lambda}_{\mu\nu} + \Gamma^\lambda_{\sigma\rho} C^{\kappa\sigma}_{\mu\nu} - \Gamma^\sigma_{\mu\rho} C^{\kappa\lambda}_{\sigma\nu} - \Gamma^\sigma_{\nu\rho} C^{\kappa\lambda}_{\mu\sigma}) \tag{5.25}$$

with the Lie derivative as defined in Equation (4.44),

$$\mathcal{L}_{\mathbf{A}} C^{\kappa\lambda}_{\mu\nu} = A^\rho C^{\kappa\lambda}_{\mu\nu,\rho} - A^\kappa_{,\sigma} C^{\sigma\lambda}_{\mu\nu} - A^\lambda_{,\sigma} C^{\kappa\sigma}_{\mu\nu} + A^\sigma_{,\mu} C^{\kappa\lambda}_{\sigma\nu} + A^\sigma_{,\nu} C^{\kappa\lambda}_{\mu\sigma} \, . \tag{5.26}$$

The two kinds of derivatives are actually identical if the contravariant vector field $\mathbf{A}$ satisfies the equation

$$A^\rho \Gamma^\kappa_{\sigma\rho} = -A^\kappa_{,\sigma} \, . \tag{5.27}$$

If the connection is symmetric, $\Gamma^\kappa_{\sigma\rho} = \Gamma^\kappa_{\rho\sigma}$, then the condition on $\mathbf{A}$ is that its covariant derivative vanishes,

$$A^\kappa_{;\sigma} = A^\kappa_{,\sigma} + \Gamma^\kappa_{\rho\sigma} A^\rho = 0 \, . \tag{5.28}$$

We say that the vector field $\mathbf{A}$ is parallel, or covariantly constant, when its covariant derivative vanishes.

5.3 Flat and curved space

Tensors from the connection

The connection Γ is not a tensor, or phrased differently, the connection coefficients $\Gamma^\lambda_{\mu\nu}$ are not tensor components. The reason is the last term in Equation (5.12), which makes the

transformation formula *inhomogeneous* when the coordinate transformation is not linear. By definition, tensor components transform *homogeneously*, which means in particular that zero transforms into zero.

Since the inhomogeneous term in Equation (5.12) is independent of the connection coefficients, it falls out of the transformation formula for the difference between two connections. Hence, if we vary the connection Γ, then the *variation* $\delta\Gamma$ is a tensor, even though Γ itself is not.

Because the inhomogeneous term is symmetric in the lower indices it also falls out of the transformation formula for the *torsion*, defined as

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} \; . \tag{5.29}$$

Thus the torsion is a tensor. We say that the connection is *symmetric* when

$$T^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu} = 0 \; . \tag{5.30}$$

This condition is independent of the choice of coordinate system, since the torsion is a tensor.

Note that the torsion $T^{\lambda}{}_{\mu\nu}$ is a tensor with three indices, not to be confused with the energy-momentum tensor $T^{\mu\nu}$, having two indices.

Commutation of covariant differentiation

The covariant derivative of a scalar function $\phi = \phi(x)$ is identical to the usual partial derivative, $\phi_{;\mu} = \phi_{,\mu}$. If we take twice the covariant derivative, we get that

$$\phi_{;\mu\nu} = (\phi_{;\mu})_{;\nu} = \phi_{,\mu\nu} - \Gamma^{\lambda}_{\mu\nu}\phi_{,\lambda} \; . \tag{5.31}$$

Interchanging the order of the two differentiations gives that

$$\phi_{;\mu\nu} - \phi_{;\nu\mu} = -\left(\Gamma^{\lambda}_{\mu\nu} - \Gamma^{\lambda}_{\nu\mu}\right)\phi_{,\lambda} = -T^{\lambda}{}_{\mu\nu}\phi_{,\lambda} \; . \tag{5.32}$$

Thus, the condition for two covariant differentiations to commute, when applied to an arbitrary scalar function ϕ, is that the torsion vanishes, $T^{\lambda}{}_{\mu\nu} = 0$.

If we take twice the covariant derivative of a contravariant vector A^{κ}, we get first that

$$A^{\kappa}{}_{;\mu} = A^{\kappa}{}_{,\mu} + \Gamma^{\kappa}_{\lambda\mu}A^{\lambda} \; , \tag{5.33}$$

and then that

$$\begin{aligned} A^{\kappa}{}_{;\mu\nu} &= (A^{\kappa}{}_{;\mu})_{;\nu} = (A^{\kappa}{}_{;\mu})_{,\nu} + \Gamma^{\kappa}_{\rho\nu}A^{\rho}{}_{;\mu} - \Gamma^{\sigma}_{\mu\nu}A^{\kappa}{}_{;\sigma} \\ &= A^{\kappa}{}_{,\mu\nu} + \Gamma^{\kappa}_{\lambda\mu,\nu}A^{\lambda} + \Gamma^{\kappa}_{\lambda\mu}A^{\lambda}{}_{,\nu} + \Gamma^{\kappa}_{\rho\nu}\left(A^{\rho}{}_{,\mu} + \Gamma^{\rho}_{\lambda\mu}A^{\lambda}\right) - \Gamma^{\sigma}_{\mu\nu}A^{\kappa}{}_{;\sigma} \; . \end{aligned} \tag{5.34}$$

Interchanging the order of the differentiations gives that

$$A^{\kappa}{}_{;\mu\nu} - A^{\kappa}{}_{;\nu\mu} = -R^{\kappa}{}_{\lambda\mu\nu}A^{\lambda} - T^{\sigma}{}_{\mu\nu}A^{\kappa}{}_{;\sigma} \; , \tag{5.35}$$

where we have defined

$$R^{\kappa}{}_{\lambda\mu\nu} = \Gamma^{\kappa}_{\lambda\nu,\mu} - \Gamma^{\kappa}_{\lambda\mu,\nu} + \Gamma^{\kappa}_{\rho\mu}\Gamma^{\rho}_{\lambda\nu} - \Gamma^{\kappa}_{\rho\nu}\Gamma^{\rho}_{\lambda\mu} \; . \tag{5.36}$$

Since all the other quantities in Equation (5.35) are tensors, and since the vector A^{κ} is arbitrary, it follows that $R^{\kappa}{}_{\lambda\mu\nu}$ is a tensor. It is called the *curvature tensor*, and we will examine it more closely in Chapter 6. We see that both the torsion tensor $T^{\sigma}{}_{\mu\nu}$ and the curvature tensor $R^{\kappa}{}_{\lambda\mu\nu}$ measure by how much covariant differentiations in different directions fail to commute.

Flat space

That spacetime (or "timespace") is flat, means by definition that there exists at least one coordinate system in which the connection is trivial, $\Gamma^\kappa_{\lambda\mu} = 0$. In this particular coordinate system, covariant derivation reduces to ordinary partial differentiation, and the torsion and curvature tensors both vanish, $T^\kappa{}_{\lambda\mu} = 0$ and $R^\kappa{}_{\lambda\mu\nu} = 0$. But the torsion and curvature tensor are *tensors*, therefore they vanish in every coordinate system if they vanish in one.

If the connection coefficients $\Gamma^\kappa_{\lambda\mu}$ do not vanish identically, we may learn whether or not spacetime is flat by looking at the torsion and the curvature tensor. The characteristic property of a flat manifold is that $T^\kappa{}_{\lambda\mu} = 0$ and $R^\kappa{}_{\lambda\mu\nu} = 0$, so that covariant differentiations in different directions commute. These two conditions are necessary for flatness, as remarked above, and are actually also sufficient, even though the sufficiency is somewhat less obvious. We will not prove here that spacetime is flat if the torsion and curvature vanish.

In order to see in a different way the significance of the two conditions $T^\kappa{}_{\lambda\mu} = 0$ and $R^\kappa{}_{\lambda\mu\nu} = 0$, consider the transformation formula for the connection coefficients, Equation (5.12). If we try to transform away a given connection $\Gamma^\kappa_{\lambda\mu}$, that is, to find new coordinates $\widetilde{x}^\mu$ such that $\widetilde{\Gamma}^\kappa_{\lambda\mu} = 0$, we have the following set of second order partial differential equations to solve,

$$\frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\Gamma^\kappa_{\sigma\lambda} - \frac{\partial^2 \widetilde{x}^\mu}{\partial x^\lambda \partial x^\sigma} = 0\,. \tag{5.37}$$

Since partial derivatives always commute, the symmetry condition $\Gamma^\kappa_{\sigma\lambda} = \Gamma^\kappa_{\lambda\sigma}$, or equivalently, $T^\kappa{}_{\sigma\lambda} = 0$, is necessary for the existence of four independent solutions $\widetilde{x}^0$, $\widetilde{x}^1$, $\widetilde{x}^2$ and $\widetilde{x}^3$.

This is however not the whole story, since we may differentiate Equation (5.37). We get then first that

$$\frac{\partial^3 \widetilde{x}^\mu}{\partial x^\rho \partial x^\lambda \partial x^\sigma} = \frac{\partial^2 \widetilde{x}^\mu}{\partial x^\rho \partial x^\kappa}\,\Gamma^\kappa_{\sigma\lambda} + \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\,\Gamma^\kappa_{\sigma\lambda,\rho}\,. \tag{5.38}$$

Here we may use Equation (5.37) to eliminate the second derivatives of $\widetilde{x}^\mu$, this gives that

$$\frac{\partial^3 \widetilde{x}^\mu}{\partial x^\rho \partial x^\lambda \partial x^\sigma} = \frac{\partial \widetilde{x}^\mu}{\partial x^\kappa}\left(\Gamma^\kappa_{\alpha\rho}\Gamma^\alpha_{\sigma\lambda} + \Gamma^\kappa_{\sigma\lambda,\rho}\right). \tag{5.39}$$

The left hand side of this equation is symmetric in all the three indices ρ, λ and σ, and so the right hand side must have the same symmetry. That the right hand side is symmetric in λ and σ, is no new condition. But a new necessary condition for solution is that the right hand side must be symmetric in ρ and λ. The symmetry must hold without restrictions on the derivatives $\partial\widetilde{x}^\mu/\partial x^\kappa$, and the resulting condition is that $R^\kappa{}_{\sigma\rho\lambda} = 0$.

Curved spacetime and Einstein's principle of equivalence

When the connection is symmetric, then it is always possible to choose the coordinates $\widetilde{x}$ such that $\widetilde{\Gamma}^\kappa_{\lambda\mu} = 0$ at any single given point. In fact, this is only a question of choosing suitable values for the first and second derivatives of the coordinate transformation $x \mapsto \widetilde{x}$ *at this one point*, so that Equation (5.37) holds there.

The postulate that such a coordinate system exists at any given point, is one possible formulation of *Einstein's principle of equivalence* in the general theory of relativity. The mathematical formulation of the principle of equivalence is that the connection must be symmetric, $\Gamma^\kappa_{\lambda\mu} = \Gamma^\kappa_{\mu\lambda}$.

5.4 Metric connection

Covariant derivation can be defined independent of the metric, but the two concepts fit neatly together if we require that covariant differentiation should commute with raising and lowering of indices, like it commutes with contraction. If for example $A_\mu = g_{\mu\nu}A^\nu$, then

$$A_{\mu;\rho} = (g_{\mu\nu}A^\nu)_{;\rho} = g_{\mu\nu;\rho}A^\nu + g_{\mu\nu}A^\nu{}_{;\rho} \,, \tag{5.40}$$

according to the Leibniz rule. We now require that we get the same result by differentiating first and lowering the index afterwards, i.e. that

$$A_{\mu;\rho} = g_{\mu\nu}A^\nu{}_{;\rho} \,. \tag{5.41}$$

If this is to hold for an arbitrary vector field A^ν, a necessary and sufficient condition is that the covariant derivative of the metric vanishes,

$$g_{\mu\nu;\rho} = g_{\mu\nu,\rho} - g_{\sigma\nu}\Gamma^\sigma_{\mu\rho} - g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} = 0 \,. \tag{5.42}$$

An equivalent condition is that

$$g^{\mu\nu}{}_{;\rho} = g^{\mu\nu}{}_{,\rho} + g^{\sigma\nu}\Gamma^\mu_{\sigma\rho} + g^{\mu\sigma}\Gamma^\nu_{\sigma\rho} = 0 \,. \tag{5.43}$$

Depending on whether we regard this as a condition on the metric or on the connection, we say either that the metric is *covariantly constant*, or that the connection is *metric*.

A metric connection is uniquely given by the metric tensor and its derivatives, and by the torsion. In order to deduce the relation we write the equation $g_{\mu\nu;\rho} = 0$ three times with the same indices in different orders, as follows,

$$\begin{aligned} g_{\mu\nu,\rho} &= g_{\sigma\nu}\Gamma^\sigma_{\mu\rho} + g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} \,, \\ g_{\rho\mu,\nu} &= g_{\sigma\mu}\Gamma^\sigma_{\rho\nu} + g_{\rho\sigma}\Gamma^\sigma_{\mu\nu} \,, \\ g_{\nu\rho,\mu} &= g_{\sigma\rho}\Gamma^\sigma_{\nu\mu} + g_{\nu\sigma}\Gamma^\sigma_{\rho\mu} \,. \end{aligned} \tag{5.44}$$

This gives that

$$g_{\rho\mu,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho} = g_{\rho\sigma}(2\Gamma^\sigma_{\mu\nu} - T^\sigma{}_{\mu\nu}) + g_{\mu\sigma}T^\sigma{}_{\rho\nu} + g_{\nu\sigma}T^\sigma{}_{\rho\mu} \,, \tag{5.45}$$

when we use the symmetry of the metric tensor and the definition of the torsion, Equation (5.29). We multiply this equation by $g^{\lambda\rho}$, and write $T_{\rho\mu\nu} = g_{\rho\sigma}T^\sigma{}_{\mu\nu}$, then we get that

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho} + T_{\rho\mu\nu} + T_{\mu\nu\rho} + T_{\nu\mu\rho}) \,. \tag{5.46}$$

We may easily verify that this connection is metric, without other restrictions on the torsion than the antisymmetry $T_{\rho\mu\nu} = -T_{\rho\nu\mu}$, which holds by definition. Note that the torsion is not just the antisymmetric part of the connection, but also contributes to the symmetric part of a metric connection, unless it is antisymmetric in the first two indices so that

$$T_{\mu\nu\rho} + T_{\nu\mu\rho} = 0 \,. \tag{5.47}$$

In this case it is completely antisymmetric,

$$T_{\rho\mu\nu} = -T_{\rho\nu\mu} = T_{\nu\rho\mu} = -T_{\nu\mu\rho} = T_{\mu\nu\rho} = -T_{\mu\rho\nu} \,. \tag{5.48}$$

If the connection is symmetric, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, as well as metric, it is uniquely given by the metric as

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}) \ . \tag{5.49}$$

This unique symmetric and metric connection is called the Christoffel connection. It is used in the general theory of relativity, where the metric tensor represents the gravitational field.

For later reference, let us compute the derivative $\Gamma^\kappa_{\lambda\mu,\nu}$ for the metric and symmetric connection. Equation (5.49) together with Equation (5.43) gives that

$$\begin{aligned}
\Gamma^\kappa_{\lambda\mu,\nu} &= \frac{1}{2}\, g^{\kappa\rho}(g_{\rho\lambda,\mu\nu} + g_{\rho\mu,\lambda\nu} - g_{\lambda\mu,\rho\nu}) + \frac{1}{2}\, g^{\kappa\rho}{}_{,\nu}(g_{\rho\lambda,\mu} + g_{\rho\mu,\lambda} - g_{\lambda\mu,\rho}) \\
&= \frac{1}{2}\, g^{\kappa\rho}(g_{\rho\lambda,\mu\nu} + g_{\rho\mu,\lambda\nu} - g_{\lambda\mu,\rho\nu}) - (g^{\sigma\rho}\Gamma^\kappa_{\sigma\nu} + g^{\kappa\sigma}\Gamma^\rho_{\sigma\nu})\, g_{\rho\tau}\, \Gamma^\tau_{\lambda\mu} \\
&= \frac{1}{2}\, g^{\kappa\rho}(g_{\rho\lambda,\mu\nu} + g_{\rho\mu,\lambda\nu} - g_{\lambda\mu,\rho\nu}) - \Gamma^\kappa_{\sigma\nu}\Gamma^\sigma_{\lambda\mu} - g^{\kappa\rho} g_{\sigma\tau}\Gamma^\sigma_{\rho\nu}\Gamma^\tau_{\lambda\mu} \ .
\end{aligned} \tag{5.50}$$

5.5 Covariant divergence

The covariant divergence of a contravariant vector field $A^\mu = A^\mu(x)$ is

$$A^\mu{}_{;\mu} = A^\mu{}_{,\mu} + \Gamma^\mu_{\rho\mu} A^\rho \ . \tag{5.51}$$

This expression can be simplified when the connection is symmetric and metric. Equation (5.49) gives that

$$\Gamma^\mu_{\rho\mu} = \frac{1}{2}\, g^{\mu\sigma} g_{\sigma\mu,\rho} = \frac{1}{2g}\, g_{,\rho} = \frac{1}{\sqrt{|g|}}\, \frac{\partial}{\partial x^\rho}\, \sqrt{|g|} = \frac{1}{2}\, \frac{\partial}{\partial x^\rho}\, \ln|g| \ , \tag{5.52}$$

when we recall the definition

$$g = \det(g_{\mu\nu}) \ , \tag{5.53}$$

and use Equation (2.79). We write $|g|$ in order to cover the case $g < 0$. This allows us to write

$$\boxed{A^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\, \frac{\partial}{\partial x^\mu} \left(\sqrt{|g|} A^\mu \right) \ .} \tag{5.54}$$

An important special case is when ϕ is a scalar field, and $A^\mu = g^{\mu\nu}\phi_{,\nu}$. Then we have that

$$g^{\mu\nu}\phi_{;\nu\mu} = A^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\, \frac{\partial}{\partial x^\mu} \left(\sqrt{|g|}\, g^{\mu\nu}\, \frac{\partial\phi}{\partial x^\nu} \right) \ . \tag{5.55}$$

This formula defines the *Laplace–Beltrami operator*,

$$\boxed{\Delta = \frac{1}{\sqrt{|g|}}\, \frac{\partial}{\partial x^\mu}\, \sqrt{|g|}\, g^{\mu\nu}\, \frac{\partial}{\partial x^\nu} \ ,} \tag{5.56}$$

which generalizes the Laplace operator defined in Equation (1.10). The present definition applies to an arbitrary metric in an arbitrary coordinate system.

The d'Alembert operator in four dimensions,

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \,, \tag{5.57}$$

is the special case of the Laplace–Beltrami operator when the metric is

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \,. \tag{5.58}$$

There is a similar simplification when we compute the covariant divergence of a contravariant antisymmetric tensor of arbitrary rank, always under the condition that the connection is both symmetric and metric. For example, for an antisymmetric tensor of rank two, $A^{\mu\nu} = -A^{\nu\mu}$, we have that

$$\boxed{A^{\mu\nu}{}_{;\nu} = A^{\mu\nu}{}_{,\nu} + \Gamma^{\mu}_{\rho\nu}A^{\rho\nu} + \Gamma^{\nu}_{\rho\nu}A^{\mu\rho} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^\nu}(\sqrt{|g|}\,A^{\mu\nu}) \,.} \tag{5.59}$$

We use here that $\Gamma^{\mu}_{\rho\nu}A^{\rho\nu} = 0$ when $\Gamma^{\mu}_{\rho\nu}$ is symmetric and $A^{\rho\nu}$ antisymmetric in the indices ρ, ν.

An important example is the four Maxwell equations with sources, having the following form in a general coordinate system,

$$F^{\mu\nu}{}_{;\nu} = -\mu_0 j^\mu \,. \tag{5.60}$$

Or equivalently,

$$\frac{\partial}{\partial x^\nu}\left(\sqrt{|g|}\,F^{\mu\nu}\right) = -\mu_0\sqrt{|g|}\,j^\mu \,. \tag{5.61}$$

5.6 Exterior derivatives of forms

The commutator of two contravariant vector fields, as defined in Equation (2.36) and (2.37), is an example showing that it is possible to make a tensor by differentiation of a vector without using covariant differentiation.

The covariant derivative is also not the only tensor we may produce from the partial derivatives $B_{\mu,\nu}$ of a covariant vector field B_μ. In fact, we get another tensor simply by antisymmetrizing in the indices μ and ν. We define the *exterior derivative* of B_μ as

$$(\mathrm{d}B)_{\mu\nu} = 2\partial_{[\mu}B_{\nu]} = 2B_{[\nu,\mu]} = B_{\nu,\mu} - B_{\mu,\nu} \,, \tag{5.62}$$

and Equation (5.13) shows that it transforms as a tensor,

$$\widetilde{B}_{[\nu,\mu]} = \frac{\partial x^\sigma}{\partial \widetilde{x}^\nu}\frac{\partial x^\rho}{\partial \widetilde{x}^\mu}B_{[\sigma,\rho]} \,. \tag{5.63}$$

For example,

$$\nabla \times \boldsymbol{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathbf{k} \tag{5.64}$$

is the exterior derivative of a three dimensional vector $\boldsymbol{B} = B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}$.

Another example is the relation between the field tensor $F_{\mu\nu}$ and the four dimensional vector potential A_μ in electromagnetism,

$$F_{\mu\nu} = (\mathrm{d}A)_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \; . \tag{5.65}$$

We may define quite generally an *exterior differentiation* of forms which is coordinate independent, and yet involves nothing but standard partial derivatives. Recall that an n-form is the same as an antisymmetric covariant tensor of rank n. The exterior derivative of an n-form $B_{\kappa\lambda\ldots\nu}$ is an $(n+1)$-form

$$(\mathrm{d}B)_{\rho\kappa\lambda\ldots\nu} = (n+1)\, B_{[\kappa\lambda\ldots\nu,\rho]} \; . \tag{5.66}$$

The right hand side of this equation is by definition antisymmetric in all its $n+1$ indices. That it transforms as a tensor, follows in a similar way as in the special case $n = 1$ above. In the special case $n = 0$ the exterior derivative equals the usual partial derivative,

$$(\mathrm{d}B)_\rho = B_{,\rho} \; . \tag{5.67}$$

We have previously introduced a special notation for forms, writing

$$\begin{aligned} \mathbf{B} &= \frac{1}{n!}\, B_{\kappa\lambda\ldots\nu}\, \mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \ldots \wedge \mathrm{d}x^\nu \; , \\ \mathrm{d}\mathbf{B} &= \frac{1}{(n+1)!}\, (\mathrm{d}B)_{\rho\kappa\lambda\ldots\nu}\, \mathrm{d}x^\rho \wedge \mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \ldots \wedge \mathrm{d}x^\nu \; . \end{aligned} \tag{5.68}$$

The definition in Equation (5.66) means that

$$\mathrm{d}\mathbf{B} = \left(\frac{1}{n!}\, B_{\kappa\lambda\ldots\nu,\rho}\, \mathrm{d}x^\rho\right) \wedge \mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \ldots \wedge \mathrm{d}x^\nu \; . \tag{5.69}$$

In particular, the exterior derivative of a 2-form $B_{\kappa\lambda}$ is

$$\begin{aligned} (\mathrm{d}B)_{\rho\kappa\lambda} = 3B_{[\kappa\lambda,\rho]} &= \frac{1}{2}\,(B_{\kappa\lambda,\rho} + B_{\rho\kappa,\lambda} + B_{\lambda\rho,\kappa} - B_{\lambda\kappa,\rho} - B_{\kappa\rho,\lambda} - B_{\rho\lambda,\kappa}) \\ &= B_{\kappa\lambda,\rho} + B_{\rho\kappa,\lambda} + B_{\lambda\rho,\kappa} \; . \end{aligned} \tag{5.70}$$

And the exterior derivative of a 3-form $B_{\kappa\lambda\mu}$ is

$$(\mathrm{d}B)_{\rho\kappa\lambda\mu} = 4B_{[\kappa\lambda\mu,\rho]} = B_{\kappa\lambda\mu,\rho} - B_{\rho\kappa\lambda,\mu} + B_{\mu\rho\kappa,\lambda} - B_{\lambda\mu\rho,\kappa} \; . \tag{5.71}$$

5.7 Poincaré's lemma, magnetic monopoles

Twice exterior differentiation gives identically zero, $\mathrm{d}^2 = 0$, because partial differentiations commute,

$$(\mathrm{d}^2B)_{\sigma\rho\kappa\lambda\ldots\nu} = (n+2)(\mathrm{d}B)_{[\rho\kappa\lambda\ldots\nu,\sigma]} = (n+2)(n+1)\, B_{[\kappa\lambda\ldots\nu,\rho\sigma]} = 0 \; . \tag{5.72}$$

In general we say that an n-form $\mathbf{B}$ is *closed* if $\mathrm{d}\mathbf{B} = 0$, and is *exact* if $\mathbf{B} = \mathrm{d}\mathbf{A}$ for some $(n-1)$-form $\mathbf{A}$. An exact form $\mathrm{d}\mathbf{A}$ is always closed, since $\mathrm{d}^2\mathbf{A} = 0$.

The converse result, that a closed form is always exact, is called *Poincaré's lemma*. It holds in the d dimensional Euclidean space $\mathbf{R}^d$. It holds also *locally* in an arbitrary manifold of dimension d, in the sense that around any given point there is always some open neighbourhood in which all closed forms are exact. However, it may not hold *globally* in a manifold with a non-trivial topology.

Poincaré's lemma will hold for example for all closed 1-forms in some region if that region is *simply connected*, which means that every continuous closed curve is continuously deformable into a point. In the case of n-forms, the condition is that every closed surface of dimension n should be continuously deformable into a point. A couple of examples will illustrate what may sometimes go wrong.

Example: $n = 1$

Puncture the (x, y) plane by removing the origin $(x, y) = (0, 0)$. The polar coordinates (r, φ) are well defined by the relations $x = r\cos\varphi$, $y = r\sin\varphi$, although the angle φ is multivalued. We have that

$$\begin{aligned} \mathrm{d}x &= \cos\varphi\,\mathrm{d}r - r\sin\varphi\,\mathrm{d}\varphi\,, \\ \mathrm{d}y &= \sin\varphi\,\mathrm{d}r + r\cos\varphi\,\mathrm{d}\varphi\,, \end{aligned} \tag{5.73}$$

and inversely,

$$\begin{aligned} \mathrm{d}r &= \quad \cos\varphi\,\mathrm{d}x + \sin\varphi\,\mathrm{d}y = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{\sqrt{x^2+y^2}}\,, \\ \mathrm{d}\varphi &= -\frac{\sin\varphi}{r}\,\mathrm{d}x + \frac{\cos\varphi}{r}\,\mathrm{d}y = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2+y^2}\,. \end{aligned} \tag{5.74}$$

The 1-form

$$\mathbf{B} = B_x\,\mathrm{d}x + B_y\,\mathrm{d}y = \frac{x}{\sqrt{x^2+y^2}}\,\mathrm{d}x + \frac{y}{\sqrt{x^2+y^2}}\,\mathrm{d}y \tag{5.75}$$

is closed, as we may verify by computation,

$$\begin{aligned} \mathrm{d}\mathbf{B} &= \left(\frac{\partial B_x}{\partial x}\,\mathrm{d}x + \frac{\partial B_x}{\partial y}\,\mathrm{d}y\right)\wedge\mathrm{d}x + \left(\frac{\partial B_y}{\partial x}\,\mathrm{d}x + \frac{\partial B_y}{\partial y}\,\mathrm{d}y\right)\wedge\mathrm{d}y \\ &= \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y = 0\,. \end{aligned} \tag{5.76}$$

It is also exact, since $\mathbf{B} = \mathrm{d}r$, and $r = \sqrt{x^2+y^2}$ is a single valued function in the punctured plane, not only continuously differentiable but even analytic.

The 1-form

$$\mathbf{C} = C_x\,\mathrm{d}x + C_y\,\mathrm{d}y = -\frac{y}{x^2+y^2}\,\mathrm{d}x + \frac{x}{x^2+y^2}\,\mathrm{d}y \tag{5.77}$$

is also closed,

$$\mathrm{d}\mathbf{C} = \left(\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right)\mathrm{d}x\wedge\mathrm{d}y = 0\,. \tag{5.78}$$

It is exact in the sense that $\mathbf{C} = \mathrm{d}\varphi$. However, φ is not a single valued function in the whole punctured plane, it is multivalued because it increases by 2π when we go once around the origin in the anticlockwise direction. We do not accept multivalued functions, therefore we do not accept $\mathbf{C}$ as an exact 1-form. Note that the derivative $\mathrm{d}\varphi$ is single valued, even though φ is multivalued.

The root of the problem is that the punctured plane is not simply connected: a loop around the origin can not be continuously deformed into a point. To obtain a simply connected region we may for example cut the plane by removing the negative x axis. Then what remains is a simply connected region where the angle $\varphi = \varphi(x, y)$ is a single valued function,

$$\varphi(x, y) = 2 \arctan\left(\frac{y}{r + x}\right), \tag{5.79}$$

and where the 1-form $\mathbf{C} = \mathrm{d}\varphi$ is exact.

Example: $n = 2$, magnetic monopole field

Consider a static (time independent) magnetic flux density in three dimensions,

$$\boldsymbol{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}\,. \tag{5.80}$$

The equation

$$\nabla \cdot \boldsymbol{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{5.81}$$

is one of Maxwell's equations. It says that no magnetic charges exist that are sources of the magnetic field. We solve it by introducing a vector potential $\boldsymbol{A}$ such that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}\,. \tag{5.82}$$

A different description of the same reality is to treat the magnetic flux density as a 2-form,

$$\mathbf{B} = B_x\,\mathrm{d}y \wedge \mathrm{d}z + B_y\,\mathrm{d}z \wedge \mathrm{d}x + B_z\,\mathrm{d}x \wedge \mathrm{d}y\,. \tag{5.83}$$

Then the same Maxwell's equation takes the form $\mathrm{d}\mathbf{B} = 0$, since

$$\begin{aligned}
\mathrm{d}\mathbf{B} &= \mathrm{d}B_x \wedge \mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}B_y \wedge \mathrm{d}z \wedge \mathrm{d}x + \mathrm{d}B_z \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial B_x}{\partial x}\,\mathrm{d}x + \frac{\partial B_x}{\partial y}\,\mathrm{d}y + \frac{\partial B_x}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}y \wedge \mathrm{d}z \\
&\quad + \left(\frac{\partial B_y}{\partial x}\,\mathrm{d}x + \frac{\partial B_y}{\partial y}\,\mathrm{d}y + \frac{\partial B_y}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}z \wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial B_z}{\partial x}\,\mathrm{d}x + \frac{\partial B_z}{\partial y}\,\mathrm{d}y + \frac{\partial B_z}{\partial z}\,\mathrm{d}z\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \\
&= \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\,.
\end{aligned} \tag{5.84}$$

Since the 2-form $\mathbf{B}$ is closed, by Poincaré's lemma it is also exact, which means that there exists a 1-form

$$\mathbf{A} = A_x\,\mathrm{d}x + A_y\,\mathrm{d}y + A_z\,\mathrm{d}z \tag{5.85}$$

such that

$$\begin{aligned}\mathbf{B} &= \mathrm{d}\mathbf{A} = \mathrm{d}A_x \wedge \mathrm{d}x + \mathrm{d}A_y \wedge \mathrm{d}y + \mathrm{d}A_z \wedge \mathrm{d}z \\ &= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \mathrm{d}y \wedge \mathrm{d}z + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \mathrm{d}z \wedge \mathrm{d}x + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \mathrm{d}x \wedge \mathrm{d}y \ . \end{aligned} \tag{5.86}$$

The Maxwell equation $\nabla \cdot \boldsymbol{B} = 0$, or $\mathrm{d}\mathbf{B} = 0$, forbids magnetic charges. But if we now puncture the three dimensional space by removing the origin, then we are free to place a magnetic monopole at the origin, so as to get the field

$$B_x = \frac{kx}{r^3} \ , \qquad B_y = \frac{ky}{r^3} \ , \qquad B_z = \frac{kz}{r^3} \ , \tag{5.87}$$

where k is a konstant, and $r = \sqrt{x^2 + y^2 + z^2}$. We easily verify that $\mathrm{d}\mathbf{B} = 0$ for $r > 0$. Hence, by Poincaré's lemma, there exists at least locally a vector potential $\mathbf{A}$ such that $\mathbf{B} = \mathrm{d}\mathbf{A}$. The equation for $\mathbf{A}$ is most easily solved in polar coordinates r, θ, φ, defined by the relations

$$x = r \sin\theta \cos\varphi \ , \qquad y = r \sin\theta \sin\varphi \ , \qquad z = r \cos\theta \ . \tag{5.88}$$

From these follows that

$$\begin{aligned} \mathrm{d}x &= \sin\theta \cos\varphi \ \mathrm{d}r + r \cos\theta \cos\varphi \ \mathrm{d}\theta - r \sin\theta \sin\varphi \ \mathrm{d}\varphi \ , \\ \mathrm{d}y &= \sin\theta \sin\varphi \ \mathrm{d}r + r \cos\theta \sin\varphi \ \mathrm{d}\theta + r \sin\theta \cos\varphi \ \mathrm{d}\varphi \ , \\ \mathrm{d}z &= \cos\theta \ \mathrm{d}r - r \sin\theta \ \mathrm{d}\theta \ , \end{aligned} \tag{5.89}$$

and altogether we find that

$$\mathbf{B} = \frac{k}{r^3} \left(x \, \mathrm{d}y \wedge \mathrm{d}z + y \, \mathrm{d}z \wedge \mathrm{d}x + z \, \mathrm{d}x \wedge \mathrm{d}y\right) = k \sin\theta \ \mathrm{d}\theta \wedge \mathrm{d}\varphi \ . \tag{5.90}$$

See Problem 6. Thus, the equation $\mathrm{d}\mathbf{A} = \mathbf{B}$ has the following three components in polar coordinates,

$$\frac{\partial A_r}{\partial \theta} - \frac{\partial A_\theta}{\partial r} = 0 \ , \qquad \frac{\partial A_r}{\partial \varphi} - \frac{\partial A_\varphi}{\partial r} = 0 \ , \qquad \frac{\partial A_\varphi}{\partial \theta} - \frac{\partial A_\theta}{\partial \varphi} = k \sin\theta \ . \tag{5.91}$$

The simplest solution is

$$\mathbf{A} = -k \cos\theta \ \mathrm{d}\varphi = \frac{kz(y \, \mathrm{d}x - x \, \mathrm{d}y)}{r(x^2 + y^2)} \ . \tag{5.92}$$

This is singular at $x^2 + y^2 = 0$, corresponding to $\theta = 0$ and $\theta = \pi$ in polar coordinates. Another solution, which is singular at $\theta = \pi$ but not at $\theta = 0$, is

$$\mathbf{A} = k(1 - \cos\theta) \ \mathrm{d}\varphi = \frac{k(r - z)(x \, \mathrm{d}y - y \, \mathrm{d}x)}{r(x^2 + y^2)} = \frac{k(x \, \mathrm{d}y - y \, \mathrm{d}x)}{r(r + z)} \ . \tag{5.93}$$

Thus, by removing the negative z axis we obtain a region in which the closed 2-form $\mathbf{B}$ is exact.

5.8 Divergence of densities

Let A^μ be a vector density in four dimensions. Then its dual,

$$ {}_\circ A_{\alpha\beta\gamma} = \epsilon_{\mu\alpha\beta\gamma}\, A^\mu \ , \tag{5.94} $$

is a 3-form, the exterior derivative of which is a 4-form,

$$ (\mathrm{d}_\circ A)_{\delta\alpha\beta\gamma} = 4({}_\circ A)_{[\alpha\beta\gamma,\delta]} = 4\epsilon_{\mu[\alpha\beta\gamma} A^\mu{}_{,\delta]} \ . \tag{5.95} $$

Finally the dual of this 4-form is a scalar density,

$$ \begin{aligned} {}^\circ(\mathrm{d}_\circ A) \ &= \ \frac{1}{24}\,\epsilon^{\delta\alpha\beta\gamma}\,(\mathrm{d}_\circ A)_{\delta\alpha\beta\gamma} = \frac{1}{6}\,\epsilon^{\delta\alpha\beta\gamma}\,\epsilon_{\mu[\alpha\beta\gamma} A^\mu{}_{,\delta]} \\ &= \ \frac{1}{6}\,\epsilon^{\delta\alpha\beta\gamma}\,\epsilon_{\mu\alpha\beta\gamma} A^\mu{}_{,\delta} = A^\delta{}_{,\delta} \ . \end{aligned} \tag{5.96} $$

We see that by taking first the dual of a vector density, next the exterior derivative and finally the dual once more, we get the divergence of the original vector density, *with standard partial derivatives* and not covariant derivatives. This divergence is a scalar density, as is proved by our computation via the exterior differentiation.

As another example, let $A^{\mu\nu}$ be a 2-density, i.e. an antisymmetric tensor density of rank two. The dual,

$$ {}_\circ A_{\alpha\beta} = \frac{1}{2}\,\epsilon_{\mu\nu\alpha\beta}\, A^{\mu\nu} \ , \tag{5.97} $$

is a 2-form, the exterior derivative of which is a 3-form,

$$ (\mathrm{d}_\circ A)_{\gamma\alpha\beta} = 3({}_\circ A)_{[\alpha\beta,\gamma]} = \frac{3}{2}\,\epsilon_{\mu\nu[\alpha\beta}\, A^{\mu\nu}{}_{,\gamma]} \ , \tag{5.98} $$

the dual of which is in turn a vector density,

$$ \begin{aligned} {}^\circ(\mathrm{d}_\circ A)^\kappa \ &= \ \frac{1}{6}\,\epsilon^{\kappa\gamma\alpha\beta}\,(\mathrm{d}_\circ A)_{\gamma\alpha\beta} = \frac{1}{4}\,\epsilon^{\kappa\gamma\alpha\beta}\,\epsilon_{\mu\nu[\alpha\beta}\, A^{\mu\nu}{}_{,\gamma]} \\ &= \ \frac{1}{4}\,\epsilon^{\kappa\gamma\alpha\beta}\,\epsilon_{\mu\nu\alpha\beta}\, A^{\mu\nu}{}_{,\gamma} = A^{\kappa\gamma}{}_{,\gamma} \ . \end{aligned} \tag{5.99} $$

In general, when $\mathbf{A}$ is an n-density, and we compute its divergence by standard partial differentiation, we get an $(n-1)$-density, which we may call $\mathrm{div}\,\mathbf{A}$. We see that the operation

$$ \mathrm{div} = \ {}^\circ\mathrm{d}\,{}_\circ \tag{5.100} $$

is invariant under arbitrary coordinate transformations, when it acts on densities, without involving either connection or metric.

The left hand side of Equation (5.61), Maxwell's equation, is the divergence of a 2-density, hence it is a 1-density. Since the right hand side is also a 1-density, the left and right hand sides both transform in the same way, and the equation is invariant under general coordinate transformations.

The Laplace–Beltrami operator rederived

When we define the divergence of a vector density as above, and in addition invoke the metric, we are in a position to rediscover the Laplace–Beltrami operator defined in Equation (5.56),

$$\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^\mu} \sqrt{|g|}\, g^{\mu\nu} \frac{\partial}{\partial x^\nu} \,. \tag{5.101}$$

When it acts on a scalar function $\phi = \phi(x) = \phi(x^0, x^1, x^2, x^3)$, the result $\Delta\phi$ is again a scalar function, and a simple proof is the following. First we have that

$$A_\nu = \phi_{,\nu} = \frac{\partial \phi}{\partial x^\nu} \tag{5.102}$$

is a covariant vector, therefore $B^\mu = g^{\mu\nu} A_\nu$ is a contravariant vector and $C^\mu = \sqrt{|g|}\, B^\mu$ a vector density. The divergence of this vector density,

$$D = C^\mu{}_{,\mu} = \frac{\partial C^\mu}{\partial x^\mu} \,, \tag{5.103}$$

is a scalar density, and finally $\Delta\phi = D/\sqrt{|g|}$ is a scalar, as we wanted to prove.

We have already derived the same operator from the covariant differentiation with a metric and symmetric connection. Later on, we are going to derive it a third time, starting from an invariant variational principle for a scalar field, the Klein–Gordon field.

Problems

1. Show that the two conditions (5.42) and (5.43) are equivalent definitions of a covariantly constant metric (or of a metric connection). That is, if one of them holds, then the other one follows.

2. Express the Laplace–Beltrami operator Δ in the plane both in the Cartesian coordinates (x, y) and in the polar coordinates (r, φ).
   Use the results from Problem 3 in Chapter 2, and compare with Equation (1.81).

3. Express the Laplace–Beltrami operator in the three dimensional Cartesian coordinates (x, y, z) and in the polar coordinates (r, θ, φ).
   Use the results from Problem 4 in Chapter 2, and compare with Problem 1 in Chapter 1.

4. Write in polar coordinates (θ, φ) the Laplace–Beltrami operator Δ on the surface of the sphere, using the metric found in Problem 5 in Chapter 2.

5. As an example of the commutation of covariant differentiation in different directions, show in general that for a covariant tensor $A_{\kappa\lambda}$ of rank two we have

$$A_{\kappa\lambda;\mu\nu} - A_{\kappa\lambda;\nu\mu} = R^{\rho}{}_{\kappa\mu\nu}\, A_{\rho\lambda} + R^{\rho}{}_{\lambda\mu\nu}\, A_{\kappa\rho} - T^{\rho}{}_{\mu\nu}\, A_{\kappa\lambda;\rho} \ .$$

   Take $A_{\kappa\lambda} = g_{\kappa\lambda}$ in this relation, and show that if the connection is metric, i.e. if $g_{\kappa\lambda;\mu} = 0$, then the curvature tensor is antisymmetric in the two first indices, i.e. $R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu}$, when we define $R_{\kappa\lambda\mu\nu} = g_{\kappa\rho}\, R^{\rho}{}_{\lambda\mu\nu}$.

6. Verify the second equality in Equation (5.90).
   Hint: to simplify the computation, use e.g. that

$$x\, \mathrm{d}y \wedge \mathrm{d}z + y\, \mathrm{d}z \wedge \mathrm{d}x = (x\, \mathrm{d}y - y\, \mathrm{d}x) \wedge \mathrm{d}z \ .$$

Chapter 6

Parallel transport and curvature

We saw in Chapter 5 that covariant differentiations in two different directions do not always commute, unless both the torsion tensor and the curvature tensor vanish. In this chapter we will study the curvature tensor from a slightly different mathematical point of view.

In the general theory of relativity the torsion in the physical spacetime is assumed to vanish identically. The physical meaning of the curvature tensor is that it is nonzero in the presence of a gravitational field, and it represents the gravitational field in Einstein's equation, as we shall see in Chapter 16.

Covariant differentiation defines in a coordinate independent way what it means that a vector field is constant along a given curve. In other words, covariant differentiation defines how to *parallel transport* a vector along a curve. Conversely, we may define covariant differentiation by means of parallel transport, so in this sense the two concepts are equivalent. The curvature tensor describes what happens when we parallel transport a vector around an infinitesimal closed curve.

Parallel transport of vectors gives meaning to the concept of *straight lines*. Quite literally, a line is straight if it has a constant direction, i.e. if the tangent vectors at all points on the curve are parallel. Straight lines are also called *geodesic lines*, or *geodesics* for short. A geodesic line on the surface of a sphere, in particular, is called a great circle. The equation saying that all tangent vectors along a curve are parallel, is called the *geodesic equation*, it involves the connection and no other geometric quantities.

Another well known characteristic of a straight line is that it is the shortest route between two points, or sometimes actually the *longest* route, in any case a route of extremal length. This property of a geodesic is in fact a variational principle from which we may derive the geodesic equation, as we will do in Chapter 15. The connection occurring in the geodesic equation as derived from the variational principle, may be defined to be symmetric, and is then uniquely defined by the metric.

If we have both a metric and a connection, then we have two ways of deriving a geodesic equation, and we have to ask whether the two methods produce the same equation. We say that the connection is metric, or that the metric is covariantly constant, when the metric and the connection are compatible in this way. It turns out that an equivalent condition is that the parallel transport of a vector always preserves the length of the vector.

6.1 Parallel transport of vectors and tensors

Let $A^\mu = A^\mu(x) = A^\mu(x^0, x^1, x^2, x^3)$ be a contravariant vector field. Given a curve $x^\mu = x^\mu(u)$, the vector field along the curve is a function of the curve parameter u, i.e. $A^\mu = A^\mu(u) = A^\mu(x(u))$, and we may differentiate the components A^μ with respect to u,

$$\frac{\mathrm{d}A^\mu}{\mathrm{d}u} = \frac{\partial A^\mu}{\partial x^\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}u} = A^\mu{}_{,\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}u} \,. \tag{6.1}$$

Here $\mathrm{d}x^\rho/\mathrm{d}u$ are the components of the tangent vector to the curve. If $\mathrm{d}A^\mu/\mathrm{d}u = 0$, then the components of the vector field are constant along the curve. However, that the *components* are constant does not necessarily mean that the *vector* is constant. In our definition of a constant vector field we want a criterion which does not depend on the coordinate system, and in this respect the criterion of constant components is unacceptable.

By definition, the vector field $A^\mu = A^\mu(x)$ is constant along the curve $x = x(u)$, or in other words, all the vectors $A^\mu = A^\mu(u) = A^\mu(x(u))$ along the curve are parallel, if

$$A^\mu{}_{;\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}u} = \left(A^\mu{}_{,\rho} + \Gamma^\mu_{\sigma\rho}A^\sigma\right)\frac{\mathrm{d}x^\rho}{\mathrm{d}u} = 0 \,. \tag{6.2}$$

This condition may be rewritten in the following way, making it clear that it involves only the vector field along the curve,

$$\boxed{\frac{\mathrm{d}A^\mu}{\mathrm{d}u} + \Gamma^\mu_{\sigma\rho}A^\sigma\frac{\mathrm{d}x^\rho}{\mathrm{d}u} = 0 \,.} \tag{6.3}$$

Given an initial condition $A^\mu(u) = C^\mu$ at one parameter value $u = u_0$ this equation has a unique solution $A^\mu(u)$, which is, by definition, the parallel transport of the vector C^μ along the curve.

By a similar reasoning we find the equation for parallel transport of a covariant vector B_μ along a curve,

$$\boxed{\frac{\mathrm{d}B_\mu}{\mathrm{d}u} - \Gamma^\sigma_{\mu\rho}B_\sigma\frac{\mathrm{d}x^\rho}{\mathrm{d}u} = 0 \,.} \tag{6.4}$$

The definition is easily extended to the parallel transport of any tensor.

The two equations (6.3) and (6.4) together imply e.g. that if the contravariant vectors $A^\mu = A^\mu(u)$ are parallel along the curve $x(u)$, and the covariant vectors $B_\mu = B_\mu(u)$ are parallel as well, then the scalar quantity $A^\mu B_\mu$ is constant along the curve,

$$\frac{\mathrm{d}(A^\mu B_\mu)}{\mathrm{d}u} = \frac{\mathrm{d}A^\mu}{\mathrm{d}u}B_\mu + A^\mu\frac{\mathrm{d}B_\mu}{\mathrm{d}u} = -\Gamma^\mu_{\sigma\rho}A^\sigma\frac{\mathrm{d}x^\rho}{\mathrm{d}u}B_\mu + A^\mu\Gamma^\sigma_{\mu\rho}B_\sigma\frac{\mathrm{d}x^\rho}{\mathrm{d}u} = 0 \,. \tag{6.5}$$

We may multiply Equation (6.3) by $\mathrm{d}u$ to rewrite it as

$$\mathrm{d}A^\mu = -\Gamma^\mu_{\sigma\rho}A^\sigma\,\mathrm{d}x^\rho \,. \tag{6.6}$$

In this form, the equation is explicitly independent of the curve parameter u, and shows thus that the parallel transport along a curve is independent of how the curve is parametrized. It says that a contravariant vector A^μ at a point x is parallel to the vector

$$\boxed{A^\mu + \mathrm{d}A^\mu = \left(\delta^\mu_\sigma - \Gamma^\mu_{\sigma\rho}\,\mathrm{d}x^\rho\right) A^\sigma} \tag{6.7}$$

at the infinitesimally close point $x + \mathrm{d}x$. Similarly, we may write Equation (6.4) in the form

$$\mathrm{d}B_\mu = \Gamma^\sigma_{\mu\rho} B_\sigma\,\mathrm{d}x^\rho\ , \tag{6.8}$$

saying that the covariant vector B_μ at x is parallel to the following vector at $x + \mathrm{d}x$,

$$\boxed{B_\mu + \mathrm{d}B_\mu = \left(\delta^\sigma_\mu + \Gamma^\sigma_{\mu\rho}\,\mathrm{d}x^\rho\right) B_\sigma\ .} \tag{6.9}$$

We have now defined what is meant by parallel vectors, whether they are contravariant or covariant. The definition is coordinate independent, but on the other hand it is *curve dependent*. In order to compare two vectors localized at separate points in space and time, we have to parallel transport the vector from one point along some curve to the other point. In a flat space the result of a parallel transport depends only on the start and end points and is independent of the curve. But in a curved space the parallel transport along different curves will give different results. Only infinitesimal parallel transport, as given by Equation (6.7) for a contravariant vector and by Equation (6.9) for a covariant vector, is uniquely defined in a curved space.

The geodesic equation

The most direct definition of a straight line is that it has a constant direction. More precisely, it has a constant unit tangent vector $\mathrm{d}x^\mu/\mathrm{d}s$, where $s = s(u)$ is a special curve parameter measuring the distance along the curve, called the arc length. Inserting $A^\mu = \mathrm{d}x^\mu/\mathrm{d}s$ in Equation (6.3) gives the *geodesic equation*,

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right) + \Gamma^\mu_{\sigma\rho}\,\frac{\mathrm{d}x^\sigma}{\mathrm{d}s}\,\frac{\mathrm{d}x^\rho}{\mathrm{d}u} = 0\ . \tag{6.10}$$

Multiplying the equation by $\mathrm{d}u/\mathrm{d}s$ gives it the following standard form,

$$\boxed{\frac{\mathrm{d}^2x^\mu}{\mathrm{d}s^2} + \Gamma^\mu_{\sigma\rho}\,\frac{\mathrm{d}x^\sigma}{\mathrm{d}s}\,\frac{\mathrm{d}x^\rho}{\mathrm{d}s} = 0\ .} \tag{6.11}$$

Note that the geodesic equation defines the curve in terms of a special parameter s which is proportional to the arc length measured along the curve from an arbitrary starting point. In fact, the parameter is unique up to two arbitrary constants, since $s' = \lambda(s - s_0)$ is an equally good parameter when s_0 is an arbitrary zero point and λ an arbitrary scale factor.

The geodesic equation determines distances along a geodesic, up to a proportionality constant, independent of the existence of a metric.

Note also that it is only the symmetric part of the connection that contributes in the geodesic equation. The torsion, which is the antisymmetric part of the connection, does not contribute.

Metric connection

Given both a connection $\Gamma^{\lambda}_{\mu\nu}$ and a metric $g_{\mu\nu}$, it is natural to ask whether the parallel transport of a vector preserves its length. Equation (6.3) implies that

$$\frac{\mathrm{d}}{\mathrm{d}u}\,(g_{\mu\nu}A^\mu A^\nu) = \Big(g_{\mu\nu,\rho} - g_{\sigma\nu}\Gamma^\sigma_{\mu\rho} - g_{\mu\sigma}\Gamma^\sigma_{\nu\rho}\Big)\,\frac{\mathrm{d}x^\rho}{\mathrm{d}u}\,A^\mu A^\nu = g_{\mu\nu;\rho}\,\frac{\mathrm{d}x^\rho}{\mathrm{d}u}\,A^\mu A^\nu\ . \tag{6.12}$$

Thus, the length of an arbitrary contravariant vector is preserved under parallel transport if and only if $g_{\mu\nu;\rho} = 0$.

We have seen this condition, the vanishing of the covariant derivative of the metric tensor, already in Chapter 5. When it holds, we say either that the metric is covariantly constant, or that the connection is metric. We found that for a given metric tensor field there exists a unique connection which is symmetric and metric. As we shall see in Chapter 15, the geodesic equation as derived from a variational principle contains again the unique symmetric and metric connection.

6.2 Covariant differentiation from parallel transport

We have used above the concept of covariant differentiation in order to define the parallel transport of vectors. We will now see how to go the other way, from parallel transport to covariant differentiation.

Partial differentiation of the components $A^\mu = A^\mu(x)$ of a contravariant vector field $\mathbf{A} = \mathbf{A}(x)$ amounts to a comparison of the components $A^\mu(x)$ and $A^\mu(x+\mathrm{d}x)$ at two infinitesimally nearby points x and $x+\mathrm{d}x$. The partial derivatives $A^\mu{}_{,\nu}$ are defined by the relation

$$A^\mu(x+\mathrm{d}x) = A^\mu(x) + A^\mu{}_{,\nu}(x)\,\mathrm{d}x^\nu\ , \tag{6.13}$$

valid for an arbitrary infinitesimal $\mathrm{d}x$. However, these derivatives are not *covariant*, because the definition is coordinate dependent, or phrased differently, $A^\mu{}_{,\nu}$ is not a tensor under non-linear coordinate transformations.

The basic problem is that we try to compare quantities that are not directly comparable: the vectors $\mathbf{A}(x)$ and $\mathbf{A}(x+\mathrm{d}x)$ localized at the two different points x and $x+\mathrm{d}x$. A way to compare them which makes better sense is to parallel transport one vector to the other point, and this is how we define the covariant derivatives $A^\mu{}_{;\nu}$. By definition, the vector $A^\mu(x) + A^\mu{}_{;\nu}(x)\,\mathrm{d}x^\nu$ is located at x, and in order to compare it with the vector $A^\mu(x+\mathrm{d}x)$ located at x, we have to parallel transport it from x to $x+\mathrm{d}x$. Thus, the equation defining the covariant differentiation looks as follows,

$$\begin{aligned} A^\mu(x+\mathrm{d}x) &= \Big(\delta^\mu_\rho - \Gamma^\mu_{\rho\sigma}\,\mathrm{d}x^\sigma\Big)\,(A^\rho(x) + A^\rho{}_{;\nu}(x)\,\mathrm{d}x^\nu) \\ &= A^\mu(x) + \Big(A^\mu{}_{;\nu}(x) - \Gamma^\mu_{\rho\nu}A^\rho(x)\Big)\,\mathrm{d}x^\nu\ . \end{aligned} \tag{6.14}$$

We put $(\mathrm{d}x)^2 = 0$ here because $\mathrm{d}x$ is infinitesimal. The present definition of $A^\mu{}_{;\nu}(x)$ means that it is tensor localized at x. Comparison between the two equations above gives that

$$\boxed{A^\mu{}_{;\nu} = A^\mu{}_{,\nu} + \Gamma^\mu_{\rho\nu} A^\rho \ .} \tag{6.15}$$

The covariant derivative $B_{\mu;\nu}$ of a covariant vector field $B_\mu = B_\mu(x)$ is defined in a similar way, by the relations

$$\begin{aligned} B_\mu(x + \mathrm{d}x) &= B_\mu(x) + B_{\mu,\nu}(x)\,\mathrm{d}x^\nu \\ &= \left(\delta^\lambda_\mu + \Gamma^\lambda_{\mu\sigma}\,\mathrm{d}x^\sigma\right)\left(B_\lambda(x) + B_{\lambda;\nu}(x)\,\mathrm{d}x^\nu\right) \\ &= B_\mu(x) + \left(B_{\mu;\nu}(x) + \Gamma^\lambda_{\mu\nu} B_\lambda(x)\right)\mathrm{d}x^\nu \ . \end{aligned} \tag{6.16}$$

This gives that

$$\boxed{B_{\mu;\nu} = B_{\mu,\nu} - \Gamma^\lambda_{\mu\nu} B_\lambda \ .} \tag{6.17}$$

The generalization to tensors of arbitrary rank should be obvious.

6.3 Torsion and curvature

The torsion and curvature tensors both have clear geometric interpretations in terms of parallel transport. The torsion measures by how much an infinitesimal parallellogram fails to be closed, whereas the curvature tensor measures how much the parallel transport of vectors between two points, or from one point back to the same point, depends on the curve.

The torsion tensor

We obtain an infinitesimal parallellogram with one corner at the point x in the following way. Let $\mathrm{d}x$ and $\mathrm{d}y$ be the two infinitesimal sides meeting at x. We parallel transport $\mathrm{d}y$ along $\mathrm{d}x$, the result is

$$\mathrm{d}\widehat{y}^\lambda = \mathrm{d}y^\lambda - \Gamma^\lambda_{\mu\nu}\,\mathrm{d}y^\mu\,\mathrm{d}x^\nu \ . \tag{6.18}$$

Then we parallel transport $\mathrm{d}x$ along $\mathrm{d}y$, with the result

$$\mathrm{d}\widehat{x}^\lambda = \mathrm{d}x^\lambda - \Gamma^\lambda_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}y^\nu \ . \tag{6.19}$$

What then lacks for the parallellogram to close, is

$$\mathrm{d}x^\lambda + \mathrm{d}\widehat{y}^\lambda - \left(\mathrm{d}y^\lambda + \mathrm{d}\widehat{x}^\lambda\right) = T^\lambda{}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}y^\nu \ . \tag{6.20}$$

The curvature tensor

Parallellity of vectors is an absolute concept only if the parallel transport from a point A to a point B along two different curves C_1 and C_2 gives the same result, dependent on the points A and B and on the vector transported, but independent of the curves C_1 and C_2. If we parallel transport first from A to B along C_1 and then along C_2 backwards from B to A, then we in effect parallel transport all the way around a closed curve starting and ending at A. The condition is that we always return from the round trip with the same vector we started with.

> *Parallellity of vectors is an absolute concept if and only if every vector is parallel to itself under parallel transport around an arbitrary closed curve.*

A necessary condition is that parallel transport of a vector around an *infinitesimal* closed curve must always give back the same vector. From infinitesimal closed curve we may build up finite closed curves. We say that the space is *simply connected* if every finite closed curve may be built up from infinitesimal closed curves, and in that case the infinitesimal condition is both necessary and sufficient. Otherwise there will be additional conditions due to the "non-trivial" closed curves that may not be built up from infinitesimal ones.

Let now $\mathrm{d}x$ and $\mathrm{d}y$ be infinitesimal, and let us study the parallel transport of a contravariant vector A^μ at the point x around the infinitesimal closed curve

$$x \to x + \mathrm{d}x \to x + \mathrm{d}x + \mathrm{d}y \to x + \mathrm{d}y \to x \; . \tag{6.21}$$

There are four different parallel transports involved, given by the following four matrices, according to Equation (6.7),

$$\begin{array}{lcll} x & \to & x + \mathrm{d}x : & P^\kappa{}_\lambda = \delta^\kappa_\lambda - \Gamma^\kappa_{\lambda\mu}\mathrm{d}x^\mu \; , \\ x & \to & x + \mathrm{d}y : & Q^\kappa{}_\lambda = \delta^\kappa_\lambda - \Gamma^\kappa_{\lambda\nu}\mathrm{d}y^\nu \; , \\ x + \mathrm{d}x & \to & x + \mathrm{d}x + \mathrm{d}y : & \widehat{Q}^\kappa{}_\lambda = \delta^\kappa_\lambda - \left(\Gamma^\kappa_{\lambda\nu} + \Gamma^\kappa_{\lambda\nu,\mu}\mathrm{d}x^\mu\right)\mathrm{d}y^\nu \; , \\ x + \mathrm{d}y & \to & x + \mathrm{d}x + \mathrm{d}y : & \widehat{P}^\kappa{}_\lambda = \delta^\kappa_\lambda - \left(\Gamma^\kappa_{\lambda\mu} + \Gamma^\kappa_{\lambda\mu,\nu}\mathrm{d}y^\nu\right)\mathrm{d}x^\mu \; . \end{array} \tag{6.22}$$

We compute here to first order in the infinitesimal sides $\mathrm{d}x$ and $\mathrm{d}y$ of the closed curve, that is, we include terms that are proportional to the product $\mathrm{d}x\,\mathrm{d}y$, but we negect the quadratic terms $(\mathrm{d}x)^2$ and $(\mathrm{d}y)^2$.

The matrix for parallel transport from x to $x + \mathrm{d}x + \mathrm{d}y$ via $x + \mathrm{d}x$ is

$$(\widehat{Q}P)^\kappa{}_\lambda = \widehat{Q}^\kappa{}_\rho P^\rho{}_\lambda = \delta^\kappa_\lambda - \Gamma^\kappa_{\lambda\mu}\mathrm{d}x^\mu - \Gamma^\kappa_{\lambda\nu}\mathrm{d}y^\nu - \Gamma^\kappa_{\lambda\nu,\mu}\mathrm{d}x^\mu\mathrm{d}y^\nu + \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\lambda\mu}\mathrm{d}x^\mu\mathrm{d}y^\nu \; . \tag{6.23}$$

Remember our agreement to ignore terms of second order in at least one of the infinitesimal quantities $\mathrm{d}x$ and $\mathrm{d}y$. By interchangin $\mathrm{d}x$ and $\mathrm{d}y$ we get the matrix for parallel transport from x to $x + \mathrm{d}x + \mathrm{d}y$ via $x + \mathrm{d}y$,

$$(\widehat{P}Q)^\kappa{}_\lambda = \delta^\kappa_\lambda - \Gamma^\kappa_{\lambda\mu}\mathrm{d}y^\mu - \Gamma^\kappa_{\lambda\nu}\mathrm{d}x^\nu - \Gamma^\kappa_{\lambda\nu,\mu}\mathrm{d}y^\mu\mathrm{d}x^\nu + \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\lambda\mu}\mathrm{d}y^\mu\mathrm{d}x^\nu \; . \tag{6.24}$$

We see that

$$(\widehat{P}Q)^\kappa{}_\lambda = (\widehat{Q}P)^\kappa{}_\lambda + R^\kappa{}_{\lambda\mu\nu}\mathrm{d}x^\mu\mathrm{d}y^\nu \; , \tag{6.25}$$

where $R^{\kappa}{}_{\lambda\mu\nu}$ is the Riemann curvature tensor as defined in Equation (5.36),

$$\boxed{R^{\kappa}{}_{\lambda\mu\nu} = \Gamma^{\kappa}_{\lambda\nu,\mu} - \Gamma^{\kappa}_{\lambda\mu,\nu} + \Gamma^{\kappa}_{\rho\mu}\Gamma^{\rho}_{\lambda\nu} - \Gamma^{\kappa}_{\rho\nu}\Gamma^{\rho}_{\lambda\mu} \, .} \tag{6.26}$$

The matrix for parallel transport all around the closed curven is, always to first order in each of $\mathrm{d}x$ and $\mathrm{d}y$,

$$(Q^{-1}\widehat{P}^{-1}\widehat{Q}P)^{\kappa}{}_{\lambda} = ((\widehat{P}Q)^{-1}\widehat{Q}P)^{\kappa}{}_{\lambda} = \delta^{\kappa}_{\lambda} - R^{\kappa}{}_{\lambda\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}y^{\nu} \, . \tag{6.27}$$

There is a small matrix computation behind the last equality. Let $U = \widehat{P}Q$, $V = \widehat{Q}P$ and $W = U - V$, that is, $W^{\kappa}{}_{\lambda} = R^{\kappa}{}_{\lambda\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}y^{\nu}$. Then to first order in $\mathrm{d}x$ and $\mathrm{d}y$ we have that

$$U^{-1}V = U^{-1}(U - W) = I - U^{-1}W = I - W \, . \tag{6.28}$$

That $R^{\kappa}{}_{\lambda\mu\nu}$ is actually a tensor, is not quite obvious from Equation (6.26). However, the very construction of the curvature tensor, that it describes parallel transport of vectors around infinitesimal closed curves, guarantees that it transforms as tensor under coordinate transformations. Here is the argument in a little more detail. The parallel transport of a given vector A^{μ} around the infitesimal closed curve gives the vector

$$B^{\mu} = A^{\mu} - R^{\mu}{}_{\nu\rho\sigma}\mathrm{d}x^{\rho}\mathrm{d}y^{\sigma}A^{\nu} \, . \tag{6.29}$$

We know that all quantities in this equation apart from $R^{\mu}{}_{\nu\rho\sigma}$ transform as vectors, and since they are *arbitrary* vectors, that is possible only if $R^{\mu}{}_{\nu\rho\sigma}$ transforms as a tensor.

The curvature tensor for a symmetric and metric connection

Equation (5.50) gives the following expression for the curvature tensor, under the assumption that the connection is symmetric and metric,

$$R^{\kappa}{}_{\lambda\mu\nu} = g^{\kappa\rho}\left(\frac{1}{2}\left(g_{\rho\nu,\lambda\mu} - g_{\lambda\nu,\rho\mu} - g_{\rho\mu,\lambda\nu} + g_{\lambda\mu,\rho\nu}\right) + g_{\sigma\tau}(\Gamma^{\sigma}_{\rho\nu}\Gamma^{\tau}_{\lambda\mu} - \Gamma^{\sigma}_{\rho\mu}\Gamma^{\tau}_{\lambda\nu})\right) . \tag{6.30}$$

Defining $R_{\kappa\lambda\mu\nu} = g_{\kappa\rho}R^{\rho}{}_{\lambda\mu\nu}$, we may write the same formula as

$$\boxed{R_{\kappa\lambda\mu\nu} = \frac{1}{2}\left(g_{\kappa\nu,\lambda\mu} - g_{\lambda\nu,\kappa\mu} - g_{\kappa\mu,\lambda\nu} + g_{\lambda\mu,\kappa\nu}\right) + g_{\sigma\tau}(\Gamma^{\sigma}_{\kappa\nu}\Gamma^{\tau}_{\lambda\mu} - \Gamma^{\sigma}_{\kappa\mu}\Gamma^{\tau}_{\lambda\nu}) \, .} \tag{6.31}$$

6.4 Symmetry properties of the curvature tensor

The expression for the curvature tensor in Equation (6.31), valid for a symmetric and metric connection, is either symmetric or antisymmetric under quite a number of different interchanges of indices. Some of these symmetries are actually valid more generally, as we shall now see.

From the definition of the curvature tensor follows directly two identities, an antisymmetry in the last two indices,

$$\boxed{R^{\kappa}{}_{\lambda\mu\nu} + R^{\kappa}{}_{\lambda\nu\mu} = 0 \ ,} \tag{6.32}$$

and a symmetry relation under a cyclic permutation of the three lower indices,

$$\begin{aligned} R^{\kappa}{}_{\lambda\mu\nu} + R^{\kappa}{}_{\nu\lambda\mu} + R^{\kappa}{}_{\mu\nu\lambda} &= T^{\kappa}{}_{\lambda\nu,\mu} + T^{\kappa}{}_{\nu\mu,\lambda} + T^{\kappa}{}_{\mu\lambda,\nu} \\ &\quad + \Gamma^{\kappa}_{\rho\mu} T^{\rho}{}_{\lambda\nu} + \Gamma^{\kappa}_{\rho\lambda} T^{\rho}{}_{\nu\mu} + \Gamma^{\kappa}_{\rho\nu} T^{\rho}{}_{\mu\lambda} \\ &= T^{\kappa}{}_{\lambda\nu;\mu} + T^{\kappa}{}_{\nu\mu;\lambda} + T^{\kappa}{}_{\mu\lambda;\nu} \\ &\quad - T^{\kappa}{}_{\rho\mu} T^{\rho}{}_{\lambda\nu} - T^{\kappa}{}_{\rho\lambda} T^{\rho}{}_{\nu\mu} - T^{\kappa}{}_{\rho\nu} T^{\rho}{}_{\mu\lambda} \ . \end{aligned} \tag{6.33}$$

Thus, for a symmetric connection, without torsion, we have that

$$\boxed{R^{\kappa}{}_{\lambda\mu\nu} + R^{\kappa}{}_{\nu\lambda\mu} + R^{\kappa}{}_{\mu\nu\lambda} = 0 \ .} \tag{6.34}$$

If the connection is metric, that means by definition that each of the four matrices P, Q, $\widehat{P}$ and $\widehat{Q}$ preserves the metric. The same is then true of the matrix $Q^{-1}\widehat{P}^{-1}\widehat{Q}P$, and since it represents parallel transport of vectors from the point x back to x, this means that it is an infinitesimal Lorentz transformation. As we shall see in Chapter 8, this means that the curvature tensor is antisymmetric in the first two indices,

$$\boxed{R_{\kappa\lambda\mu\nu} + R_{\lambda\kappa\mu\nu} = 0 \ .} \tag{6.35}$$

Problem 5 in Chapter 5 gives another derivation of this antisymmetry.

By combining all the three symmetry relations (6.32), (6.34) and (6.35) we get a fourth useful relation,

$$\begin{aligned} R_{\kappa\lambda\mu\nu} &= -R_{\kappa\nu\lambda\mu} - R_{\kappa\mu\nu\lambda} = R_{\nu\kappa\lambda\mu} + R_{\mu\kappa\nu\lambda} \\ &= -R_{\nu\mu\kappa\lambda} - R_{\nu\lambda\mu\kappa} - R_{\mu\lambda\kappa\nu} - R_{\mu\nu\lambda\kappa} \\ &= \ \ R_{\mu\nu\kappa\lambda} + R_{\lambda\nu\mu\kappa} + R_{\lambda\mu\kappa\nu} + R_{\mu\nu\kappa\lambda} \\ &= 2R_{\mu\nu\kappa\lambda} - R_{\lambda\kappa\nu\mu} \\ &= 2R_{\mu\nu\kappa\lambda} - R_{\kappa\lambda\mu\nu} \ , \end{aligned} \tag{6.36}$$

or,

$$\boxed{R_{\kappa\lambda\mu\nu} = R_{\mu\nu\kappa\lambda} \ .} \tag{6.37}$$

All these symmetry relations mean that the rank 4 curvature tensor has not $4^4 = 256$ independent components, but only 20, when the connection is symmetric and metric. In order to count independent components of $R_{\kappa\lambda\mu\nu}$ we may count independent index combinations.

– The first index pair $K \equiv \kappa\lambda$ may be either 01, 02, 03, 12, 13 or 23. All other values are related to these 6, or give $R_{\kappa\lambda\mu\nu} = 0$, due to the antisymmetry.

– The same holds for last index pair $M \equiv \mu\nu$. Only the 6 combinations 01, 02, 03, 12, 13 or 23 give independent values of $R_{\kappa\lambda\mu\nu}$, because of the antisymmetry.

– Thus, the independent components may be arranged into a 6×6 matrix R_{KM}. This matrix is symmetric, $R_{KM} = R_{MK}$, by Equation (6.37), and this symmetry reduces the number of independent components from $6 \times 6 = 36$ to $6 \times 7/2 = 21$.

– Equation (6.34), which we may write as

$$R_{\kappa\lambda\mu\nu} + R_{\kappa\nu\lambda\mu} + R_{\kappa\mu\nu\lambda} = 0 \,, \tag{6.38}$$

gives one single extra relation, reducing the number of independent components from 21 til 20. In fact, this equation gives nothing new if two of the indices are equal, because it then follows from the symmetry relations we have already used. The only new relation is that

$$R_{0123} + R_{0312} + R_{0231} = 0 \,. \tag{6.39}$$

6.5 Scalar curvature, the Ricci and Weyl tensors

The *Ricci curvature tensor* is defined as

$$\boxed{R_{\lambda\nu} = R^{\kappa}{}_{\lambda\kappa\nu} \,.} \tag{6.40}$$

We may make two other contractions, $R^{\kappa}{}_{\lambda\mu\kappa}$ and $R^{\kappa}{}_{\kappa\mu\nu}$, but they give little new. First, we have that $R^{\kappa}{}_{\lambda\mu\kappa} = -R_{\lambda\mu}$, because of Equation (6.32). Second, we assume that the connection is metric, and then we get from Equation (6.35) that

$$R^{\kappa}{}_{\kappa\mu\nu} = \frac{1}{2}\, g^{\kappa\lambda} \left(R_{\kappa\lambda\mu\nu} + R_{\lambda\kappa\mu\nu}\right) = 0 \,. \tag{6.41}$$

The geometrical meaning of this equation is that the parallel transport around a closed curve preserves volume.

For a symmetric connection Equation (6.34) gives by contraction that

$$0 = R^{\kappa}{}_{\lambda\kappa\nu} + R^{\kappa}{}_{\nu\lambda\kappa} + R^{\kappa}{}_{\kappa\nu\lambda} = R_{\lambda\nu} - R_{\nu\lambda} + R^{\kappa}{}_{\kappa\nu\lambda} \,. \tag{6.42}$$

This implies that the Ricci tensor is symmetric if the connection is both symmetric and metric,

$$\boxed{R_{\mu\nu} = R_{\nu\mu} \,.} \tag{6.43}$$

The symmetry relations for the curvature tensor give no further relations for the Ricci tensor. That it is symmetric, means that it has $4 \times 5/2 = 10$ independent components,

compared to the 20 independent components of the full Riemann curvature tensor. The Ricci tensor contains in a sense exactly half of the information about the curvature, in four dimensions.

The scalar curvature is defined as

$$\boxed{R = g^{\mu\nu} R_{\mu\nu} \; .} \tag{6.44}$$

Given the Ricci tensor and the metric tensor, what remains of the Riemann curvature tensor $R_{\kappa\lambda\mu\nu}$ is the *Weyl curvature tensor* $C_{\kappa\lambda\mu\nu}$, defined by the relation

$$\begin{aligned} R_{\kappa\lambda\mu\nu} \; = \; C_{\kappa\lambda\mu\nu} & - \frac{1}{2} \left(g_{\kappa\nu} R_{\lambda\mu} + g_{\lambda\mu} R_{\kappa\nu} - g_{\kappa\mu} R_{\lambda\nu} - g_{\lambda\nu} R_{\kappa\mu} \right) \\ & - \frac{1}{6} \left(g_{\kappa\mu} g_{\lambda\nu} - g_{\kappa\nu} g_{\lambda\mu} \right) R \; . \end{aligned} \tag{6.45}$$

This definition is so designed that the Weyl tensor has the same symmetries as $R_{\kappa\lambda\mu\nu}$, and does not contribute to the Ricci tensor, since

$$g^{\kappa\mu} C_{\kappa\lambda\mu\nu} = R_{\lambda\nu} + \frac{1}{2} \left(R_{\lambda\nu} + R_{\lambda\nu} - 4 R_{\lambda\nu} - g_{\lambda\nu} R \right) + \frac{1}{6} \left(4 g_{\lambda\nu} - g_{\lambda\nu} \right) R = 0 \; . \tag{6.46}$$

In fact, the Weyl tensor is completely traceless, in the sense that a contraction of any two indices gives zero. For example, $g^{\kappa\lambda} C_{\kappa\lambda\mu\nu} = 0$ because $C_{\kappa\lambda\mu\nu} = -C_{\lambda\kappa\mu\nu}$.

The Einstein gravitational equation for a gravitational field in vacuum is $R_{\mu\nu} = 0$, and it implies that the curvature tensor equals the Weyl-tensor, which need not vanish. Thus we may use the Weyl-tensor e.g. to characterize gravitational waves in an otherwise flat spacetime, and quite generally as a measure of how much the geometry of spacetime in a vacuum region is deformed by a gravitational field.

6.6 The Bianchi identity

The *Bianchi identity* is an identity for the covariant derivatives of the curvature tensor,

$$\boxed{R^{\kappa}{}_{\lambda\mu\nu;\rho} + R^{\kappa}{}_{\lambda\rho\mu;\nu} + R^{\kappa}{}_{\lambda\nu\rho;\mu} = 0 \; .} \tag{6.47}$$

It holds under the condition that the connection is symmetric, otherwise we have the relation

$$R^{\kappa}{}_{\lambda\mu\nu;\rho} + R^{\kappa}{}_{\lambda\rho\mu;\nu} + R^{\kappa}{}_{\lambda\nu\rho;\mu} = T^{\sigma}{}_{\mu\rho} \, R^{\kappa}{}_{\lambda\nu\sigma} + T^{\sigma}{}_{\rho\nu} \, R^{\kappa}{}_{\lambda\mu\sigma} + T^{\sigma}{}_{\nu\mu} \, R^{\kappa}{}_{\lambda\rho\sigma} \; . \tag{6.48}$$

The Bianchi identity is proved very easily by a standard trick. First, it is true that a tensor equation is valid in all coordinate systems if it is valid in one. Second, picking one arbitrary point we may choose our coordinate system such that the connection coefficients vanish, $\Gamma^{\kappa}_{\lambda\mu} = 0$, at this single point. That is always possible when the connection is symmetric. In this particular coordinate system and at this particular point we then have that

$$R^{\kappa}{}_{\lambda\mu\nu;\rho} = R^{\kappa}{}_{\lambda\mu\nu,\rho} = \Gamma^{\kappa}_{\lambda\nu,\mu\rho} - \Gamma^{\kappa}_{\lambda\mu,\nu\rho} \; . \tag{6.49}$$

Since partial differentiations in different directions commute, the Bianchi identity follows. But when it holds at one arbitrary point, in one particular coordinate system, then it holds everywhere in all coordinate systems.

From the Bianchi identity follows that

$$g^{\lambda\nu}\left(R^{\kappa}{}_{\lambda\kappa\nu;\rho} + R^{\kappa}{}_{\lambda\rho\kappa;\nu} + R^{\kappa}{}_{\lambda\nu\rho;\kappa}\right) = 0\ . \tag{6.50}$$

We now make the additional assumption that the metric is covariantly constant. Then the first term in this equation is the partial derivative of the scalar curvature,

$$g^{\lambda\nu}R^{\kappa}{}_{\lambda\kappa\nu;\rho} = \left(g^{\lambda\nu}R^{\kappa}{}_{\lambda\kappa\nu}\right)_{;\rho} = R_{;\rho} = R_{,\rho}\ . \tag{6.51}$$

Recall that covariant differentiation of a scalar is the same as ordinary partial differentiation. Furthermore, the second term in the equation is

$$g^{\lambda\nu}R^{\kappa}{}_{\lambda\rho\kappa;\nu} = -g^{\lambda\nu}R^{\kappa}{}_{\lambda\kappa\rho;\nu} = -g^{\lambda\nu}R_{\lambda\rho;\nu}\ , \tag{6.52}$$

and the third term is the same,

$$g^{\lambda\nu}R^{\kappa}{}_{\lambda\nu\rho;\kappa} = g^{\lambda\nu}g^{\kappa\sigma}R_{\sigma\lambda\nu\rho;\kappa} = -g^{\lambda\nu}g^{\kappa\sigma}R_{\lambda\sigma\nu\rho;\kappa} = -g^{\kappa\sigma}R^{\nu}{}_{\sigma\nu\rho;\kappa} = -g^{\kappa\sigma}R_{\sigma\rho;\kappa}\ . \tag{6.53}$$

Thus, contraction of the Bianchi identity gives for s symmetric and metric connection that

$$\boxed{R_{,\rho} = 2g^{\lambda\nu}R_{\lambda\rho;\nu}\ .} \tag{6.54}$$

6.7 A method for computing the Ricci tensor

We derive here, for later use, a method that may reduce a little bit the work of computing the Ricci tensor $R_{\mu\nu}$ for a given metric $g_{\mu\nu}$. The method is suitable for use in a computer program, e.g. in Mathematica or Maple. See Appendix A for an example of a Maple implementation.

Since we assume that the connection is both symmetric and metric, and consequently that the Ricci tensor is symmetric, it is enough to compute $R_{\mu\nu}A^{\mu}A^{\nu}$ as a function of an arbitrary vector field A^{μ}. From Equation (5.35), and from the definition of the Ricci tensor, we have that

$$\begin{aligned} R_{\lambda\nu}A^{\lambda}A^{\nu} &= A^{\nu}A^{\kappa}{}_{;\nu\kappa} - A^{\nu}A^{\kappa}{}_{;\kappa\nu} \\ &= B^{\kappa}{}_{;\kappa} - A^{\nu}{}_{;\kappa}A^{\kappa}{}_{;\nu} - A^{\nu}A^{\kappa}{}_{;\kappa\nu}\ , \end{aligned} \tag{6.55}$$

We define $B^{\kappa} = A^{\nu}A^{\kappa}{}_{;\nu}$, and the first term on the right hand side is

$$R^{(\mathrm{I})} = B^{\kappa}{}_{;\kappa} = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^{\kappa}}\left(\sqrt{|g|}\,B^{\kappa}\right) = B^{\kappa}{}_{,\kappa} + \gamma_{,\kappa}\,B^{\kappa}\ , \tag{6.56}$$

with

$$\gamma = \frac{1}{2}\,\ln|g|\ . \tag{6.57}$$

We have that

$$\begin{aligned} B^\kappa &= A^\nu A^\kappa{}_{;\nu} = g^{\kappa\rho} A^\nu A_{\rho;\nu} = g^{\kappa\rho} A^\nu (A_{\rho,\nu} - \Gamma^\sigma_{\rho\nu} A_\sigma) \\ &= g^{\kappa\rho} A^\nu \left(A_{\rho,\nu} - A_\sigma \frac{1}{2} g^{\sigma\lambda} (g_{\lambda\rho,\nu} + g_{\lambda\nu,\rho} - g_{\rho\nu,\lambda}) \right) \\ &= g^{\kappa\rho} A^\nu \left(A_{\rho,\nu} - \frac{1}{2} A^\lambda g_{\lambda\nu,\rho} \right) . \end{aligned} \tag{6.58}$$

A^μ is an arbitrary vector field, which we may as well choose to have constant components, implying that

$$B^\kappa = g^{\kappa\rho} A^\nu \left(A_{\rho,\nu} - \frac{1}{2} A_{\nu,\rho} \right) . \tag{6.59}$$

The second term on the right hand side in Equation (6.55) is

$$R^{(\mathrm{II})} = -A^\mu{}_{;\kappa} A^\kappa{}_{;\mu} . \tag{6.60}$$

Here we may also choose $A^\kappa{}_{,\mu} = 0$, to get that

$$A^\kappa{}_{;\mu} = \Gamma^\kappa_{\lambda\mu} A^\lambda = A^\lambda \frac{1}{2} g^{\kappa\rho} (g_{\rho\lambda,\mu} + g_{\rho\mu,\lambda} - g_{\lambda\mu,\rho}) = \frac{1}{2} g^{\kappa\rho} \left(A_{\rho,\mu} - A_{\mu,\rho} + A^\lambda g_{\rho\mu,\lambda} \right) . \tag{6.61}$$

This gives that $R^{(\mathrm{II})} = R^{(\mathrm{IIa})} + R^{(\mathrm{IIb})}$, with

$$\begin{aligned} R^{(\mathrm{IIa})} &= \frac{1}{4} g^{\mu\sigma} g^{\kappa\rho} (A_{\kappa,\sigma} - A_{\sigma,\kappa})(A_{\rho,\mu} - A_{\mu,\rho}) , \\ R^{(\mathrm{IIb})} &= -\frac{1}{4} g^{\mu\sigma} g^{\kappa\rho} (A^\nu g_{\kappa\sigma,\nu})(A^\lambda g_{\rho\mu,\lambda}) . \end{aligned} \tag{6.62}$$

The third term on the right hand side in Equation (6.55) is

$$R^{(\mathrm{III})} = -A^\nu A^\kappa{}_{;\kappa\nu} = -A^\nu \frac{\partial}{\partial x^\nu} A^\kappa{}_{;\kappa} , \tag{6.63}$$

since the divergence $A^\kappa{}_{;\kappa} = A^\kappa{}_{,\kappa} + \gamma_{,\kappa} A^\kappa$ is a scalar. If all $A^\kappa{}_{,\nu} = 0$, we then have that

$$R^{(\mathrm{III})} = -\gamma_{,\lambda\nu} A^\lambda A^\nu . \tag{6.64}$$

Problems

1. The curvature tensor $R_{\kappa\lambda\mu\nu}$ of a symmetric and metric connection has 20 independent components in four dimensions.
   What is the number of independent components in one dimension?
   In two dimensions? In three dimensions?
   Hint: show that you need only consider the symmetry relations (6.32), (6.35) and (6.37).

2. In two dimensions, if we have given any two tensor fields A_{ijkl} and B_{ijkl} with the same symmetry properties as the Riemann curvature tensor R_{ijkl} of a symmetric and metric connection, then they must be proportional, by the counting of independent components in Problem 1. That is, we must have $B_{ijkl} = \alpha A_{ijkl}$, where α is a scalar field.
   In particular, this implies that the curvature tensor R_{ijkl} is given in terms of the metric tensor g_{ij} as
   $$R_{ijkl} = \lambda(g_{ik}g_{jl} - g_{il}g_{jk}) \ ,$$
   or equivalently,
   $$R^i{}_{jkl} = \lambda\,(\delta^i_k\, g_{jl} - \delta^i_l\, g_{jk}) \ ,$$
   where λ is a scalar field (not necessarily constant).
   Use this formula to compute Ricci's curvature tensor $R_{jl} = R^i{}_{jil}$ and the curvature scalar $R = g^{jl}\, R_{jl}$.

3. Given dimensionless coordinates x and y, and the two dimensional line element
   $$\mathrm{d}s^2 = (1 + \cos^2 x)\,\mathrm{d}x^2 + 2\cos x \cos y\ \mathrm{d}x\,\mathrm{d}y + (1 + \cos^2 y)\,\mathrm{d}y^2 \ .$$

   a) Show that there is a two dimensional suface $z = f(x, y)$ in the three dimensional space with coordinates (x, y, z) such that the above line element has the form $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$.
   Sketch what the surface looks like, e.g. using a computer plotting program such as Maple.

   b) What are the covariant components g_{xx}, g_{xy}, g_{yx} and g_{yy} of the metric tensor?
   What are the contravariant components g^{xx}, g^{xy}, g^{yx} and g^{yy}?
   Compute explicitly the $2^3 = 8$ components of the Christoffel connection, Equation (5.49).
   How many independent components does the connection have when we take into account the symmetry $\Gamma^i_{jk} = \Gamma^i_{kj}$?

   c) Verify the formula from Problem 2, $R^i{}_{jkl} = \lambda\,(\delta^i_k\, g_{jl} - \delta^i_l\, g_{jk})$, in the present example, e.g. by computing some components (or all the 16 components!) of the curvature tensor. Find an explicit expression for the scalar λ as a function of x and y.

   d) Compute Ricci's curvature tensor $R_{jl} = R^i{}_{jil}$ and the curvature scalar $R = g^{jl}\, R_{jl}$.
   Examine the sign of the curvature scalar e.g. at those points that are maxima, minima and saddle points of the function $f(x, y)$ defined under point a).
   It is common terminology to say that the curvature of a surface is positive or negative, depending on whether the curvature scalar is positive or negative.

4. We defined the curvature tensor from what happens to a contravariant vector under parallel transport around an infinitesimal closed curve.
   What happens to a covariant vector? To a tensor in general?

5. Use the metric on the surface of a sphere found in Problem 2.5, and compute the Christoffel connection, Equation (5.49).
   What happens to a contravariant tangent vector to the surface of the sphere when it is parallel transported (in such a way that it is all the time tangent to the surface) once around the circle $\theta = \theta_0 =$ constant?
   More precisely: what happens to the direction and the length of the vector?
   Show that if we picture the tangent vectors to the surface of the sphere as lying in a plane which is tangent to the surface, then we may picture the parallel transport of tangent vectors as a sliding of the tangent plane along the surface, with no rotation in the plane.

6. The equation $z = a - \sqrt{a^2 - x^2 - y^2}$ where a is a positive constant, defines a two dimensional surface embedded in the three dimensional Euclidean space with the metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. It is the surface of a sphere with radius a and centre at $(x, y, z) = (0, 0, a)$.
   Study this surface, using (x, y) as coordinates.
   Find an explicit expression for the metric $\mathrm{d}s^2$ on the surface.
   Then transform to polar coordinates (r, φ) such that $x = r\cos\varphi$, $y = r\sin\varphi$.
   Find a new radial coordinate $q = q(r)$ such that $\mathrm{d}s^2 = \mathrm{d}q^2 + f(q)\,\mathrm{d}\varphi^2$.
   What is the function $f(q)$?
   Compute the scalar curvature R using one of the coordinate systems (x, y), (r, φ) or (q, φ) (or all three). Is it positive or negative?

7. Study the two dimensional surface $z = \sqrt{a^2 + x^2 + y^2} - a$ with the metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}z^2$, in a similar way as in Problem 6.
   Is this three dimensional metric positive definit?
   Is the two dimensional metric on the surface positive definit?
   Can the surface be embedded as a surface of the form $X = X(x, y)$, $Y = Y(x, y)$, $Z = Z(x, y)$ in a three dimensional space with the metric $\mathrm{d}S^2 = \mathrm{d}X^2 + \mathrm{d}Y^2 + \mathrm{d}Z^2$?

8. ("Cut and paste" geometry). Draw a closed curve on a sheet of paper, and draw parallel vectors at various points around the curve. The sheet is flat: a vector which is parallel transported around any closed curve returns with its original length and direction.
   Now draw two straight lines on the sheet of paper, intersecting at one point at some angle α. Cut out the wedge of angle α, and glue the two edges together. This produces a cone. The cone is still flat everywhere, except at the top point. In fact, draw a closed curve not encircling the top point, and draw parallel vectors at some points around the curve, to see that there is no change in length and direction of a vector parallel transported around the curve. Try what happens to a vector parallel transported around the top point. The curvature there is infinite, in the sense that parallel transport around an arbitrarily small curve produces a finite rotation angle. We may make the curvature finite by rounding off, for example cutting the cone at the top and replacing the cut off section by the cap of a sphere. Is the curvature positive or negative?
   Next, take a new sheet of paper, draw a wedge as before, of opening angle α, say. But

do not cut out the wedge, until you have narrowed it in the following way. Draw two new straight lines that are parallel, starting from points on the two legs, at the same distance from the top of the wedge, and intersecting the first two lines at angles both equal to $\alpha/2$. Cut out the narrowed wedge. Gluing together the two edges of the wedge should produce a surface with two singular points. Parallel transporting vectors around one or both singular points you should observe that there is infinite curvature at both singular points, positive at one point and negative at the other, in such a way that the curvature at one point cancels the curvature at the other point.

So is there any remaining effect of the curvature, when you parallel transport vectors around a closed curve encircling both singularities? Yes, indeed, there is torsion: if you try to draw a parallelogram around the two singularities, it will not be closed.

In a crystal, we may define two vectors as parallel if they have the same length and the same direction relative to the crystal lattice. Then the two examples considered here correspond to crystal defects. One singularity like the top of a cone corresponds to a defect called a *disclination*. Two disclinations at different places in the crystal may cancel each other, and the result is then a *dislocation*. A disclination is clearly a more serious defect than a dislocation.

Note that if there exists curvature and torsion in a region, one may observe it by going around the region, *without ever coming close to the region*. The same is true of crystal defects.

Chapter 7

Integration

It is the privilege of n-forms, the antisymmetric covariant tensor fields of rank n, to be candidates for invariant integration over n dimensional surfaces in spacetime. However, they have to share this privilege with their duals, the $(4-n)$-densities. The integration is invariant in the sense that the value of the integral of an n-form or a $(4-n)$-density is independent of the coordinate system and of the parametrization of the surface. The only exception to the invariance is the sign of the integral, which may depend on the orientation of the surface, which in turn depends on the parametrization.

In the present chapter we will discuss in some detail the simplest examples of integration, with $n = 1$ and $n = 2$. The generalization to $n > 2$ is afterwards relatively straightforward.

7.1 Line integrals

Our first example will be a covariant vector field $A_\mu = A_\mu(x)$, and a curve C, parametrized by a parameter u such that $x^\mu = x^\mu(u)$, with $a \le u \le b$. The line integral of A_μ along C is defined as

$$I_1 = \int_a^b A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}u}\,\mathrm{d}u = \int_a^b A_\mu(x(u)) \frac{\mathrm{d}x^\mu(u)}{\mathrm{d}u}\,\mathrm{d}u\ . \tag{7.1}$$

The sign convention is that the integral goes from a to b when $a < b$.

If we use another parameter $v = v(u)$ along the curve, we have to replace the integration limits a and b by $c = v(a)$ and $d = v(b)$. The reparametrization $u \to v = v(u)$ either preserves the orientation of the curve, this means that $\mathrm{d}v/\mathrm{d}u > 0$ for all $u \in [a, b]$, or else it reverses the orientation, this means that $\mathrm{d}v/\mathrm{d}u < 0$ for all $u \in [a, b]$.

If the orientation is preserved, then $c < d$, hence by definition we integrate over v from c to d, and the integral is invariant,

$$J_1 = \int_c^d A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}v}\,\mathrm{d}v = I_1\ . \tag{7.2}$$

If the orientation is reversed, then $d < c$, hence we must integrate over v from d to c, and the integral changes sign,

$$J_1 = \int_d^c A_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}v}\,\mathrm{d}v = -I_1\ . \tag{7.3}$$

As we see, the integral is invariant under a reparametrization $u \mapsto v$ of the curve C, except that its sign depends on the orientation of the curve. A complete definition of the curve C includes a definition of how it is oriented, that is, in which direction along the curve we integrate. To a curve C with a given orientation corresponds a curve that we may call $-C$, identical to C except that its orientation is opposite.

Since the integral is invariant under an arbitrary reparametrization preserving the orientation of the curve, we write

$$I_1 = \int_C A_\mu \, \mathrm{d}x^\mu = \int_C \mathbf{A} = -\int_{-C} \mathbf{A} \, . \tag{7.4}$$

This notation shows at the same time that the integral is invariant under an arbitrary coordinate transformation $x^\mu \mapsto \widetilde{x}^\mu$, since

$$\mathbf{A} = A_\mu \, \mathrm{d}x^\mu = \widetilde{A}_\mu \, \mathrm{d}\widetilde{x}^\mu \, . \tag{7.5}$$

7.2 Surface integrals

Our second example is a 2-form $A_{\mu\nu} = A_{\mu\nu}(x) = -A_{\nu\mu}(x)$. We write also

$$\mathbf{A} = \frac{1}{2} A_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \sum_{\mu<\nu} A_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \, . \tag{7.6}$$

We want to integrate it over a two dimensional surface Ω, parametrized by two parameters $(u^1, u^2) \in U$ such that we have $x^\mu = x^\mu(u^1, u^2)$ on the surface. Here U is the region of variation for the parameters (u^1, u^2).

Recall the definition of the exterior poduct, Equation (3.8),

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu - \mathrm{d}x^\nu \otimes \mathrm{d}x^\mu \, . \tag{7.7}$$

This suggests that we define the surface integral of $\mathbf{A}$ over Ω as

$$\begin{aligned} I_2 &= \frac{1}{2} \int_U A_{\mu\nu}(x) \left(\frac{\partial x^\mu}{\partial u^1} \frac{\partial x^\nu}{\partial u^2} - \frac{\partial x^\nu}{\partial u^1} \frac{\partial x^\mu}{\partial u^2} \right) \mathrm{d}u^1 \mathrm{d}u^2 \\ &= \frac{1}{2} \int_U A_{\mu\nu}(x) \left(\epsilon^{ij} \frac{\partial x^\mu}{\partial u^i} \frac{\partial x^\nu}{\partial u^j} \right) \mathrm{d}u^1 \mathrm{d}u^2 \, , \end{aligned} \tag{7.8}$$

where ϵ^{ij} is the two dimensional Levi–Civita symbol. Again the sign of the integral is defined by the convention that the integrals over u^1 and u^2 are both to be taken in the positive direction. Because of the antisymmetry $A_{\mu\nu} = -A_{\nu\mu}$, we have also that

$$I_2 = \int_U A_{\mu\nu}(x) \, \frac{\partial x^\mu}{\partial u^1} \frac{\partial x^\nu}{\partial u^2} \, \mathrm{d}u^1 \mathrm{d}u^2 \, . \tag{7.9}$$

In the most general case, one single parameter region U will not be sufficient to cover the whole surface Ω, and then we have to sum over two or more parameter regions $U_1, U_2, \ldots$. It is essential that the orientation is the same for all the parameter regions. Recall our definition, that two overlapping maps have the same orientation if the coordinate transformation between them has a positive Jacobi determinant everywhere.

A consequence of this definition is that the surface integral is invariant up to a sign under a reparametrization of the surface, $(u^1, u^2) \mapsto (v^1, v^2)$. In terms of the parameters (v^1, v^2) the integral is defined as

$$J_2 = \frac{1}{2} \int_V A_{\mu\nu} \left(\epsilon^{ij} \frac{\partial x^\mu}{\partial v^i} \frac{\partial x^\nu}{\partial v^j} \right) \mathrm{d}v^1 \mathrm{d}v^2 \ , \tag{7.10}$$

where V is the region of variation for (v^1, v^2), and where both v^1 and v^2 are to be integrated in the positive direction. We have that

$$\epsilon^{ij} \frac{\partial x^\mu}{\partial v^i} \frac{\partial x^\nu}{\partial v^j} = \epsilon^{ij} \frac{\partial u^k}{\partial v^i} \frac{\partial u^l}{\partial v^j} \frac{\partial x^\mu}{\partial u^k} \frac{\partial x^\nu}{\partial u^l} = \det\left(\frac{\partial u}{\partial v} \right) \epsilon^{kl} \frac{\partial x^\mu}{\partial u^k} \frac{\partial x^\nu}{\partial u^l} \ . \tag{7.11}$$

If the Jacobi determinant $\det(\partial u / \partial v)$ is positive, it means by definition that the new parameters (v^1, v^2) and the original parameters (u^1, u^2) define the same orientation of the surface. We see that in this case the integral is invariant, $J_2 = I_2$. If the Jacobi determinant is negative, it means that the parameters (v^1, v^2) define the opposite orientation of the surface, relative to (u^1, u^2). In this case the integral changes sign, $J_2 = -I_2$.

Because the integral is independent of the parametrization, up to a sign that depends on the orientation, we are justified in writing

$$I_2 = \frac{1}{2} \int_\Omega A_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \int_\Omega \mathbf{A} \ . \tag{7.12}$$

It is to be understood that the orientation of the surface Ω is part of its definition. The same surface with the opposite orientation we call, quite naturally, $-\Omega$, and we have that

$$\int_{-\Omega} \mathbf{A} = - \int_\Omega \mathbf{A} \ . \tag{7.13}$$

Again, the notation shows clearly that the integral is invariant under a coordinate transformation $x^\mu \mapsto \widetilde{x}^\mu$. In fact,

$$\mathbf{A} = \frac{1}{2} A_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \frac{1}{2} \widetilde{A}_{\mu\nu} \, \mathrm{d}\widetilde{x}^\mu \wedge \mathrm{d}\widetilde{x}^\nu \ . \tag{7.14}$$

We translate from the compact notation in Equation (7.12) to the more explicit notation in Equation (7.8) or (7.9) by the mechanical substitution

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \to \left(\epsilon^{ij} \frac{\partial x^\mu}{\partial u^i} \frac{\partial x^\nu}{\partial u^j} \right) \mathrm{d}u^1 \mathrm{d}u^2 = 2 \frac{\partial x^\mu}{\partial u^1} \frac{\partial x^\nu}{\partial u^2} \mathrm{d}u^1 \mathrm{d}u^2 \ . \tag{7.15}$$

The last equality is valid because of the antisymmetry $A_{\mu\nu} = -A_{\nu\mu}$ of the 2-form $\mathbf{A}$ to be integrated.

7.3 Duality

By means of the two examples $n = 1$ and $n = 2$ we have seen how to define the integral $\int_\Omega \mathbf{A}$ of an n-form $\mathbf{A}$ over an n dimensional surface Ω. We have seen previously that a $(d-n)$-density $\mathbf{B}$, in d dimensions, is the dual of an n-form ${}_\circ\mathbf{B}$. Hence it is very natural to define the integral of $\mathbf{B}$ as the integral of ${}_\circ\mathbf{B}$,

$$I = \int_\Omega \mathbf{B} = \int_\Omega {}_\circ\mathbf{B} \ . \tag{7.16}$$

Examples in dimension $d = 4$

For example, with $d = n = 4$, $\mathbf{B} = B$ is a scalar density, ${}_\circ\mathbf{B}$ is a 4-form, and

$$I = \frac{1}{24}\int_\Omega B\,\epsilon_{\kappa\lambda\mu\nu}\,\mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \int_\Omega B\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3\ . \tag{7.17}$$

According to our recipe, to compute the last integral we should introduce parameters $u = (u^1, u^2, u^3, u^4)$ such that $x^\mu = x^\mu(u) = x^\mu(u^1, u^2, u^3, u^4)$, and then substitute

$$\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \to \left(\epsilon^{ijkl}\,\frac{\partial x^0}{\partial u^i}\,\frac{\partial x^1}{\partial u^j}\,\frac{\partial x^2}{\partial u^k}\,\frac{\partial x^3}{\partial u^l}\right)\mathrm{d}u^1\mathrm{d}u^2\mathrm{d}u^3\mathrm{d}u^4\ . \tag{7.18}$$

The natural choice of parameters in this case are the coordinates themselves, $u^1 = x^0$, $u^2 = x^1$, $u^3 = x^2$, $u^4 = x^3$, this gives that

$$I = \int_\Omega B\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \int_\Omega B\,\mathrm{d}^4x\ . \tag{7.19}$$

As another example, let $d = 4$ and $n = 3$. Then $\mathbf{B}$ is a vector density, ${}_\circ\mathbf{B}$ is a 3-form, and

$$I = \frac{1}{6}\int_\Omega B^\kappa\,\epsilon_{\kappa\lambda\mu\nu}\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \int_\Omega B^\kappa\,\mathrm{d}^3S_\kappa\ . \tag{7.20}$$

The three dimensional surface element in four dimensions is

$$\mathrm{d}^3S_\kappa = \frac{1}{6}\,\epsilon_{\kappa\lambda\mu\nu}\,\mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\ . \tag{7.21}$$

More explicitly, this means that

$$\begin{aligned} I = \ & \int_\Omega B^0\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 - \int_\Omega B^1\,\mathrm{d}x^0 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ & + \int_\Omega B^2\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \int_\Omega B^3\,\mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2\ . \end{aligned} \tag{7.22}$$

A tensor density $\mathbf{B}$ is often made from an antisymmetric contravariant tensor $\mathbf{C}$ as $\mathbf{B} = \sqrt{|g|}\,\mathbf{C}$. For example, if $\mathbf{B} = \sqrt{|g|}\,f$, where f is a scalar function, then

$$I = \frac{1}{24}\int_\Omega \sqrt{|g|}\,f\,\epsilon_{\kappa\lambda\mu\nu}\,\mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = \int_\Omega \sqrt{|g|}\,f\ \mathrm{d}^4x\ . \tag{7.23}$$

Here $\mathrm{d}V = \sqrt{|g|}\,\mathrm{d}^4x$ is an *invariant volume element*. In fact, a coordinate transformation $x^\mu \mapsto \widetilde{x}^\mu$ gives that

$$\sqrt{|g|}\,\mathrm{d}^4x \mapsto \sqrt{|\widetilde{g}|}\,\mathrm{d}^4\widetilde{x} = \left(\left|\det\left(\frac{\partial x}{\partial \widetilde{x}}\right)\right|\sqrt{|g|}\right)\left(\left|\det\left(\frac{\partial \widetilde{x}}{\partial x}\right)\right|\mathrm{d}^4x\right) = \sqrt{|g|}\,\mathrm{d}^4x\ . \tag{7.24}$$

See Equation (3.30).

An example of physical relevance is a generally covariant field theory, with field equations having the same form in all coordinate systems. The Lagrange density defining such a theory is most often a scalar density of the form

$$\mathcal{L} = \sqrt{|g|}\,\mathcal{L}_s\ , \tag{7.25}$$

where $\mathcal{L}_s$ is a scalar function depending on the fields and their derivatives. We may call $\mathcal{L}_s$ the Lagrange *scalar*, to distinguish it from the Lagrange *density* $\mathcal{L}$.

7.4 Length, area and volume

The metric $g_{\mu\nu}$ defines such concepts as length, area and volume in a way which is invariant under coordinate transformations. We define the length of a curve, the area of a two dimensional surface, or the volume of a three or four dimensional region to be always non-negative, independent of the orientation.

Length is defined by the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\ . \tag{7.26}$$

To be explicit, if the curve C is parametrized as $x^\mu = x^\mu(u)$ with $a \le u \le b$, then its length is

$$L = \int_C \mathrm{d}s = \int_a^b \mathrm{d}u\,\sqrt{\left|g_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}u}\right|}\ . \tag{7.27}$$

Equation (7.23) gives an invariant definition of the integral of an arbitrary scalar function f, in four dimensions, over a four dimensional region Ω. Taking $f = 1$ we get the invariant four dimensional *volume* of Ω,

$$V = \int_\Omega \sqrt{|g|}\,\mathrm{d}x^0\mathrm{d}x^1\mathrm{d}x^2\mathrm{d}x^3 = \int_\Omega \sqrt{|g|}\,\mathrm{d}^4x\ . \tag{7.28}$$

We may define in a similar way the *area* of a two dimensional surface, or the *volume* of a three dimensional region. Assume e.g. that the two dimensional surface Ω is parametrised as $x^\mu = x^\mu(u^1, u^2)$, with $(u^1, u^2) \in U$. Then the line element on the surface may be written as

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = h_{ij}\,\mathrm{d}u^i\,\mathrm{d}u^j\ , \tag{7.29}$$

where h_{ij} is a 2×2 matrix,

$$h_{ij} = g_{\mu\nu}\,\frac{\partial x^\mu}{\partial u^i}\,\frac{\partial x^\nu}{\partial u^j}\ . \tag{7.30}$$

Define now $h = \det(h_{ij})$, and define the area as

$$A = \int_U \sqrt{|h|}\,\mathrm{d}u^1\mathrm{d}u^2\ . \tag{7.31}$$

7.5 Differentiation and integration

Stokes's theorem

The connection between differentiation and integration is the fundamental theorem of calculus,

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)\ , \tag{7.32}$$

where $f = f(x)$ is a function of one variabel x, such that its derivative $f' = f'(x)$ exists and is continuous for all $x \in [a, b]$. It is possible to relax the conditions imposed, e.g. allowing f' to be undefined at isolated points, if only f is continuous at the same points. We may relax

even the condition of continuity of f, if we allow f' to have singularities of the same type as the Dirac δ-function.

The same fundamental theorem applies to higher dimensional integrals, with only such modifications as are necessary because there are more integration variables. For example, if $f = f(x, y)$ is a function of two variables x and y, and we integrate the partial derivative $\partial f/\partial x$ over the rectangle $x \in [a, b]$, $y \in [c, d]$, then the x-integration can be performed by means of the fundamental theorem, and we get that

$$\int_a^b \mathrm{d}x \int_c^d \mathrm{d}y \, \frac{\partial}{\partial x} f(x, y) = \int_c^d \mathrm{d}y \, \left(f(b, y) - f(a, y) \right) . \tag{7.33}$$

If we need to integrate not over the rectangle but over the circle $x^2 + y^2 \leq R^2$, we get equally easily that

$$\int_{-R}^{R} \mathrm{d}y \int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \mathrm{d}x \, \frac{\partial f}{\partial x} = \int_{-R}^{R} \mathrm{d}y \, \left(f\left(\sqrt{R^2 - y^2}, y \right) - f\left(-\sqrt{R^2 - y^2}, y \right) \right) . \tag{7.34}$$

Stokes's theorem is the fundamental theorem of calculus in n dimensions,

$$\int_\Omega \mathrm{d}\mathbf{A} = \int_{\partial\Omega} \mathbf{A} \, . \tag{7.35}$$

In principle it is just as easy to understand as the two dimensional example above, although the compact formulation may be somewhat less transparent. The surface Ω to be integrated over is n dimensional, hence its boundary $\partial\Omega$ is $n - 1$ dimensional. $\mathbf{A}$ is an $(n - 1)$-form,

$$\mathbf{A} = \frac{1}{(n-1)!} \, A_{\kappa\lambda\ldots\rho} \, \mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \cdots \wedge \mathrm{d}x^\rho \, , \tag{7.36}$$

and $\mathrm{d}\mathbf{A}$ is the exterior derivative of $\mathbf{A}$, which is an n-form,

$$\mathrm{d}\mathbf{A} = \frac{1}{(n-1)!} \, A_{\kappa\lambda\ldots\rho,\sigma} \, \mathrm{d}x^\sigma \wedge \mathrm{d}x^\kappa \wedge \mathrm{d}x^\lambda \wedge \cdots \wedge \mathrm{d}x^\rho \, . \tag{7.37}$$

For the theorem to hold, the orientation of the boundary $\partial\Omega$ must be correctly defined relative to the orientation of Ω. The definition is most easily demonstrated in an example. In general we can not treat the whole boundary simultaneously, so we consider one local patch of the integration region Ω where we can use one single set of parameters $(u^1, u^2, \ldots, u^n)$. By definition, the orientation of Ω defined by these parameters is positive. We assume that the parameters are defined in such a a way that the part of the boundary we consider, is given by an equation $u^1 = C =$ constant, and that the interiour of Ω, close to the boundary, is given by the inequality $u^1 < C$. After we fix $u^1 = C$, the remaining parameters $(u^2, u^3, \ldots, u^n)$ define a parametrization of the local patch of the boundary, and by definition this specifies the positive orientation of the boundary.

Take an example that illustrates Stokes's theorem and outlines the general proof. Let Ω be a two dimensional surface in four dimensional spacetime, parametrized as $x^\mu = x^\mu(u^1, u^2)$ with $(u^1, u^2) \in U$, where the parameter region U is a rectangle,

$$U = [a, b] \times [c, d] = \{ \, (u^1, u^2) \mid a \leq u^1 \leq b \, , \;\; c \leq u^2 \leq d \, \} \, . \tag{7.38}$$

Thus $n = 2$, and $\mathbf{A} = A_\mu \, \mathrm{d}x^\mu$ is a 1-form, whereas $\mathrm{d}\mathbf{A} = A_{\mu,\sigma} \, \mathrm{d}x^\sigma \wedge \mathrm{d}x^\mu$ is a 2-form. The left hand side of Equation (7.35) is, by definition,

$$\begin{aligned} \int_\Omega \mathrm{d}\mathbf{A} &= \int_a^b \mathrm{d}u^1 \int_c^d \mathrm{d}u^2 \, A_{\mu,\sigma} \left(\epsilon^{ij} \frac{\partial x^\sigma}{\partial u^i} \frac{\partial x^\mu}{\partial u^j} \right) \\ &= \int_a^b \mathrm{d}u^1 \int_c^d \mathrm{d}u^2 \, \epsilon^{ij} \frac{\partial}{\partial u^i} \left(A_\mu \frac{\partial x^\mu}{\partial u^j} \right) \\ &= \int_a^b \mathrm{d}u^1 \int_c^d \mathrm{d}u^2 \left(\frac{\partial}{\partial u^1} \left(A_\mu \frac{\partial x^\mu}{\partial u^2} \right) - \frac{\partial}{\partial u^2} \left(A_\mu \frac{\partial x^\mu}{\partial u^1} \right) \right) . \end{aligned} \tag{7.39}$$

In the last double integral one integration can be carried out, with the result

$$\begin{aligned} \int_\Omega \mathrm{d}\mathbf{A} = \quad & \int_c^d \mathrm{d}u^2 \left(A_\mu \frac{\partial x^\mu}{\partial u^2} \right) \Big|_{u^1=b} - \int_c^d \mathrm{d}u^2 \left(A_\mu \frac{\partial x^\mu}{\partial u^2} \right) \Big|_{u^1=a} \\ & - \int_a^b \mathrm{d}u^1 \left(A_\mu \frac{\partial x^\mu}{\partial u^1} \right) \Big|_{u^2=d} + \int_a^b \mathrm{d}u^1 \left(A_\mu \frac{\partial x^\mu}{\partial u^1} \right) \Big|_{u^2=c} . \end{aligned} \tag{7.40}$$

It is not too difficult to see that this is identical to the right hand side of Equation (7.35). The boundary $\partial\Omega$ consists of the four curves $u^1 = a$, $u^1 = b$, $u^2 = c$ and $u^2 = d$, so that $\int_{\partial\Omega} \mathbf{A} = \int_{\partial\Omega} A_\mu \, \mathrm{d}x^\mu$ may be split into four line integrals, as in Equation (7.40). All we have to check is that our convention for the orientation of the boundary gives the correct sign to each of the four line integrals.

The reasoning goes as follows. The parameter u^2 defines a positive orientation of the curve $u^1 = b$, but a negative orientation of the curve $u^1 = a$, because the interior of Ω is given by the inequalities $a < u^1 < b$. The parameter u^1, on the other hand, defines a negative orientation of the curve $u^2 = d$, but a positive orientation of the curve $u^2 = c$, because the interchange of parameters, $(u^1, u^2) \mapsto (u^2, u^1)$, is a transformation with a negative Jacobi determinant, and so inverts the orientation of Ω.

Gauss's theorem (the divergence theorem)

This is simply the dual version of Stokes's teorem,

$$\int_\Omega \operatorname{div} \mathbf{B} = \int_{\partial\Omega} \mathbf{B} \, . \tag{7.41}$$

Like above, Ω is an n dimensional surface with boundary $\partial\Omega$. But $\mathbf{B}$ is here a $(4 - n + 1)$-density, or more generally a $(d - n + 1)$-density in a d dimensional space, with components $B^{\kappa\ldots\nu\rho}$, wheras the divergence of $\mathbf{B}$, $\operatorname{div} \mathbf{B}$, is a $(4-n)$-density, or in general a $(d-n)$-density, with components $B^{\kappa\ldots\nu\rho}{}_{,\rho}$.

Problems

1. In the Euclidean plane the area of a region Ω is

$$A = \int_\Omega \mathrm{d}x\,\mathrm{d}y = \int_\Omega \mathrm{d}x \wedge \mathrm{d}y \ .$$

Show that the area may also be computed as a line integral around the boundary $\partial\Omega$,

$$A = \int_{\partial\Omega} x\,\mathrm{d}y = \frac{1}{2}\int_{\partial\Omega} (x\,\mathrm{d}y - y\,\mathrm{d}x) \ .$$

Use this e.g. to compute the area of a triangle and a circle.
Similarly, in the three dimensional Euclidean space the volume of a region Ω is

$$V = \int_\Omega \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int_\Omega \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \ .$$

Can the volume be computed as a surface integral over the boundary $\partial\Omega$?

Chapter 8

The special theory of relativity

8.1 The speed of light

The special and the general theories of relativity are both based on the important experimental result that the speed of light in vacuum,

$$c = 299\,792\,458 \text{ m/s} , \tag{8.1}$$

is a universal constant, independent of the velocities of the light source and the observer.

The above value of c, measured in meter per second, is exact, because it defines the *meter* as a unit of length. The time unit *second* is defined such that the hyperfine splitting of the ground state of ^{133}Cs, the only stable isotope of the element cesium, is exactly

$$\nu_0 = 9\,192\,631\,770 \text{ /s} = 9\,192\,631\,770 \text{ Hz} . \tag{8.2}$$

This definition is directly useful, since it is possible to tune an oscillating electronic circuit very precisely to be in resonance with cesium atoms, and then to measure time simply by counting the number of oscillations. The radio waves produced by the oscillator and inducing hyperfine transitions in the cesium atoms have a wavelength of

$$\lambda_0 = \frac{c}{\nu_0} = 3.261\,225\,571\,749\ldots \text{ cm} . \tag{8.3}$$

If light is of the nature of waves in a medium, the so called ether, then one would expect the observed velocity of light to depend on the velocity of the observer relative to the ether, but not on the velocity of the source. If light consists of particles, called photons, one would expect their observed velocity to depend at least on the relative velocity of the observer and the source.

The famous experiment of Albert Michelson and Edward Morley in 1887 demonstrated that the observed speed of light does not depend on the motion of the observer. Other observations and experiments show that the observed speed of light also does not depend on the motion of the light source. For example, according to an argument of Willem de Sitter, if the speed of light depended on the motion of the source, then we would see strange effects when observing double stars orbiting each other, at distances of hundreds of light years away from us. Direct observations of the velocities of photons from the disintegration of neutral π mesons confirm that there is no dependence on the velocity of the source. The speed of a π meson as a light source may come very close to the vacuum speed of light.

Since c is a universal conversion factor between units for time and length, it is quite natural to use the same unit for time and length, thus defining $c = 1$. This is indeed a common practice in those parts of physics where relativistic effects are important. When in addition the reduced Planck's constant $\hbar$ is set to 1, we have the so called *natural unit system*. According to John Wheeler, measuring time in second and length in meter, so that $c \neq 1$, is just as unnatural as measuring the length of a road in miles and the width in feet.

8.2 The principle of relativity

The Michelson–Morley experiment was an attempt to measure the absolute motion of the Earth, by comparing the speed of light in different directions. More precisely, it was an attempt to measure the motion relative to the ether, which would be an absolute rest frame. The null result of the experiment showed that absolute motion can not be detected by a measurement of the speed of light.

The *principle of relativity* is the more general statement that motion with constant velocity has no absolute meaning, because there is no possible experiment which an observer can make in order to decide whether he is at rest, or is moving with a constant velocity relative to an absolute rest frame.

A basic assumption in the *special* theory of relativity, as opposed to the *general* theory, is that there exist special coordinate systems, so called *inertial frames*, in which particles that are not subject to external forces move in straight lines with constant velocity. Relative to one given inertial frame, another inertial frame may be displaced in time and space: its origin may be in a different place and at a different time. It may be rotated: its three spatial axes may point in different directions. And it may move with a constant velocity relative to the first frame.

According to the principle of relativity, *all inertial frames are equivalent*. The special theory or relativity is based on the principle that the laws of physics are the same in all inertial frames.

It follows that a physicist in an inertial frame, unable to see the external world because his laboratory has no windows, is also unable to identify his inertial frame. There is no clever physical experiment he may do in order to determine e.g. which direction is up or down, or whether the laboratory is at rest or is moving with constant velocity. In particular, such information can not be obtained by any measurement of the speed of light.

Ideally, inertial frames can exist only far away from gravitating masses. Nevertheles, special relativity is a very good approximation to reality for example in a physics laboratory on the Earth, and on the atomic scale it may be applicable even in the extremely strong gravitational field of a neutron star.

8.3 The Minkowski metric

In an inertial coordinate system we may replace the time coordinate t by a coordinate $x^0 = ct$ having the dimension of length. The Euclidean spatial coordinates we call $x^1 = x$, $x^2 = y$ and $x^3 = z$, and we write $x = x^\mu = (x^0, x^1, x^2, x^3)$. That the speed of light is c, means that the equation of motion for a narrow beam of light has the form

$$c^2\,(\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2 = 0\,. \tag{8.4}$$

This equation says that $\mathrm{d}s = 0$, when we introduce the constant metric tensor

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{8.5}$$

and the line element

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = (\mathrm{d}x^0)^2 - (\mathrm{d}x^1)^2 - (\mathrm{d}x^2)^2 - (\mathrm{d}x^0)^2\ . \tag{8.6}$$

The notation here is slightly unfortunate, since an upper index may be either a power or a coordinate index. Another possible source of confusion is that we write formally the line element $\mathrm{d}s^2 = (\mathrm{d}s)^2$ as a square, even though it may well become negative.

The idea of regarding time and space as a four dimensional "spacetime" with such a metric, goes back to Herman Minkowski, in 1908. The transition from the special to the general theory of relativity consists in allowing the components $g_{\mu\nu}$ of the metric to vary in space and time.

The fact that the metric tensor $g_{\mu\nu}$ is constant in the special theory of relativity, means that the connection coefficients vanish, according to Equation (5.49). Hence covariant differentiation is identical to ordinary partial differentiation, and there is no need to distinguish between them.

Another observer, using other coordinates $\widetilde{x} = (\widetilde{x}^0, \widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3) = (\widetilde{ct}, \widetilde{x}, \widetilde{y}, \widetilde{z})$, will describe the same light signal by the same equation $\mathrm{d}s = 0$. In the new coordinates we have that

$$\mathrm{d}s^2 = \widetilde{g}_{\mu\nu}\,\mathrm{d}\widetilde{x}^\mu\,\mathrm{d}\widetilde{x}^\nu = \widetilde{g}_{\mu\nu}\,\frac{\partial\widetilde{x}^\mu}{\partial x^\rho}\,\frac{\partial\widetilde{x}^\nu}{\partial x^\sigma}\,\mathrm{d}x^\rho\,\mathrm{d}x^\sigma\ , \tag{8.7}$$

and comparison with Equation (8.6) shows that

$$\widetilde{g}_{\mu\nu}\,\frac{\partial\widetilde{x}^\mu}{\partial x^\rho}\,\frac{\partial\widetilde{x}^\nu}{\partial x^\sigma} = g_{\rho\sigma}\ . \tag{8.8}$$

That the speed of light c is the same for all observers, means that the equation $\mathrm{d}s = 0$ may also be written as

$$c^2\,(\mathrm{d}\widetilde{t})^2 - (\mathrm{d}\widetilde{x})^2 - (\mathrm{d}\widetilde{y})^2 - (\mathrm{d}\widetilde{z})^2 = 0\ . \tag{8.9}$$

It follows that

$$\widetilde{g}_{\mu\nu} = \lambda\,\eta_{\mu\nu} = \lambda\,g_{\mu\nu}\ , \tag{8.10}$$

where $\lambda = \lambda(x) = \lambda(x^0, x^1, x^2, x^3)$ is a scale factor which may in principle depend on time and place. Coordinate transformations having the property that the transformed metric is $\widetilde{g}_{\mu\nu} = \lambda\,g_{\mu\nu}$, are called *conformal transformations*. They are interesting for example because they leave Maxwell's equations invariant. Indeed, light is electromagnetic waves, and Maxwell's equations are the field equations of electromagnetism.

However, we will not study here the most general class of conformal transformations. In the special theory of relativity we restrict ourselves to the special case $\lambda = 1$, which means that the allowed coordinate transformations leave the metric invariant,

$$\widetilde{g}_{\mu\nu} = g_{\mu\nu}\ , \tag{8.11}$$

or, by Equation (8.8),

$$g_{\mu\nu}\,\frac{\partial\widetilde{x}^\mu}{\partial x^\rho}\,\frac{\partial\widetilde{x}^\nu}{\partial x^\sigma} = g_{\rho\sigma}\ . \tag{8.12}$$

8.4 The Poincaré group

Coordinate transformations satisfying Equation (8.12), with the constant Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, are called *Poincaré transformations*. It is easy to show that they have to be linear. In fact, differentiation of (8.12) gives directly that

$$g_{\mu\nu}\left(\frac{\partial^2\widetilde{x}^\mu}{\partial x^\lambda \partial x^\rho}\,\frac{\partial\widetilde{x}^\nu}{\partial x^\sigma} + \frac{\partial\widetilde{x}^\mu}{\partial x^\rho}\,\frac{\partial^2\widetilde{x}^\nu}{\partial x^\lambda \partial x^\sigma}\right) = 0\ , \tag{8.13}$$

since $g_{\mu\nu}$ is assumed to be constant. The same equation may be written in the following two ways, with the free indices λ, ρ, σ interchanged,

$$\begin{aligned} g_{\mu\nu}\left(\frac{\partial^2\widetilde{x}^\mu}{\partial x^\rho \partial x^\lambda}\,\frac{\partial\widetilde{x}^\nu}{\partial x^\sigma} + \frac{\partial\widetilde{x}^\mu}{\partial x^\lambda}\,\frac{\partial^2\widetilde{x}^\nu}{\partial x^\rho \partial x^\sigma}\right) &= 0\ , \\ g_{\mu\nu}\left(\frac{\partial^2\widetilde{x}^\mu}{\partial x^\sigma \partial x^\rho}\,\frac{\partial\widetilde{x}^\nu}{\partial x^\lambda} + \frac{\partial\widetilde{x}^\mu}{\partial x^\rho}\,\frac{\partial^2\widetilde{x}^\nu}{\partial x^\sigma \partial x^\lambda}\right) &= 0\ . \end{aligned} \tag{8.14}$$

Adding the last two equatione and subtracting the first one, we get that

$$2g_{\mu\nu}\,\frac{\partial\widetilde{x}^\mu}{\partial x^\lambda}\,\frac{\partial^2\widetilde{x}^\nu}{\partial x^\rho \partial x^\sigma} = 0\ . \tag{8.15}$$

Since the matrices $g_{\mu\nu}$ and $\partial\widetilde{x}^\mu/\partial x^\lambda$ must both be nonsingular, it follows further that

$$\frac{\partial^2\widetilde{x}^\nu}{\partial x^\rho \partial x^\sigma} = 0\ . \tag{8.16}$$

Thus, the most general Poincaré transformation has the form

$$\boxed{x^\mu \mapsto \widetilde{x}^\mu = d^\mu + \Lambda^\mu{}_\nu x^\nu\ ,} \tag{8.17}$$

or in a slightly more compact notation, $x \mapsto \widetilde{x} = d + \Lambda x$. It is made up of a *Lorentz transformation* $x \mapsto \Lambda x$, where Λ is a constant 4×4-matrix,

$$\Lambda = \Lambda^\mu{}_\nu = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix}, \tag{8.18}$$

followed by a *translation* $x \mapsto d + x$, where $d = d^\mu = (d^0, d^1, d^2, d^3)$ is a constant vector.

The Poincaré transformations form a group, the Poincaré group, also known as the inhomogeneous Lorentz group. It is an example of a Lie group. Two subgroups are the translation group $\mathbf{R}^4$, consisting of all translations in time and space, and the homogeneous Lorentz group $\mathrm{O}(1,3)$, often called simply the Lorentz group, consisting of all Lorentz transformations. Together these two subgroups generate the whole Poincaré group. A third subgroup is the rotation group, $\mathrm{O}(3)$, which is a subgroup of the Lorentz group, consisting of all rotations in three dimensional space. See Appendix B for a very brief introduction to groups and Lie groups in general.

The condition in Equation (8.12) is that

$$\boxed{g_{\kappa\lambda}\Lambda^{\kappa}{}_{\mu}\Lambda^{\lambda}{}_{\nu} = g_{\mu\nu}\ ,} \tag{8.19}$$

or if written as a matrix equation,

$$\Lambda^{\mathrm{T}} g \Lambda = g\ . \tag{8.20}$$

Here Λ^{T} is the transposed of the matrix Λ,

$$\Lambda^{\mathrm{T}} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^1{}_0 & \Lambda^2{}_0 & \Lambda^3{}_0 \\ \Lambda^0{}_1 & \Lambda^1{}_1 & \Lambda^2{}_1 & \Lambda^3{}_1 \\ \Lambda^0{}_2 & \Lambda^1{}_2 & \Lambda^2{}_2 & \Lambda^3{}_2 \\ \Lambda^0{}_3 & \Lambda^1{}_3 & \Lambda^2{}_3 & \Lambda^3{}_3 \end{pmatrix}\ . \tag{8.21}$$

Equation (8.20) may be written in the form $g^{-1}\Lambda^{\mathrm{T}} g\Lambda = I =$ the identity matrix. Equivalently, $\Lambda^{-1} = g^{-1}\Lambda^{\mathrm{T}} g$, or explicitly with indices,

$$(\Lambda^{-1})^{\mu}{}_{\nu} = g^{\mu\rho}\Lambda^{\sigma}{}_{\rho} g_{\sigma\nu} = \Lambda_{\nu}{}^{\mu}\ . \tag{8.22}$$

Taking the determinant of Equation (8.20) we find that

$$(\det \Lambda^{\mathrm{T}})(\det g)(\det \Lambda) = \det g\ . \tag{8.23}$$

Since $\det \Lambda^{\mathrm{T}} = \det \Lambda$, this shows that Lorentz transformations preserve volume,

$$\boxed{\det \Lambda = \pm 1\ .} \tag{8.24}$$

When the metric is constant, independent of time and place, then the infinitesimal line element $\mathrm{d}s = \sqrt{|\mathrm{d}s^2|}$ may be integrated along straight lines, so as to define explicitly finite and not only infinitesimal distances. The distance between two arbitrary points $x = x^{\mu} = (x^0, x^1, x^2, x^3)$ and $y = y^{\mu} = (y^0, y^1, y^2, y^3)$ is the relativistic "length" $s = \sqrt{|s^2|}$ of a straight line joining them, where

$$s^2 = g_{\mu\nu}(x^{\mu} - y^{\mu})(x^{\nu} - y^{\nu})\ . \tag{8.25}$$

Note that s^2 may be positive, negative or zero. We say that the separation of the two points is *timelike* if $s^2 > 0$, *spacelike* if $s^2 < 0$, and *lightlike* if $s^2 = 0$.

An equivalent definition of a Poincaré transformation is that it preserves the four dimensional distance between arbitrary points $x = x^{\mu}$ and $y = y^{\mu}$, i.e. that

$$g_{\mu\nu}(\widetilde{x}^{\mu} - \widetilde{y}^{\mu})(\widetilde{x}^{\nu} - \widetilde{y}^{\nu}) = g_{\mu\nu}(x^{\mu} - y^{\mu})(x^{\nu} - y^{\nu})\ . \tag{8.26}$$

Connected components of the Poincaré group

Translations are rather trivial transformations, and the interesting part of the Poincaré group is the Lorentz group.

Four special Lorentz transformations are the identity transformation I, space inversion P ("P" for "parity"), time reversal T, and spacetime inversion $PT = TP = -I$. P and T are defined by the matrices

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8.27}$$

Both P, T and PT are discrete transformations, this means that none of them can be generated as the end result of a series of small Lorentz transformations.

The transformations I, P, T and PT define four separate *connected components* of the Lorentz-group. This means that every Lorentz transformation may be deformed continuously into exactly one of the four transformations, in such a way that every one of the continuous sequence of intermediate transformations is a Lorentz transformation. Note that

$$\det I = \det(-I) = 1 \,, \qquad \det P = \det T = -1 \,. \tag{8.28}$$

These facts alone are enough to prove that e.g. I and P can not be continuously deformed into each other via intermediate Lorentz transformations. In fact, assume that $\Lambda(u)$ depends continuously on the parameter u, that every $\Lambda(u)$ is a Lorentz transformation, and that $\Lambda(0) = I$. Then we must have $\det(\Lambda(u)) = 1$ for every value of u, since the only alternative is that $\det(\Lambda(u)) = -1$ for some values of u, and that is excluded because $\det(\Lambda(u))$ is a continuous function of u.

The connection component of the Lorentz group containing the identity transformation I, but not the discrete transformations P, T and PT, is itself a group. It is a subgroup of the full Lorentz group, called the *proper* Lorentz group. It is characterized by the two conditions that

$$\det \Lambda = +1 \qquad \text{and} \qquad \Lambda^0{}_0 > 0 \,. \tag{8.29}$$

8.5 Continuous Lorentz transformations

The proper Lorentz group, not containing the discrete transformations P, T and PT, contains every infinitesimal Lorentz transformation, of the form $\Lambda^\mu{}_\nu = \delta^\mu_\nu + \omega^\mu{}_\nu$, i.e.

$$\Lambda = I + \omega = \begin{pmatrix} 1+\omega^0{}_0 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3 \\ \omega^1{}_0 & 1+\omega^1{}_1 & \omega^1{}_2 & \omega^1{}_3 \\ \omega^2{}_0 & \omega^2{}_1 & 1+\omega^2{}_2 & \omega^2{}_3 \\ \omega^3{}_0 & \omega^3{}_1 & \omega^3{}_2 & 1+\omega^3{}_3 \end{pmatrix}, \tag{8.30}$$

where ω er infinitesimal. Such a transformation is infinitesimally close to the identity I, and it must have determinant 1, since the determinant must be infinitesimally close to 1, and ± 1 are the only existing possibilities for a Lorentz-transformation. The determinant is, to first order in ω,

$$\det \Lambda = 1 + \omega^\mu{}_\mu \,. \tag{8.31}$$

Thus $\omega^{\mu}{}_{\mu} = 0$, the trace of ω must vanish.

Equation (8.19) gives the following condition on ω, neglecting terms of order ω^2,

$$g_{\kappa\nu}\omega^{\kappa}{}_{\mu} + g_{\mu\lambda}\omega^{\lambda}{}_{\nu} = 0 \; . \tag{8.32}$$

In other words, ω must be antisymmetric,

$$\boxed{\omega_{\mu\nu} = -\omega_{\nu\mu} \; ,} \tag{8.33}$$

when we define $\omega_{\mu\nu} = g_{\mu\kappa}\omega^{\kappa}{}_{\nu}$. From the antisymmetry follows that

$$\omega^{\mu}{}_{\nu} = \omega_{\rho\sigma}g^{\mu\rho}\delta^{\sigma}_{\nu} = \frac{1}{2}\,\omega_{\rho\sigma}\left(g^{\mu\rho}\delta^{\sigma}_{\nu} - g^{\mu\sigma}\delta^{\rho}_{\nu}\right) \; , \tag{8.34}$$

and once more the vanishing of the trace, $\omega^{\mu}{}_{\mu} = 0$.

The most general form ω can have, is

$$\omega = \omega^{\mu}{}_{\nu} = \begin{pmatrix} 0 & \chi_x & \chi_y & \chi_z \\ \chi_x & 0 & -\alpha_z & \alpha_y \\ \chi_y & \alpha_z & 0 & -\alpha_x \\ \chi_z & -\alpha_y & \alpha_x & 0 \end{pmatrix} = \boldsymbol{\alpha}\cdot\boldsymbol{J} + \boldsymbol{\chi}\cdot\boldsymbol{K} \; , \tag{8.35}$$

where $\boldsymbol{\alpha} = \alpha_x\,\mathbf{i} + \alpha_y\,\mathbf{j} + \alpha_z\,\mathbf{k}$ and $\boldsymbol{\chi} = \chi_x\,\mathbf{i} + \chi_y\,\mathbf{j} + \chi_z\,\mathbf{k}$ are three dimensional vectors. Likewise, $\boldsymbol{J}$ and $\boldsymbol{K}$ are vectors, with components that are 4×4-matrices,

$$J_x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad J_y = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad J_z = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{8.36}$$

$$K_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{8.37}$$

That ω is infinitesimal, means that $\boldsymbol{\alpha}$ and $\boldsymbol{\chi}$ are infinitesimal. According to Appendix B we generalize from infinitesimal to finite transformations by means of the exponential function. Thus, with finite values for the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\chi}$, the matrix

$$\Lambda = \mathrm{e}^{\boldsymbol{\alpha}\cdot\boldsymbol{J}+\boldsymbol{\chi}\cdot\boldsymbol{K}} = I + \boldsymbol{\alpha}\cdot\boldsymbol{J} + \boldsymbol{\chi}\cdot\boldsymbol{K} + \frac{1}{2}\left(\boldsymbol{\alpha}\cdot\boldsymbol{J} + \boldsymbol{\chi}\cdot\boldsymbol{K}\right)^2 + \ldots \tag{8.38}$$

is a Lorentz transformation. Equation (B.29) gives that

$$\det\left(\mathrm{e}^{\boldsymbol{\alpha}\cdot\boldsymbol{J}+\boldsymbol{\chi}\cdot\boldsymbol{K}}\right) = \mathrm{e}^{\mathrm{Tr}(\boldsymbol{\alpha}\cdot\boldsymbol{J}+\boldsymbol{\chi}\cdot\boldsymbol{K})} = 1 \; . \tag{8.39}$$

A Lorentz transformation which is such an exponential function, may be continuously connected to the identity I, by a continuous variation of the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\chi}$. The most general Lorentz transformation which can be continuously connected to the identity may be written as a product of exponential functions.

Let us examine some more specific examples.

Example: rotation about the x axis

Take as an example $\alpha_x = \alpha$ and $\alpha_y = \alpha_z = 0$. Since

$$ {J_x}^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad {J_x}^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = -J_x \ , \tag{8.40} $$

we have then that

$$ \begin{aligned} \Lambda \ &= \ \mathrm{e}^{\alpha J_x} = I + \alpha J_x + \frac{1}{2}\,\alpha^2 {J_x}^2 - \frac{1}{6}\,\alpha^3 J_x - \frac{1}{24}\,\alpha^4 {J_x}^2 + \ldots \\ &= \ I + \sin\alpha\, J_x + (1-\cos\alpha)\, {J_x}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\alpha & -\sin\alpha \\ 0 & 0 & \sin\alpha & \cos\alpha \end{pmatrix}. \end{aligned} \tag{8.41} $$

This matrix represents a rotation by an angle α about the x axis. We may choose to interpret it either as an active or as a passive transformation, like in Figure 1.2 or in Figure 1.4. Rotation angles about a fixed axis are additive, for example, two rotations about the x axis, first by an angle α_1 and next by an angle α_2, will together produce a rotation about the x axis by an angle $\alpha_2 + \alpha_1$,

$$ \mathrm{e}^{\alpha_2 J_x}\, \mathrm{e}^{\alpha_1 J_x} = \mathrm{e}^{(\alpha_2+\alpha_1) J_x} \ . \tag{8.42} $$

Example: a pure Lorentz transformation along the x axis

Take as another example $\chi_x = \chi$, $\chi_y = \chi_z = 0$. Since

$$ {K_x}^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad {K_x}^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = K_x \ , \tag{8.43} $$

we have then that

$$ \begin{aligned} \Lambda \ &= \ \mathrm{e}^{\chi K_x} = I + \chi K_x + \frac{1}{2}\,\chi^2 {K_x}^2 + \frac{1}{6}\,\chi^3 K_x + \frac{1}{24}\,\chi^4 {K_x}^2 + \ldots \\ &= \ I + \sinh\chi\, K_x + (\cosh\chi - 1)\, {K_x}^2 = \begin{pmatrix} \cosh\chi & \sinh\chi & 0 & 0 \\ \sinh\chi & \cosh\chi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{8.44} $$

This matrix represents a pure Lorentz transformation along the x axis. It transforms the coordinates $x^0 = ct$ and $x^1 = x$, but not $x^2 = y$ and $x^3 = z$. Figure 8.1 shows how it can be interpreted as a passive transformation, i.e. as a relation between two different coordinate systems in spacetime.

The physical interpretation of this passive transformation is that the transformed coordinates describe the world as seen by an observer moving with a constant velocity relative to the original coordinate system. We may see that by setting $\widetilde{x} =$ constant, $\widetilde{y} =$ constant and

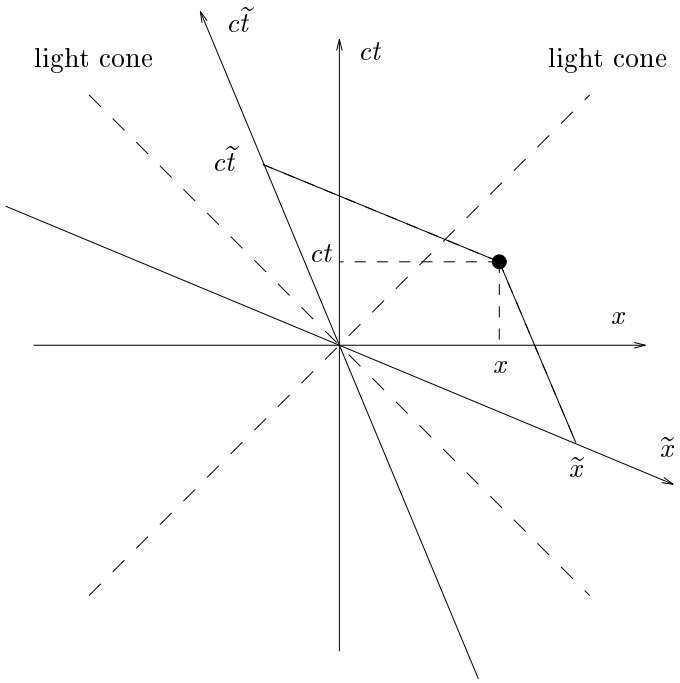

Figure 8.1: The passive interpretation of a pure Lorentz transformation, Equation (8.44) with $\chi > 0$.

$\widetilde{z} =$ constant, these are in fact the equations of motion for a particle at rest in the transformed coordinate system. That

$$\widetilde{x} = (\cosh\chi)x + (\sinh\chi)ct = \text{constant} \ , \tag{8.45}$$

means that

$$x = \frac{\widetilde{x}}{\cosh\chi} - (\tanh\chi)ct = \text{constant} - vt \ , \tag{8.46}$$

with $v = \beta c = (\tanh\chi)c$. Thus, the velocity of the observer is $-v$ along the x axis.

Alternatively, we may interpret the Lorentz transformation as an active transformation, which does not change the coordinate system, but which moves a particle from the place (x, y, z) at time t to another place $(\widetilde{x}, \widetilde{y}, \widetilde{z})$ at another time $\widetilde{t}$. For example, a particle lying at rest originally, will be transformed so that

$$\widetilde{x} = (\cosh\chi)x + (\sinh\chi)ct \ , \qquad c\widetilde{t} = (\cosh\chi)ct + (\sinh\chi)x \ . \tag{8.47}$$

Or equivalently,

$$\widetilde{x} = \frac{x}{\cosh^2\chi} + (\tanh\chi)c\widetilde{t} = \text{constant} + v\widetilde{t} \ , \tag{8.48}$$

with $v = \beta c = (\tanh\chi)c$, as above. In this example, the active Lorentz transformation of a particle at rest will give the particle a velocity v in the x direction.

The parameter χ is called *rapidity*. It is additive, in the same way as the rotation angle about a fixed axis. As we have seen, the relation between the rapidity χ and the velocity $v = \beta c$ is that

$$\beta = \tanh\chi = \frac{\sinh\chi}{\cosh\chi} = \frac{\mathrm{e}^{\chi} - \mathrm{e}^{-\chi}}{\mathrm{e}^{\chi} + \mathrm{e}^{-\chi}} \,, \tag{8.49}$$

which is the same as

$$\mathrm{e}^{\chi} = \sqrt{\frac{1+\beta}{1-\beta}} \,. \tag{8.50}$$

By defining

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\chi \,, \tag{8.51}$$

we may write

$$\Lambda = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} . \tag{8.52}$$

The Lie algebra of the Lorentz group

The group multiplication in the proper Lorentz group, which is the connection component of the Lorentz group containing the identity transformation I but not the discrete transformations P, T and PT, is given by the Lie algebra. In other words, the composition of proper Lorentz transformations is given by the commutation relations of the generators $\boldsymbol{J} = (J_x, J_y, J_z)$ and $\boldsymbol{K} = (K_x, K_y, K_z)$,

$$\begin{aligned} [J_x, J_y] &= -[K_x, K_y] = J_z \,, \qquad [J_x, K_y] = [K_x, J_y] = K_z \,, \\ [J_y, J_z] &= -[K_y, K_z] = J_x \,, \qquad [J_y, K_z] = [K_y, J_z] = K_x \,, \\ [J_z, J_x] &= -[K_z, K_x] = J_y \,, \qquad [J_z, K_x] = [K_z, J_x] = K_y \,, \end{aligned} \tag{8.53}$$

$$[J_x, K_x] = [J_y, K_y] = [J_z, K_z] = 0 \,. \tag{8.54}$$

The same commutation relations may be expressed more compactly in terms of the complex matrices

$$M_k^{\pm} = \frac{1}{2}\left(J_k \pm \mathrm{i}K_k\right) . \tag{8.55}$$

We see that $[M_j^+, M_k^-] = 0$ for $j, k = x, y, z$, and that

$$\begin{aligned} [M_x^+, M_y^+] &= M_z^+ \,, \qquad [M_y^+, M_z^+] = M_x^+ \,, \qquad [M_z^+, M_x^+] = M_y^+ \,, \\ [M_x^-, M_y^-] &= M_z^- \,, \qquad [M_y^-, M_z^-] = M_x^- \,, \qquad [M_z^-, M_x^-] = M_y^- \,. \end{aligned} \tag{8.56}$$

The full Lorentz group

In order to complete the multiplication table of the Lorentz group, we have to include the following relations involving the discrete transformations P, T and $PT = TP$,

$$P^2 = T^2 = (PT)^2 = I \,, \tag{8.57}$$

and

$$P\boldsymbol{J} = \boldsymbol{J}P \,, \qquad T\boldsymbol{J} = \boldsymbol{J}T \,, \qquad P\boldsymbol{K} = -\boldsymbol{K}P \,, \qquad T\boldsymbol{K} = -\boldsymbol{K}T \,. \tag{8.58}$$

The last relations imply e.g. that

$$P\,(\boldsymbol{\alpha}\cdot\boldsymbol{J} + \boldsymbol{\chi}\cdot\boldsymbol{K})^2 = (\boldsymbol{\alpha}\cdot\boldsymbol{J} - \boldsymbol{\chi}\cdot\boldsymbol{K})\,P\,(\boldsymbol{\alpha}\cdot\boldsymbol{J} + \boldsymbol{\chi}\cdot\boldsymbol{K}) = (\boldsymbol{\alpha}\cdot\boldsymbol{J} - \boldsymbol{\chi}\cdot\boldsymbol{K})^2\,P \,. \tag{8.59}$$

More generally, $P\,(\boldsymbol{\alpha}\cdot\boldsymbol{J} + \boldsymbol{\chi}\cdot\boldsymbol{K})^n = (\boldsymbol{\alpha}\cdot\boldsymbol{J} - \boldsymbol{\chi}\cdot\boldsymbol{K})^n\,P$ when $n \geq 0$, and

$$P\,\mathrm{e}^{\boldsymbol{\alpha}\cdot\boldsymbol{J}+\boldsymbol{\chi}\cdot\boldsymbol{K}} = \mathrm{e}^{\boldsymbol{\alpha}\cdot\boldsymbol{J}-\boldsymbol{\chi}\cdot\boldsymbol{K}}\,P \,, \qquad T\,\mathrm{e}^{\boldsymbol{\alpha}\cdot\boldsymbol{J}+\boldsymbol{\chi}\cdot\boldsymbol{K}} = \mathrm{e}^{\boldsymbol{\alpha}\cdot\boldsymbol{J}-\boldsymbol{\chi}\cdot\boldsymbol{K}}\,T \,. \tag{8.60}$$

8.6 The law of cosmic laziness

A point particle not influenced by external forces will move in a straight line with constant velocity. This is a basic postulate of the special theory of relativity, and it serves both as a defintion of the inertial frames that are the privileged coordinate systems, and as a law of motion.

A remarkable property of the inertial motion of a particle, according to the special theory of relativity, is that it maximizes the *proper time* τ, which is the time measured by a clock following the motion. This property may be used as a variational principle determining the equation of motion. Bertrand Russell gave this principle a highly appropriate name, calling it the *law of cosmic laziness*.

The effect that the proper time τ differs from the coordinate time t measured in an inertial system, is called *time dilatation*. The relation between τ and t is that

$$\mathrm{d}\tau = \mathrm{d}t\,\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} \,, \tag{8.61}$$

for a particle moving with velocity

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \,. \tag{8.62}$$

The time dilatation is experimentally verified both with an atomic clock that was flown around the Earth, and by comparison of the lifetimes of unstable particles moving with different velocities, from zero velocity up to velocities extremely close to the speed of light.

Instead of the proper time τ we may introduce the "relativistic length" $s = c\tau$, defined in general by the relation

$$\mathrm{d}s^2 = c^2\,\mathrm{d}\tau^2 = c^2\,\mathrm{d}t^2 - |\mathrm{d}\boldsymbol{r}|^2 = g_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu \,. \tag{8.63}$$

When a particle follows an orbit $\boldsymbol{r} = \boldsymbol{r}(t)$ from A to B, that is, from a point $\boldsymbol{r}_A$ at time t_A, to a point $\boldsymbol{r}_B$ at time t_B, then the relativistic length of the orbit is

$$s_{AB} = c\tau_{AB} = c\int_A^B \mathrm{d}\tau = c\int_A^B \mathrm{d}t\,\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} = \int_A^B \mathrm{d}t\,\sqrt{g_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\,. \tag{8.64}$$

Note that the length s_{AB} is *reparametrization invariant*, it does not depend on whether we parametrize the orbit by the coordinate time t or by any other parameter $u = u(t)$. In fact, a change of integration variable gives that

$$s_{AB} = \int_A^B \mathrm{d}t\,\sqrt{g_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}t}} = \int_A^B \mathrm{d}u\,\sqrt{g_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}u}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}u}}\,. \tag{8.65}$$

In particular, if the particle moves with constant velocity from A to B, then the relativistic length of the straight line in spacetime is

$$s_{AB} = \sqrt{c^2(t_B - t_A)^2 - (\boldsymbol{r}_B - \boldsymbol{r}_A)^2}\,. \tag{8.66}$$

We assume of course that the expression under the square root is nonnegative, i.e. that the velocity of the particle is not greater than the speed of light.

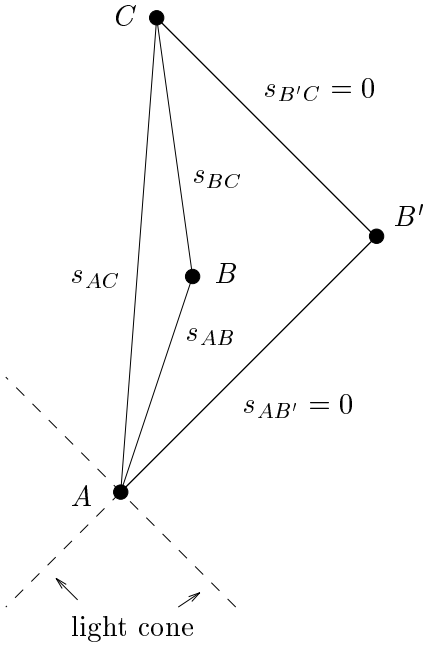

Figure 8.2: The reverse triangle inequality $s_{AC} > s_{AB} + s_{BC}$. That the length s_{AC} of the straight line is maximal, not minimal, should be obvious, since $s_{AB'} = s_{B'C} = 0$.

Assume now that the particle moves first with constant velocity from A to B, and after that with another constant velocity from B to C. We may then show the reverse triangle inequality

$$s_{AC} > s_{AB} + s_{BC}\,. \tag{8.67}$$

See Figure 8.2. The inversion of the inequality sign compared to what we are used to from Euclidean geometry is due to the minus signs under the square root sign in Equation (8.66). An extreme example is the triangle $AB'C$ in Figure 8.2, where two sides have zero length, because they represent light signals.

The simplest way to prove Equation (8.67) is to use the Lorentz invariance of the relativistic length s, implying that it is sufficient to prove the special case $\boldsymbol{r}_A = \boldsymbol{r}_C \neq \boldsymbol{r}_B$. In this case we see directly that

$$s_{AC} = c(t_C - t_A) = c(t_B - t_A) + c(t_C - t_B) > s_{AB} + s_{BC} \ . \tag{8.68}$$

We may use the reverse triangle inequality for an arbitrary orbit, which we divide into small or infinitesimal pieces. We realize that by straightening out the orbit we will increase its length, which is c times the proper time along the orbit. Thus, the general conclusion is that when a particle moves in a straight line with constant velocity, this is the motion that makes the proper time τ maximal.

The twin paradox

The famous so called twin paradox is nothing but the reverse triangle inequality in terms of the proper time $\tau = s/c$. The effect is that an astronaut travelling far out in space with a high velocity, then turning around and returning, will be younger on his return than his twin who remained stationary. What may seem paradoxical is the asymmetry of the effect. Both twins see the other one moving away and coming back, but when they meet again, one will definitely be older and the other one younger. There should be no paradox here, once we realize that the one who will be oldest, is the one who has followed a straight line, also called a *geodesic path*, in spacetime. The twin paradox is really no more paradoxical than the fact that a straight line in Euclidean space is shorter than any other curve joining two given points.

8.7 Lorentz contraction

Two observers A and B moving with a constant velocity $v = \beta c$ relative to each other may choose their x, y and z axes such that the transformation between the two coordinate systems is a pure Lorentz transformation in the x direction, as in Equation (8.52).

Imagine now a measuring rod of length L lying at rest relative to the observer B and moving relative to A with a velocity $v = \beta c$ in the negative x direction. We ask how long the observer A will measure the rod to be.

Since the transversal coordinates y and z are unchanged by the transformation, we predict that A will measure the same length L if the rod is oriented in the transversal direction. This conclusion is also the only one possible, based on the principle of relativity, that there should be a complete symmetry between the two observers.

In fact, A may in principle measure the length of the moving rod by arranging for the two ends of the rod to make scratch marks at the moment when the rod passes by. But in the same way, the stationary measuring rod used by A, of length L and transversely oriented, may in principle produce scratch marks on the moving rod. By such experiments the two observers will observe the same unambiguous fact, either that the two rods are equally long, or that one of them is shorter than the other one. But the rod which is moving relative to

observer A is stationary relative to B, and vice versa. Hence the only result consistent with the symmetry between the two observers, is that the measured lengths are equal.

If the measuring rod moving relative to A is oriented in the x direction, parallel to the velocity, then the situation is different. The observer B sees the two ends of the rod lying at rest, with constant $\widetilde{x}$ coordinates $\widetilde{x}_a$ and $\widetilde{x}_b$. According to the Lorentz transformation,

$$\widetilde{x}_a = \gamma(x_a + vt_a) \ , \qquad \widetilde{x}_b = \gamma(x_b + vt_b) \ , \tag{8.69}$$

with the obvious notation. Thus, if the observer A looks where the two ends are located at one given time $t_a = t_b$, then we have

$$L = \widetilde{x}_b - \widetilde{x}_a = \gamma(x_b - x_a + v(t_b - t_a)) = \gamma(x_b - x_a) \ . \tag{8.70}$$

Thus, A observes a distance between the two end points which is

$$L_A = x_b - x_a = \frac{L}{\gamma} = L\sqrt{1 - \frac{v^2}{c^2}} \ . \tag{8.71}$$

Using this particular measuring procedure, A finds a length shorter than L. This contraction of the length of a moving object in the direction of motion is called *Lorentz contraction*. That it is a real physical effect, like the time dilatation, and not just an illusion, is illustrated by the thought experiment described in Problem 8.

8.8 Addition of velocities, constant acceleration

When adding velocities in the theory of relativity, it seems that we have to modify the standard rules for addition. In particular, whatever velocity we add to that of a light signal, the sum must always be the velocity of light. The light from a lamp moves with the velocity c relative to the lamp, but even if the lamp moves with velocity v in the same direction as it is shining, the velocity of the light is still c and not $c + v$. Thus, for velocities in a fixed direction, the unorthodox rule $c + v = c$ must hold. The speed of light is the ultimate speed limit, it can never be exceeded no matter how much we add velocities.

The general formula for the addition of velocities in the x direction is derived from the Lorentz transformation. Assume that A moves with velocity u relative to B, and that B moves with velocity v relative to C. Then B sees A moving a distance $\Delta x = u\Delta t$ during a small time interval Δt, whereas C observes A moving a distance

$$\Delta\widetilde{x} = \gamma(\Delta x + \beta c\Delta t) \tag{8.72}$$

during the time interval

$$\Delta\widetilde{t} = \gamma\left(\Delta t + \beta\,\frac{\Delta x}{c}\right). \tag{8.73}$$

Here we have

$$\beta = \frac{v}{c} \ , \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \ . \tag{8.74}$$

Thus, C sees A moving with the velocity

$$w = \frac{\Delta\widetilde{x}}{\Delta\widetilde{t}} = \frac{\Delta x + \beta c \Delta t}{\Delta t + \beta \frac{\Delta x}{c}} = \frac{u+v}{1+\frac{uv}{c^2}} \,. \tag{8.75}$$

It follows from this formula that the rapidity is additive, instead of the velocity. Introducing the rapidities χ_1 and χ_2 such that $u = c\tanh\chi_1$ and $v = c\tanh\chi_2$, we get that

$$w = \frac{u+v}{1+\frac{uv}{c^2}} = \frac{c(\tanh\chi_1 + \tanh\chi_2)}{1+(\tanh\chi_1)(\tanh\chi_2)} = c\tanh(\chi_1+\chi_2) \,. \tag{8.76}$$

We may use the velocity addition formula to find the motion of a particle accelerating with a constant acceleration a. The reasonable way to define constant acceleration is that the particle itself at any time feels an acceleration equal to a constant value a. Thus, the acceleration should be a in the instantaneous rest system, which is an inertial reference system in which the particle lies at rest at the given moment.

Let τ denote the proper time of the particle, defined such that the particle is at rest at the proper time $\tau = 0$. As seen from the instantaneous rest system, the particle increases its velocity from zero to $a\,\mathrm{d}\tau$ during the infinitesimal time interval $\mathrm{d}\tau$. Thereby it jumps of course into another rest system, but let us stay in the same inertial frame. As seen from the reference system in which the particle was initially at rest, at $\tau = 0$, the velocity increases in the proper time interval $\mathrm{d}\tau$ from v to

$$v + \mathrm{d}v = \frac{v + a\,\mathrm{d}\tau}{1+\frac{va\,\mathrm{d}\tau}{c^2}} = (v + a\,\mathrm{d}\tau)\left(1 - \frac{va\,\mathrm{d}\tau}{c^2}\right) = v + a\,\mathrm{d}\tau\left(1-\frac{v^2}{c^2}\right). \tag{8.77}$$

We may write this equation as

$$a\,\mathrm{d}\tau = \frac{c^2\,\mathrm{d}v}{c^2-v^2} = \frac{\mathrm{d}v}{2}\left(\frac{c}{c+v} + \frac{c}{c-v}\right). \tag{8.78}$$

Integration gives that

$$a\tau = c\ln\sqrt{\frac{c+v}{c-v}} \,. \tag{8.79}$$

Or equivalently,

$$v = c\tanh\left(\frac{a\tau}{c}\right). \tag{8.80}$$

We see that constant acceleration in the theory of relativity means that the rapidity $\chi = \mathrm{arctanh}(v/c)$ increases linearly with the proper time τ.

The relation between the proper time τ and the time t in a fixed reference system is that

$$\mathrm{d}t = \frac{\mathrm{d}\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \mathrm{d}\tau\cosh\left(\frac{a\tau}{c}\right) \,. \tag{8.81}$$

Defining $t = 0$ at $\tau = 0$, we get that

$$t = \frac{c}{a}\sinh\left(\frac{a\tau}{c}\right) \,, \tag{8.82}$$

and hence,

$$v = \frac{at}{\sqrt{1 + \frac{(at)^2}{c^2}}} \ . \tag{8.83}$$

Let us choose the coordinate system such that $x = 0$ at $t = 0$, then integration of the equation $\mathrm{d}x/\mathrm{d}t = v$ gives that

$$x = \frac{c^2}{a}\left(\sqrt{1 + \frac{(at)^2}{c^2}} - 1\right) = \frac{c^2}{a}\left(\cosh\left(\frac{a\tau}{c}\right) - 1\right) \ . \tag{8.84}$$

Note that with $a > 0$, the particle moves in such a way that the following inequality always holds,

$$x > ct - \frac{c^2}{a} \ . \tag{8.85}$$

Even though it is impossible for anything other than light (plus gravitational waves and maybe neutrinos) to reach the speed of light, it is theoretically possible to outrun a light signal, if one can get a sufficient head start and run with a constant acceleration.

One consequence of this startling fact is that a runner with constant acceleration $a > 0$ along the x axis runs in front of a *horizon* moving with the speed of light. If the runner starts at $x = 0$ at time $t = 0$, the position of the horizon at time t is

$$x = ct - \frac{c^2}{a} \ . \tag{8.86}$$

This is a horizon because it is impossible for the runner to see past it. The runner can never get any information about any event taking place along the x axis at the place x' at the time t', if $x' < ct' - (c^2/a)$.

8.9 Relativistic conservation laws

Any relativistic conservation law must necessarily be *local*, because it must hold in all inertial frames, and because simultaneity is a relative concept. For example, energy conservation *in all inertial frames* does not allow one joule of energy to move instantaneously from one place A to another place B, one kilometer away. The two events, the energy disappearing at A and reappearing at B, would not look simultaneous to all observers. Some observers would see energy not conserved, because they would see it appearing at B either before or after it disappeared at A. Because charge and energy are locally conserved quantities, they have to be carried along through a cable from the power station to the light bulb.

Consider the conservation of electric charge as a concrete example, illustrated in Figure 8.3. The charge inside a three dimensional region Ω_3 at time t is the integral over Ω_3 of the charge density $\rho = \rho(\boldsymbol{r}, t)$,

$$q(t) = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}, t) \ . \tag{8.87}$$

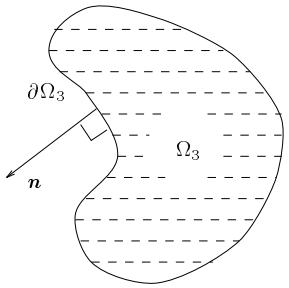

Figure 8.3: A two dimensional figure illustrating what a three dimensional region Ω_3 might look like.

The time derivative of the charge is

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\, \frac{\partial \rho}{\partial t} \,. \tag{8.88}$$

Local conservation of charge means that a change in the total charge within the region Ω_3 must be due to a net current of charge in or out through the surface $\partial\Omega_3$. The total net current equals the surface integral over $\partial\Omega_3$ of a current density $\boldsymbol{j}$. That is,

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\int_{\partial\Omega_3} \mathrm{d}^2S\, \boldsymbol{n}\cdot\boldsymbol{j} \,, \tag{8.89}$$

where d^2S is the surface element and $\boldsymbol{n}$ the unit normal of the surface $\partial\Omega_3$ in the outward direction. By means of Gauss's divergence theorem we may transform the surface integral into a volume integral,

$$\int_{\partial\Omega_3} \mathrm{d}^2S\, \boldsymbol{n}\cdot\boldsymbol{j} = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\, \nabla\cdot\boldsymbol{j} \,. \tag{8.90}$$

In this way, every local conservation law may be written in the form

$$\int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\frac{\partial \rho}{\partial t} + \nabla\cdot\boldsymbol{j} \right) = 0 \,, \tag{8.91}$$

where the region Ω_3 is arbitrary. In order for the integral to vanish, no matter how we choose our integration region Ω_3, the integrand must vanish everywhere,

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0 \,. \tag{8.92}$$

This is the *continuity equation* for electric charge. It may be written in the relativistic form

$$j^\mu{}_{,\mu} = \frac{\partial j^\mu}{\partial x^\mu} = 0 \,, \tag{8.93}$$

where we have introduced the four dimensional current density

$$j^\mu = (j^0, j^1, j^2, j^3) = (\rho c, \boldsymbol{j}) = (\rho c, j_x, j_y, j_z) \,. \tag{8.94}$$

The relativistic, four dimensional form of the continuity equation may be interpreted geometrically in its own way. Take a four dimensional region $\Omega = [t_A, t_B] \times \Omega_3$, where $[t_A, t_B]$

is a time interval, and Ω_3 is the same three dimensional region as above. The continuity equation $j^\mu{}_{,\mu} = 0$ in the interior of the region Ω implies that the boundary integral of j^μ vanishes, according to the four dimensional version of the divergence theorem,

$$\int_{\partial\Omega} \mathrm{d}^3S\, n_\mu j^\mu = \int_\Omega \mathrm{d}^4x\, j^\mu{}_{,\mu} = 0\ . \tag{8.95}$$

The surface integral over the boundary $\partial\Omega$ has three parts, as illustrated in Figure 8.4,

$$\begin{aligned}\int_{\partial\Omega} \mathrm{d}^3S\, n_\mu j^\mu &= \text{(I)} + \text{(II)} + \text{(III)} \\ &= \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}, t_B) - \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}, t_A) + \int_{t_A}^{t_B} \mathrm{d}t \int_{\partial\Omega_3} \mathrm{d}^2S\, \boldsymbol{n}\cdot\boldsymbol{j}(\boldsymbol{r}, t)\ .\end{aligned} \tag{8.96}$$

This shows that Equation (8.95), the vanishing of the surface integral over $\partial\Omega$, is just the time integral from t_A to t_B of Equation (8.89). If we assume that both the charge density and the current density vanish outside a finite region in space, or at least go to zero "sufficiently fast" when $|\boldsymbol{r}| \to \infty$, then we may move the boundary $\partial\Omega_3$ so far out that the boundary integral (III) in Figure 8.4 vanishes. In that case, Equation (8.95) expresses the conservation of the total charge, which is the charge density ρ integrated over all of the three dimensional space,

$$\int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}, t_B) = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}, t_A)\ . \tag{8.97}$$

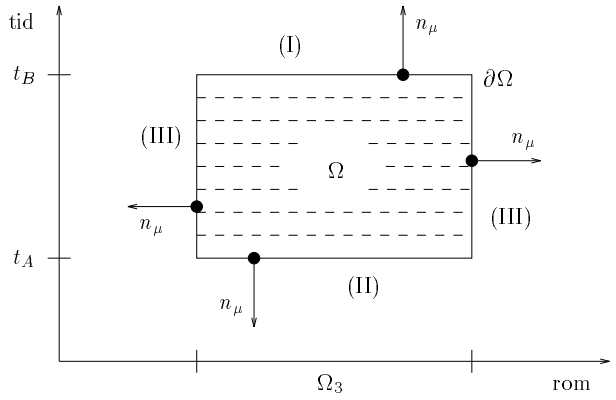

Figure 8.4: The four dimensional region $\Omega = [t_A, t_B] \times \Omega_3$, schematically drawn.

Another equally important consequence of the continuity equation $j^\mu{}_{,\mu} = 0$ is that the total charge is Lorentz invariant. We may again see that geometrically, as illustrated in Figure 8.5. A similar argument as above gives here that

$$\int \mathrm{d}^3\boldsymbol{r}\, \rho(\boldsymbol{r}, t_B) - \int \mathrm{d}^3\widetilde{\boldsymbol{r}}\, \widetilde{\rho}(\widetilde{\boldsymbol{r}}, \widetilde{t}_A) = \int_\Omega \mathrm{d}^4x\, j^\mu{}_{,\mu} - \int_{\Omega'} \mathrm{d}^4x\, j^\mu{}_{,\mu} = 0\ . \tag{8.98}$$

We have used here electric charge as an example, but the arguments are generally valid. Every local conservation law, and hence every relativistic conservation law, must have the form of a continuity equation.

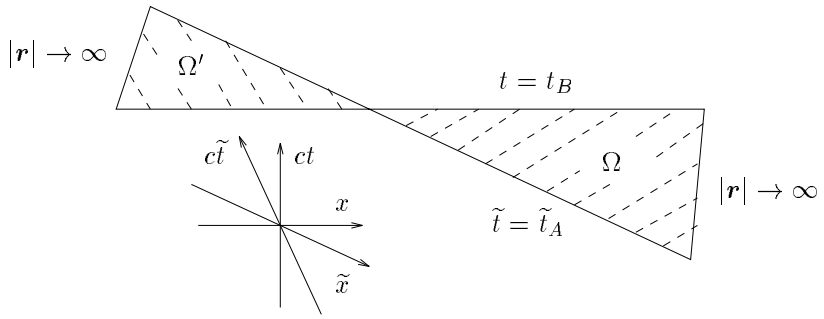

Figure 8.5: Lorentz invariance of the total charge.

The general theory of relativity

A local conservation law for a scalar quantity such as electric charge is invariant under general coordinate transformations when written in the following form,

$$j^{\mu}{}_{;\mu} = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{|g|}\, j^{\mu} \right) = 0 \, . \tag{8.99}$$

The only change necessary is to replace the partial derivatives by covariant derivatives. This is the form a conservation law for a scalar must have in the general theory of relativity.

The same general relativistic conservation law may be read as

$$\frac{\partial}{\partial x^{\mu}} \left(\sqrt{|g|}\, j^{\mu} \right) = 0 \, . \tag{8.100}$$

Then we may call j^{μ} a current *vector*, whereas $\sqrt{|g|}\, j^{\mu}$ is a current *density*, or more precisely a current vector density. The left hand side of the last equation is the covariant divergence of the current density. Because the divergence of a vector density, with nothing more than ordinary partial derivatives, is a scalar density, the equation is invariant under general coordinate transformations.

Problems

1. Find the 4×4 matrices describing:

   a) rotation by a given angle about the y axis;

   b) rotation about the z axis;

   c) a pure Lorentz transformation along the y axis;

   d) a pure Lorentz transformation along the z axis;

   e) a rotation about the x axis, followed by a pure Lorentz transformation along the y axis;

   f) a pure Lorentz transformation along the y axis, followed by a rotation about the x axis.

2. Compute the matrix $\mathrm{e}^{\alpha(J_x+K_y)}$, representing a somewhat special Lorentz transformation which is a combination of a rotation and a pure Lorentz transformation.
   Show explicitly that this is a Lorentz transformation, i.e. preserves the metric $\eta_{\mu\nu}$.
   Show that it leaves the null vector $A^\mu = (1, 0, 0, 1)$ invariant.
   We call A^μ a null (or lightlike) vector because $\eta_{\mu\nu} A^\mu A^\nu = 0$.

3. Derive the formula for time dilatation, Equation (8.61).

4. The distance to Sirius, one of the closest stars, is 9 light years. How long time does a round trip to Sirius and back take with a velocity of 180 000 km/s ($0.6\,c$), if we disregard the time it takes to accelerate to this velocity and to brake down again? To be more precise, how long time does it take for those who travel, and for those who wave goodbye and sit at home waiting for the astronauts to return?
   Is it a realistic project to travel through space at a velocity of $0.6\,c$? In fact, even the "empty" interstellar space is not quite empty, but contains a certain amount of gas, dust and radiation.

5. Is it realistic to neglect the acceleration and deceleration periods in the previous problem? In other words, how long time (coordinate time and proper time) does it take to accelerate up to a velocity of 180 000 km/s, with an acceleration of 10 m/s^2? The acceleration should be measured relative to the instantaneous rest system, in which the accelerated object is momentarily at rest.
   The acceleration of gravity on the surface of the Earth, $g = 9,81$ m/s^2, is roughly the maximum acceleration that the human body can support.

6. How much energy is needed (at least) to accelerate a space ship with a mass of 10^5 kg up to 180 000 km/s? The relativistic formula for the energy of a particle of mass (i.e. rest mass) m and velocity v is

$$E = \frac{mc^2}{\sqrt{1 - \dfrac{v^2}{c^2}}} \,.$$

   In order to find a lower bound to the energy needed, we may neglect any details of the acceleration process. If such a project should ever become a reality, the acceleration would presumably be provided by a rocket engine working over an extended period of

time. This would presumably cost much more than the minimum amount of energy, because the rocket engine at any time would have to accelerate also the remaining rocket fuel. It turns out that the efficiency of a rocket engine depends very much on the speed of its exhaust gas.

7. If an astronaut starts on his (or her) 30th birthday, and accelerates all the time in the same direction with a constant acceleration of 10 $\mathrm{m/s^2}$, how far away will he (she) be on his (her) 40th birthday? 50th birthday? 60th birthday?

8. Two space ships lie at rest one kilometer away from each other, and are connected by a rope which is pulled tight but not stressed. They start simultaneously to accelerate in the same direction. One space ship accelerates in the direction away from the second, which follows behind on the same course. Both accelerate by 10 $\mathrm{m/s^2}$, until reaching a velocity of exactly $0.6\,c$ relative to the original rest system. Then they stop accelerating and move on at constant velocity.
   What will happen to the rope connecting them?
   This is a famous thought experiment, rather more paradoxical than the twin paradox.

Chapter 9

Particle mechanics

The basic idea of the Lagrange and Hamilton formalism is to derive the equations of motion describing a physical system from a *variational principle*. This turns out to be a very powerful method, in particle mechanics as in field theory, both classical and quantum. One obvious reason is that the variational principle is a very compact formulation of the equations of motion. Furthermore, it usually exhibits very clearly the symmetries of the theory, and through Noether's theorem it gives a fundamental relation between symmetries and conservation laws.

We consider in this chapter mechanical systems with a finite number of degrees of freedom, typically systems consisting of idealized point particles. The equations of motion, in the form in which they are usually solved, are then ordinary differential equations for the particle coordinates as functions of time t.

9.1 Newton's second law

The basic equation in classical mechanics is Newton's second law for a non-relativistic point particle,

$$\boldsymbol{F} = m\boldsymbol{a} \, . \tag{9.1}$$

Here $\boldsymbol{F}$ is the force acting on the particle, and the mass m is a proportionality constant characteristic of the particle. The position of the particle is a function of time, $\boldsymbol{r} = \boldsymbol{r}(t)$, and the velocity $\boldsymbol{v}$ and acceleration $\boldsymbol{a}$ are the time derivatives of the position,

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \, , \qquad \boldsymbol{a} = \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} \, . \tag{9.2}$$

Newton's second law, written in the form

$$\frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \frac{\boldsymbol{F}}{m} \, , \tag{9.3}$$

is an equation of motion that may be integrated to give the position $\boldsymbol{r}(t)$ at any time t if we know the position $\boldsymbol{r}(t_0)$ and velocity $\boldsymbol{v}(t_0)$ at a given time t_0, and if we also know the force $\boldsymbol{F}$ acting on the particle at all times.

Newton's second law belongs to classical physics and does not apply to a quantum mechanical system like the electron in the hydrogen atom. On the other hand, it holds in a

suitably modified form even if the particle is relativistic, moving with a velocity approaching the speed of light. For example, the classical relativistic equation of motion applies to electrons and protons in particle accelerators, where the electrons may reach velocities so close to the speed of light that they lose less than a centimeter per second. We will return to the relativistic generalization of Newton's second law at the end of the next chapter.

The force $\boldsymbol{F}$ acting on a particle will usually vary along the orbit of the particle. The work performed by the force, from one point A at time t_A to another point B at time t_B, is

$$W_{AB} = \int_{t_A}^{t_B} \mathrm{d}t\ \boldsymbol{v} \cdot \boldsymbol{F} = \int_A^B \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{F}\ . \tag{9.4}$$

This work changes the kinetic energy of the particle, defined as

$$E_K = \frac{1}{2}\, m\boldsymbol{v}^2\ . \tag{9.5}$$

In fact, it follows from Newton's second law that

$$W_{AB} = \int_{t_A}^{t_B} \mathrm{d}t\ \boldsymbol{v} \cdot (m\boldsymbol{a}) = \frac{1}{2}\, m \int_{t_A}^{t_B} \mathrm{d}t\ \frac{\mathrm{d}\boldsymbol{v}^2}{\mathrm{d}t} = \frac{1}{2}\, m{\boldsymbol{v}_B}^2 - \frac{1}{2}\, m{\boldsymbol{v}_A}^2\ . \tag{9.6}$$

In many cases of interest the particle is moving in a *force field*, for example a gravitational field or an electromagnetic field, where the force is minus the gradient of a potential energy function,

$$\boldsymbol{F} = -\nabla V\ , \tag{9.7}$$

at the point where the particle is located. The potential energy $V = V(\boldsymbol{r}, t)$ is a function of the position $\boldsymbol{r}$ and the time t. In an electromagnetic field we may need to introduce also a velocity dependence in the potential energy, as we will see below.

When the potential energy is not explicitly time dependent, i.e. $V = V(\boldsymbol{r})$, then the system is said to be *conservative*, because the total energy $E = E_K + V$ of the particle, the sum of its kinetic and potential energy, is conserved. In fact, we have that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\, m\boldsymbol{v}^2 + V\right) = m\boldsymbol{v} \cdot \boldsymbol{a} + (\nabla V) \cdot \boldsymbol{v} = (m\boldsymbol{a} + \nabla V) \cdot \boldsymbol{v} = 0\ . \tag{9.8}$$

In this case we have, at least in principle, a clear procedure for measuring the potential energy V, or more precisely the difference in potential energy from one point $\boldsymbol{r}_A$ to another point $\boldsymbol{r}_B$. We may move the particle from $\boldsymbol{r}_A$ to $\boldsymbol{r}_B$ along any prescribed path $\boldsymbol{r} = \boldsymbol{r}(t)$ by applying an external force $\boldsymbol{F}_1$ chosen such that

$$\boldsymbol{F}_1 - \nabla V = m\, \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2}\ . \tag{9.9}$$

First we keep the particle at rest at $\boldsymbol{r}_A$, so that it has no kinetic energy. Next we move it to $\boldsymbol{r}_B$, where we again keep it at rest so that it has no kinetic energy. Since the kinetic energy is zero to begin with and zero at the end of the whole process, the total work done on the particle is zero,

$$0 = \int_A^B \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{F} = \int_A^B \mathrm{d}\boldsymbol{r} \cdot (\boldsymbol{F}_1 - \nabla V)\ . \tag{9.10}$$

Or equivalently,

$$\int_A^B \mathrm{d}\boldsymbol{r} \cdot \boldsymbol{F}_1 = \int_A^B \mathrm{d}\boldsymbol{r} \cdot (\nabla V) = \int_A^B \mathrm{d}V = V(\boldsymbol{r}_B) - V(\boldsymbol{r}_A) \; . \tag{9.11}$$

The line integral of the external force $\boldsymbol{F}_1$ is measurable. We see that it is independent of the path along which the particle is moved, it depends only on the end points $\boldsymbol{r}_A$ and $\boldsymbol{r}_B$ of the path, and it is equal to the difference in potential energy between the two points.

9.2 Hamilton's principle

From the example of one particle in three dimensions we generalize to a classical mechanical system with d degrees of freedom, i.e. a system described by d coordinates x^i, with $i = 1, 2, \ldots, d$. We call them *generalized coordinates*, because we need not choose them as the standard Eucliden coordinates, we may use polar coordinates instead, or any other coordinates suitable for the system we study. The coordinates need not have the physical dimension of length, or even the same dimension. The Lagrange and Hamilton formalism makes it easy to change coordinates, and this is one of its great advantages, especially when we want to formulate equations of motion that are generally covariant, i.e. invariant under general coordinate transformations.

To simplify the notation, we often do not write the coordinate index i, but write x for all coordinates simultaneously,

$$x = (x^1, x^2, \ldots, x^d) \; . \tag{9.12}$$

In other words, x denotes a point in a d-dimensional *configuration space*.

The motion of the system follows an *orbit* such that the position, or configuration, at time t is $x = x(t)$. The velocity is the time derivative of the position,

$$\dot{x} = (\dot{x}^1, \dot{x}^2, \ldots, \dot{x}^d) \; , \qquad \text{with} \qquad \dot{x}^i = \frac{\mathrm{d}x^i}{\mathrm{d}t} \; . \tag{9.13}$$

Thus, our notation is ambiguous in that the same symbol x may denote either a single point in the configuration space, or a complete time dependent orbit.

The *Lagrange function* for a non-relativistic system is usually the difference between the kinetic energy E_K and the potential energy V,

$$L = E_K - V \; . \tag{9.14}$$

The Lagrange function for a system of relativistic particles has a different form. In what follows, we will allow L to have a quite general dependence on position and velocity,

$$L = L(x, \dot{x}, t) = L(x^1, x^2, \ldots, x^d, \dot{x}^1, \dot{x}^2, \ldots, \dot{x}^d, t) \; . \tag{9.15}$$

Note that L is an implicit function of time t, since $x = x(t)$ and $\dot{x} = \dot{x}(t)$, but we allow also for the possibility that L may depend explicitly on t. Below we will consider briefly the more general case when L depends on the acceleration $\ddot{x}$, i.e. $L = L(x, \dot{x}, \ddot{x}, t)$. We never include in the Lagrange function higher than second order time derivatives.

The *action integral* for an orbit $x(t)$ from $t = t_A$ to $t = t_B$ is

$$S = \int_{t_A}^{t_B} \mathrm{d}t\, L\,. \tag{9.16}$$

Since S depends on the complete orbit $x = x(t)$, which is itself a function of time t from $t = t_A$ to $t = t_B$, we call S a *functional*. We write $S = S[x]$ in order to emphasize that S is a functional of the function x, and not just a function of the position $x(t)$ at one given time t. The Lagrange function $L = L(x(t), \dot{x}(t), t)$, on the other hand, is no functional, it is a quite ordinary function of the function values $x(t)$ and $\dot{x}(t)$, and of the time t.

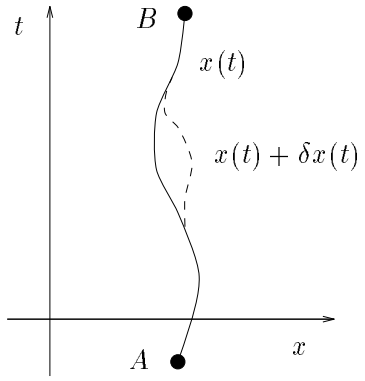

Figure 9.1: An infinitesimal variation of the orbit $x(t)$ within a limited time interval.

Imagine now an infinitesimal variation of the orbit, as indicated in Figure 9.1. We replace the function $x = x(t)$ by $x(t) + \delta x(t)$, where $\delta x(t)$ is infinitesimal for every t, and we ask what variation δS this causes in S. With $L = L(x, \dot{x}, t)$ we have that

$$\delta S = S[x + \delta x] - S[x] = \int_{t_A}^{t_B} \mathrm{d}t\, \delta L = \int_{t_A}^{t_B} \mathrm{d}t \left(\frac{\partial L}{\partial x^i}\, \delta x^i + \frac{\partial L}{\partial \dot{x}^i}\, \delta \dot{x}^i \right). \tag{9.17}$$

As usual, we follow the summation convention that when i occurs once as an upper and once as a lower index in a product of one or more factors, then a sum is understood, running over all possible values of i. In the last part of the integral above we perform a partial integration,

$$\int_{t_A}^{t_B} \mathrm{d}t\, \frac{\partial L}{\partial \dot{x}^i}\, \delta \dot{x}^i = \left. \frac{\partial L}{\partial \dot{x}^i}\, \delta x^i \right|_{t_A}^{t_B} - \int_{t_A}^{t_B} \mathrm{d}t \left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \right) \delta x^i\,. \tag{9.18}$$

We get rid of the boundary term in the partial integration by taking $\delta x(t_A) = \delta x(t_B) = 0$, this means that we keep the end points $x(t_A)$ and $x(t_B)$ fixed and vary the orbit only in the interior of the time interval $[t_A, t_B]$. Then we have that

$$\delta S = \int_{t_A}^{t_B} \mathrm{d}t \left(\frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \dot{x}^i} \right) \right) \delta x^i\,. \tag{9.19}$$

If the time t had been discrete, then we would have defined the partial derivative $\partial S / \partial x^i(t)$ by the relation

$$\delta S = \sum_{t=t_A}^{t_B} \sum_{i=1}^{d} \frac{\partial S}{\partial x^i(t)}\, \delta x^i(t)\,. \tag{9.20}$$

When t is a continuous variable, so that the sum over t has to be replaced by an integral, then we define the *functional derivative* $\delta S/\delta x^i(t)$ by the relation

$$\delta S = \int_{t_A}^{t_B} \mathrm{d}t \sum_{i=1}^{d} \frac{\delta S}{\delta x^i(t)}\, \delta x^i(t) \; . \tag{9.21}$$

This definition gives that

$$\frac{\delta S}{\delta x^i} = \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right) . \tag{9.22}$$

The equation of motion for the system may be derived from the action integral by means of

> **Hamilton's principle.** The motion of the system from position $x(t_A)$ at time t_A to position $x(t_B)$ at time t_B follows an orbit $x(t)$ such that the action integral S is extremal. That is,
>
> $$\delta S = 0 \tag{9.23}$$
>
> for every infinitesimal variation $\delta x(t)$ from the orbit $x(t)$ such that $\delta x(t_A) = \delta x(t_B) = 0$.

The equation of motion derived from Hamilton's principle is the *Euler–Lagrange equation*,

$$\boxed{\frac{\delta S}{\delta x^i} = \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0 \; .} \tag{9.24}$$

It consists of d equations at every value of t, one equation for each coordinate $x^i(t)$ which is varied.

In order to prove that the Euler–Lagrange equation follows from Hamilton's principle, assume that the orbit $x(t)$ is such that the Euler–Lagrange equation does not hold. We want to prove that we may then choose the variation δx in Equation (9.19) in such a way that we get $\delta S \neq 0$. A typical way for the Euler–Lagrange equation to be violated is e.g. that

$$\frac{\partial L}{\partial x^1} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^1}\right) > 0 \tag{9.25}$$

at some time time $t = t_0$, with $t_A < t_0 < t_B$. We always assume that we are dealing with functions that are continuously differentiable to "sufficiently high" order. Hence, when the inequality (9.25) holds at $t = t_0$, it must hold in a small interval around $t = t_0$, say in the interval $|t - t_0| < \epsilon$. Choose then all $\delta x^i(t) = 0$, except that for $|t - t_0| < \epsilon$ we choose $\delta x^1(t) > 0$. Such a variation is possible even if we require $\delta x^1(t)$ to be a $\mathcal{C}^\infty$ function for all t, and it will give $\delta S > 0$.

Hamilton's principle involves a philosophical issue which may be confusing at first sight. It looks superficially like a global principle, as if the system decides to move from a point $x(t_A)$ at time t_A to a point $x(t_B)$ at time t_B, and then adjusts its orbit in between so as to minimize, maximize, or in general extremize the action. How come then that the resulting

equation of motion, the Euler–Lagrange equation, is purely local in time? Perhaps the answer lies in the proof sketched in the previous paragraph. Since the variations $\delta x(t)$ used in the proof are purely local, we may well regard Hamilton's principle as being local in time. The essence of Hamilton's principle is that the action integral is required to be stationary under infinitesimal *local* variations of the orbit. Figure 9.1 illustrates the basic kind of variation, a local variation $\delta x(t)$ vanishing outside a small time interval.

As an example, let us return to our example of one particle, still using the Euclidean coordinates $\boldsymbol{r} = (x^1, x^2, x^3) = (x, y, z)$. The Lagrange function is

$$L = \frac{1}{2}\, m\boldsymbol{v}^2 - V(\boldsymbol{r}, t) = \frac{1}{2}\, m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z, t)\ . \tag{9.26}$$

To the coordinate $x^1 = x$ corresponds the Euler–Lagrange equation

$$0 = \frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) = -\frac{\partial V}{\partial x} - m\ddot{x}\ , \tag{9.27}$$

and to the two coordinates $x^2 = y$ and $x^3 = z$ correspond similar Euler–Lagrange equations. We summarize the three equations in one vector equation,

$$0 = \frac{\partial L}{\partial \boldsymbol{r}} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right) = -\frac{\partial V}{\partial \boldsymbol{r}} - m\dot{\boldsymbol{v}}\ , \tag{9.28}$$

which we recognize as Newton's second law $-\nabla V = m\boldsymbol{a}$.

Lagrange function depending on the acceleration

If the Lagrange function depends upon the acceleration $\ddot{x}$, then Equation (9.17) acquires the additional term

$$\int_{t_A}^{t_B} \mathrm{d}t\, \frac{\partial L}{\partial \ddot{x}^i}\, \delta\ddot{x}^i = \left(\frac{\partial L}{\partial \ddot{x}^i}\, \delta\dot{x}^i - \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \ddot{x}^i}\right)\right) \delta x^i\right)\Big|_{t_A}^{t_B} + \int_{t_A}^{t_B} \mathrm{d}t \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{x}^i}\right)\right) \delta x^i\ . \tag{9.29}$$

We use twice partial integration to transform $\delta\ddot{x}$ into δx. Again we ignore the boundary terms, since we require that both $\delta x(t_A) = \delta x(t_B) = 0$ and $\delta\dot{x}(t_A) = \delta\dot{x}(t_B) = 0$. That gives the following Euler–Lagrange equation,

$$\boxed{\frac{\delta S}{\delta x^i} = \frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial L}{\partial \ddot{x}^i}\right) = 0\ .} \tag{9.30}$$

It is easy enough to generalize to Lagrange functions containing third and higher order time derivatives. Note that the Euler–Lagrange equation will normally involve time derivatives of twice as high an order as the Lagrange function. But no rule without exceptions: a term in the Lagrange function of the form $f(x)\ddot{x}$, for example, gives no time derivatives in the equation of motion higher than $\ddot{x}$. In fact, such a term is equivalent to a term with only first order time derivatives, because

$$\int_{t_A}^{t_B} \mathrm{d}t\ f(x)\ddot{x} = f(x)\dot{x}\ |_{t_A}^{t_B} - \int_{t_A}^{t_B} \mathrm{d}t\ f'(x)\dot{x}^2\ . \tag{9.31}$$

Here the boundary terms from the partial integration do not contribute to the Euler–Lagrange equations, since they are constant under variations of the orbit that are local in time.

9.3 Hamilton's equations

To every coordinate x^i corresponds a *canonically conjugate momentum*, defined as

$$p_i = \frac{\partial L}{\partial \dot{x}^i} \,. \tag{9.32}$$

The *Hamilton function* is defined as

$$H = p_i \, \dot{x}^i - L \,. \tag{9.33}$$

A variation δx of the orbit x leads to variations of both $\dot{x}$ and p, and thereby of H,

$$\delta H = \delta p_i \, \dot{x}^i + p_i \, \delta \dot{x}^i - \frac{\partial L}{\partial x^i} \, \delta x^i - \frac{\partial L}{\partial \dot{x}^i} \, \delta \dot{x}^i = \dot{x}^i \, \delta p_i - \frac{\partial L}{\partial x^i} \, \delta x^i. \tag{9.34}$$

The last equality follows from the definition of the canonical momentum, Equation (9.32).

Because of this formula for δH it is natural to regard H as a function of x and p, so that

$$\delta H = \frac{\partial H}{\partial p_i} \, \delta p_i + \frac{\partial H}{\partial x^i} \, \delta x^i. \tag{9.35}$$

Comparison shows that

$$\frac{\partial H}{\partial p_i} = \dot{x}^i \,, \qquad \frac{\partial H}{\partial x^i} = -\frac{\partial L}{\partial x^i} \,, \tag{9.36}$$

and these two equations hold independent of the equation of motion, the Euler–Lagrange equation. Note that the partial derivatives on the left and right hand sides of the last equality sign have different meanings. In fact, we differentiate H at constant p, but L at constant $\dot{x}$, which is sometimes but not always the same. To be precise we should therefore write

$$\left.\frac{\partial H}{\partial x^i}\right|_p = -\left.\frac{\partial L}{\partial x^i}\right|_{\dot{x}} \,. \tag{9.37}$$

The Euler–Lagrange equation determines the time derivative of p, giving *Hamilton's equations*,

$$\boxed{\dot{x}^i = \frac{\partial H}{\partial p_i} \,, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i} \,.} \tag{9.38}$$

Since they are of first order in the time derivative, they determine the motion completely as soon as x and p are given at one given time $t = t_0$. Therefore we say that the position $x = (x^1, x^2, \ldots, x^d)$ and the momentum $p = (p_1, p_2, \ldots, p_d)$ together describe the *state* of the system at a given time. The $2d$ coordinates

$$(x, p) = (x^1, x^2, \ldots, x^d, p_1, p_2, \ldots, p_d) \tag{9.39}$$

specify a point in a $2d$ dimensional space which we call the *phase space*.

In our standard example, Equation (9.26), we write $\boldsymbol{p} = (p_1, p_2, p_3) = (p_x, p_y, p_z)$. In vector notation,

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = m\boldsymbol{v} \ , \tag{9.40}$$

and

$$H = \boldsymbol{p} \cdot \boldsymbol{v} - L = \frac{1}{2}\, m\boldsymbol{v}^2 + V(\boldsymbol{r}, t) = \frac{\boldsymbol{p}^2}{2m} + V(\boldsymbol{r}, t) \ . \tag{9.41}$$

Hamilton's equations are then altogether six equations, which we write as two vector equations,

$$\dot{\boldsymbol{r}} = \frac{\partial H}{\partial \boldsymbol{p}} = \frac{\boldsymbol{p}}{m} \ , \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{r}} = -\frac{\partial V}{\partial \boldsymbol{r}} \ . \tag{9.42}$$

Elimination of the momentum $\boldsymbol{p}$ from these equationes gives again Newton's second law, $m\boldsymbol{a} = -\nabla V$.

9.4 Poisson brackets

A general *observable* F is a function of the state, and posssibly also an explicit function of time,

$$F = F(x, p, t) = F(x^1, x^2, \ldots, x^d, p_1, p_2, \ldots, p_d, t) \ . \tag{9.43}$$

Coordinate and momentum components are special examples of observables. The value of an observable is time dependent, implicitly through x and p, and possibly also explicitly. The absolute time derivative of F, taking into account both the implicit and the explicit time dependence, is

$$\dot{F} \equiv \frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial F}{\partial x^i}\, \dot{x}^i + \frac{\partial F}{\partial p_i}\, \dot{p}_i + \frac{\partial F}{\partial t} = \frac{\partial F}{\partial x^i}\, \frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\, \frac{\partial H}{\partial x^i} + \frac{\partial F}{\partial t} \ . \tag{9.44}$$

We write this as

$$\dot{F} = \{F, H\} + \frac{\partial F}{\partial t} \ , \tag{9.45}$$

thereby defining the *Poisson bracket* between two observables F and G as

$$\{F, G\} = \frac{\partial F}{\partial x^i}\, \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\, \frac{\partial G}{\partial x^i} \ . \tag{9.46}$$

The Poisson bracket is *bilinear*,

$$\begin{aligned} \{\alpha A + \beta B, C\} &= \alpha\{A, C\} + \beta\{B, C\} \ , \\ \{A, \alpha B + \beta C\} &= \alpha\{A, B\} + \beta\{A, C\} \ , \end{aligned} \tag{9.47}$$

for arbitrary constants α and β, it is *antisymmetric*,

$$\{A, B\} + \{B, A\} = 0 \ , \tag{9.48}$$

and it satisfies the *Jacobi identity*,

$$\{A,\{B,C\}\}+\{B,\{C,A\}\}+\{C,\{A,B\}\}=0\,. \tag{9.49}$$

The antisymmetry of the Poisson bracket implies e.g. that $\{H,H\}=0$, and hence

$$\frac{\mathrm{d}H}{\mathrm{d}t}=\frac{\partial H}{\partial t}\,. \tag{9.50}$$

We see that the Hamilton function H is a constant of motion if and only if it is not explicitly time dependent. An equivalent condition is that the Lagrange function is not explicitly time dependent.

Classical mechanics and quantum mechanics

The transition from classical mechanics to quantum mechanics consists formally in the replacement of the Poisson bracket $\{,\}$ by the commutator $[,]$, or more precisely,

$$\{F,G\}\to\frac{1}{\mathrm{i}\hbar}\,[F,G]=\frac{FG-GF}{\mathrm{i}\hbar}\,, \tag{9.51}$$

for two arbitrary observables F and G. Take for example one coordinate x and its canonically conjugate momentum p, then

$$\{x,p\}=1\to\frac{1}{\mathrm{i}\hbar}\,[x,p]=1\,. \tag{9.52}$$

Because x and p do not commute in quantum mechanics, there arise *operator ordering problems*. Replacing the rule $xp=px$ by $xp=px+\mathrm{i}\hbar$ implies for example that the 6 operator products x^2p^2, p^2x^2, $xpxp$, $pxpx$, xp^2x and px^2p are all different (except that accidentally $xp^2x=px^2p$), and we have to choose e.g. some mean value of them to represent the classical quantity x^2p^2. It is certainly not entirely obvious which choice is the best one, and we have to introduce some more or less well motivated convention.

9.5 Constraints

The Lagrange formalism is an indispensable tool for treating systems in which the positions and the velocities are subject to constraints. There exist a great variety of constrained physical systems, and we will not go into the theory in any detail here, except to mention the Lagrange multiplier method, which is one important method for dealing with constraints in the form of equalities.

The obvious approach would be to eliminate the constraints together with the same number of variables. Afterwards, one is left with no constraints and only independent coordinates and velocities. This is a powerful method when it works. When it does not work, the Lagrange multiplier method may come to our rescue.

The easiest way to understand the method is perhaps by a model example. Assume first that we want to find an extremum, not of a functional such as the action S, but of a function of n independent variables, $f=f(x_1,x_2,\ldots,x_n)$. We do so by solving the n equations

$$\frac{\partial f}{\partial x_i}=0 \tag{9.53}$$

for the n variables $x_1, x_2, \ldots, x_n$. Equivalently, we require that the variation of f vanishes,

$$\delta f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\, \delta x_i = 0 \ , \tag{9.54}$$

for arbitrary infinitesimal variations δx_i of the n variables x_i.

Assume next that the n variables are not all free, because they have to satisfy m constraints of the form

$$g_k(x_1, x_2, \ldots, x_n) = 0 \ . \tag{9.55}$$

Then we can no longer require that $\delta f = 0$ for arbitrary variations δx_i, we can only require that $\delta f = 0$ for variations δx_i that satisfy the constraints $\delta g_k = 0$. We may still want to perform completely arbitrary variations δx_i, but then the most we can hope for is that there exist coefficients λ_k, called Lagrange multipliers, one for each constraint equation $g_k = 0$, such that

$$\delta f = \sum_{k=1}^{m} \lambda_k\, \delta g_k \ . \tag{9.56}$$

In fact, this condition will ensure that $\delta f = 0$ whenever all $\delta g_k = 0$. Since

$$\delta g_k = \sum_{i=1}^{n} \frac{\partial g_k}{\partial x_i}\, \delta x_i \ , \tag{9.57}$$

we have the following $n + k$ equations to solve for the $n + k$ variables x_i and λ_k. First the n equations

$$\frac{\partial f}{\partial x_i} - \sum_{k=1}^{m} \lambda_k\, \frac{\partial g_k}{\partial x_i} = 0 \ , \tag{9.58}$$

and then the k equations

$$g_k = 0 \ . \tag{9.59}$$

All of these $n + k$ equations may be derived as the extremal conditions for the function

$$F(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m) = f(x_1, x_2, \ldots, x_n) - \sum_{k=1}^{m} \lambda_k\, g_k(x_1, x_2, \ldots, x_n) \ , \tag{9.60}$$

where we treat the $n + k$ variables $x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m$ as independent.

To see how these ideas may be applied in the Lagrange formalism for classical mechanics, consider the circular pendulum as an example. Let x be the horizontal and z the vertical coordinate of a point mass m, suspended so that $x^2 + z^2 = a^2$, with a the constant length of a thin wire. The gravitational potential energy is mgz, where g is the constant acceleration of gravity. The Lagrange function is

$$L = \frac{1}{2}\, m(\dot{x}^2 + \dot{z}^2) - mgz \ . \tag{9.61}$$

We may easily eliminate the constraint by writing $x = a\sin\varphi$, $z = -a\cos\varphi$, then the remaining coordinate is φ, and the Lagrange function in terms of φ is

$$L = \frac{1}{2}\,ma^2\dot{\varphi}^2 + mga\cos\varphi\ . \tag{9.62}$$

The Euler–Lagrange equation is

$$0 = \frac{\partial L}{\partial\varphi} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) = -mga\sin\varphi - ma^2\ddot{\varphi}\ . \tag{9.63}$$

The alternative is the Lagrange multiplier method. Since there is an independent constraint $x^2 + z^2 - a^2 = 0$ at every instant of time, we need one Lagrange multiplier $\lambda(t)$ for every value of t. The action integral to be extremalized, including the Lagrange multipliers, is

$$S' = \int_{t_A}^{t_B}\mathrm{d}t\ L' = \int_{t_A}^{t_B}\mathrm{d}t\ \left(L - \lambda(x^2 + z^2 - a^2)\right)\ . \tag{9.64}$$

Hence, the Euler–Lagrange equations for x and z, taking the constraints into account, are

$$\begin{aligned} 0 &= \frac{\partial L'}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L'}{\partial\dot{x}}\right) = -2\lambda x - m\ddot{x}\ , \\ 0 &= \frac{\partial L'}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L'}{\partial\dot{z}}\right) = -mg - 2\lambda z - m\ddot{z}\ . \end{aligned} \tag{9.65}$$

Remember that λ is time dependent. Eliminating λ, we get the equation

$$mgx + m(x\ddot{z} - z\ddot{x}) = 0\ . \tag{9.66}$$

This is the same equation as before, since

$$x\ddot{z} - z\ddot{x} = \frac{\mathrm{d}}{\mathrm{d}t}(x\dot{z} - z\dot{x}) = a^2\ddot{\varphi}\ . \tag{9.67}$$

9.6 Equivalent Lagrange functions

Obviously, knowing the extremal points of a function is not nearly enough in order to know the whole function. For example, the two functions

$$f(x) = x^3 - 3x \qquad \text{and} \qquad g(x) = \frac{x}{1+x^2} \tag{9.68}$$

are both extremal at $x = \pm 1$. In the same way there are in general an unlimited choice of different action functionals giving the same Euler–Lagrange equation. There is a certain limitation in the assumption that an action functional must be the integral of a Lagrange function, nevertheless it should come as no surprise that many different Lagrange-functions are equivalent in the sense that they give the same Euler–Lagrange equation.

We do not ask here for the most general Lagrange function $\widetilde{L}$ equivalent to a given Lagrange function L, our only ambition is to point out one important example,

$$\widetilde{L} = CL + \frac{\mathrm{d}M}{\mathrm{d}t}\ , \tag{9.69}$$

where C is a constant ($C \neq 0$), and $M = M(x, \dot{x}, t)$. The corresponding action integral is

$$\widetilde{S} = \int_{t_A}^{t_B} \mathrm{d}t\ \widetilde{L} = CS + M(x(t_B), \dot{x}(t_B), t_B) - M(x(t_A), \dot{x}(t_A), t_A)\ . \tag{9.70}$$

For any variation $\delta x(t)$ which does not vary the last two terms, the boundary contributions due to M, we have that

$$\delta\widetilde{S} = C\ \delta S\ , \tag{9.71}$$

implying that the two conditions $\delta S = 0$ and $\delta\widetilde{S} = 0$ are equivalent. In other words, the two functionals S and $\widetilde{S}$ give the same Euler–Lagrange equation, and the two Lagrange functions L and $\widetilde{L}$ are equivalent.

It is legitimate to ask if the class of variations giving

$$\delta M(x(t_A), \dot{x}(t_A), t_A) = \delta M(x(t_B), \dot{x}(t_B), t_B) = 0 \tag{9.72}$$

is large enough for the purpose of deriving the Euler–Lagrange equation from Hamilton's principle. We derived the Euler–Lagrange equation for an orbit $x = x(t)$ by requiring that $\delta S = 0$ for every variation δx with $\delta x(t_A) = \delta x(t_B) = 0$. If M depends on $\dot{x}(t)$, then we have to impose as a further restriction on the variation δx that $\delta\dot{x}(t_A) = \delta\dot{x}(t_B) = 0$. There is indeed no harm in this extra restriction, our derivation of the Euler–Lagrange equation will still work.

We may prove even more directly that an absolute derivative

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\partial M}{\partial x^j}\,\dot{x}^j + \frac{\partial M}{\partial \dot{x}^j}\,\ddot{x}^j + \frac{\partial M}{\partial t} \tag{9.73}$$

in the Lagrange function does not contribute to the Euler–Lagrange equation. Since such a term depends on the acceleration $\ddot{x}$, we have to use Equation (9.30). What we have to prove, is that the equation

$$\left(\frac{\partial}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial}{\partial \dot{x}^i} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\frac{\partial}{\partial \ddot{x}^i}\right)\frac{\mathrm{d}M}{\mathrm{d}t} = 0 \tag{9.74}$$

is an identity holding for an arbitrary orbit $x(t)$. This is so because

$$\begin{aligned}
\frac{\partial}{\partial x^i}\,\frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{\partial^2 M}{\partial x^i\,\partial x^j}\,\dot{x}^j + \frac{\partial^2 M}{\partial x^i\,\partial \dot{x}^j}\,\ddot{x}^j + \frac{\partial^2 M}{\partial x^i\,\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial M}{\partial x^i}\ , \\
\frac{\partial}{\partial \dot{x}^i}\,\frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{\partial M}{\partial x^i} + \frac{\partial^2 M}{\partial \dot{x}^i\,\partial x^j}\,\dot{x}^j + \frac{\partial^2 M}{\partial \dot{x}^i\,\partial \dot{x}^j}\,\ddot{x}^j + \frac{\partial^2 M}{\partial \dot{x}^i\,\partial t} = \frac{\partial M}{\partial x^i} + \frac{\mathrm{d}}{\mathrm{d}t}\,\frac{\partial M}{\partial \dot{x}^i}\ , \\
\frac{\partial}{\partial \ddot{x}^i}\,\frac{\mathrm{d}M}{\mathrm{d}t} &= \frac{\partial M}{\partial \dot{x}^i}\ .
\end{aligned} \tag{9.75}$$

Problems

1. Check that the two terms in the Euler–Lagrange equation (9.24) have the same dimension.

2. The Lagrange function for a free particle on a line is

$$L = \frac{1}{2}\, m\dot{x}^2 \ .$$

   m is the mass and x the position. The action integral is non-negative, since $L \geq 0$. Given three times $t_A < t_B < t_C$. Let $x(t) = x_A =$ constant for $t < t_B$ and $x(t) = x_C =$ constant for $t \geq t_B$, with $x_A \neq x_C$. The action integral from t_A to t_C is then

$$S = \int_{t_A}^{t_C} \mathrm{d}t\ L = \int_{t_A}^{t_B} \mathrm{d}t\ L + \int_{t_B}^{t_C} \mathrm{d}t\ L = 0 \ .$$

   Hence S is minimal for this orbit.
   The conclusion is obviously wrong, but what is wrong with the argument?

3. The Lagrange function for a one dimensional harmonic oscillator of angular frequency ω is

$$L = \frac{1}{2}\, m\dot{x}^2 - \frac{1}{2}\, m\omega^2 x^2 \ .$$

   a) Derive and solve the Euler–Lagrange equation.

   b) Given an arbitrary orbit $x(t)$ from time $t = 0$ to $t = T > 0$, with $x(0) = x(T) = 0$. Expand $x(t)$ in a Fourier series.
   Use Hamilton's principle to derive a set of equations for the Fourier coefficients.
   Do the equations always have a solution, and if a solution exists, is it unique?
   When the action is extremal, is the extremum a minimum, a maximum or a saddle point?

4. A point B on a roller-coaster lies 10 meter lower than a point A and at a horizontal distance of 12 meter from A. The track is constructed in such a way that the wagon moves from A to B in the shortest possible time, given that it starts at A with zero velocity, and falls freely, except that it follows the track. We neglect friction.
   Find the curve from A to B, and find the minimal time.
   Find the minimal time if the horizontal distance is 40 meter.
   This general problem, known as the *brachistocron problem*, inspired John Bernoulli to invent the variational calculus.

5. A closed curve in the (x, y) plane, with the standard Euclidean metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, has a given length a. What is the maximum area it can enclose?
   Hint: the area inside a closed curve may be written as a line integral, see Problem 7.1.

Chapter 10

Symmetries and conservation laws

The concept of *symmetry* or *invariance* is fundamental in physics. An obvious reason is that symmetry often simplifies a problem very much, and the problems that we are able to solve exactly are those with the highest degree of symmetry. A less obvious reason, no less important, is the connection between symmetries and conservation laws, through Noether's theorem. In this chapter we will demonstrate the connection in the context of particle machanics.

First we need to give the concept of symmetry a precise mathematical meaning in terms of transformations. We have introduced the notation $x = (x^1, x^2, \ldots, x^d)$ for the configuration of a general physical system described by d coordinates. The motion of the system is described by an orbit $x(t)$, giving the configuration at time t. A *transformation* of orbits,

$$x(t) \mapsto \widetilde{x}(t) \ , \tag{10.1}$$

is a rule assigning to any given orbit $x(t)$, whether it is a solution of the equation of motion or not, another orbit $\widetilde{x}(t)$. The transformed orbit $\widetilde{x}(t)$ could be the same as the original orbit $x(t)$, then we say, naturally, that it is invariant under the transformation. The important point is that to every orbit $x(t)$ there corresponds a uniquely defined transformed orbit $\widetilde{x}(t)$.

Most of the time we will also assume that the transformation $x \mapsto \widetilde{x}$ is invertible, so that there exists an inverse transformation $\widetilde{x} \mapsto x$.

10.1 Transformations of space and time

It is possible to imagine very general ways of transforming orbits, but we will restrict our attention here to transformations of space and time, of the form

$$(x,t) \mapsto (X(x,t), T(x,t)) \ . \tag{10.2}$$

For an orbit $x = x(t)$ we have then that

$$(x(t),t) \mapsto (\widetilde{x}(\widetilde{t}), \widetilde{t}) = (X(x(t),t), T(x(t),t)) \ , \tag{10.3}$$

that is,

$$\widetilde{x}(T(x(t),t)) = X(x(t),t) \ . \tag{10.4}$$

An important special case is the infinitesimal transformation

$$X = x + d(x,t) \ , \qquad T = t + b(x,t) \ , \tag{10.5}$$

with d and b infinitesimal. It is automatically invertible, as long as we may neglect terms of second order in d and b, in fact the inverse transformation is

$$x = X - d(x,t) = X - d(X,T) \ , \qquad t = T - b(x,t) = T - b(X,T) \ . \tag{10.6}$$

For the transformed orbit $\widetilde{x} = \widetilde{x}(t)$ we have that

$$\widetilde{x}(T) = x(t) + d = x(T-b) + d, \tag{10.7}$$

or equivalently,

$$\widetilde{x}(t) = x(t-b) + d = x(t) - b\dot{x}(t) + d \ . \tag{10.8}$$

Written explicitly with indices this equation reads

$$\widetilde{x}^{\,i}(t) = x^i(t) - b(x(t),t))\,\dot{x}^i(t) + d^i(x(t),t) \ . \tag{10.9}$$

Defining $\Delta x = d - b\dot{x}$, we write

$$\widetilde{x}^{\,i}(t) = x^i(t) + \Delta x^i(t) \ . \tag{10.10}$$

Let us give some examples of finite and infinitesimal transformations.

Space and time translation. $(\boldsymbol{r},t) \mapsto (\boldsymbol{r}+\boldsymbol{d}, t+b)$, where $\boldsymbol{d}$ and b are finite or infinitesimal constants. Then $\widetilde{\boldsymbol{r}}(t+b) = \boldsymbol{r}(t) + \boldsymbol{d}$, or

$$\widetilde{\boldsymbol{r}}(t) = \boldsymbol{r}(t-b) + \boldsymbol{d} \ . \tag{10.11}$$

If b is infinitesimal, then

$$\widetilde{\boldsymbol{r}}(t) = \boldsymbol{r}(t) - b\dot{\boldsymbol{r}}(t) + \boldsymbol{d} \ . \tag{10.12}$$

Space inversion (parity) in three dimensions. $\boldsymbol{r} \mapsto -\boldsymbol{r}$, giving that $\widetilde{\boldsymbol{r}}(t) = -\boldsymbol{r}(t)$.

Time reversal. $t \mapsto -t$, giving that $\widetilde{x}^i(-t) = x^i(t)$, or $\widetilde{x}^i(t) = x^i(-t)$.

Rotation in three dimensions.

$$\widetilde{\boldsymbol{r}}(t) = \boldsymbol{r}(t) + \sin\alpha\,\boldsymbol{n}\times\boldsymbol{r}(t) + (1-\cos\alpha)\,\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{r}(t)) \ , \tag{10.13}$$

where α is the rotation angle, and the unit vector $\boldsymbol{n}$ is the rotation axis. For an infinitesimal rotation angle α we define $\boldsymbol{\alpha} = \alpha\boldsymbol{n}$, and we get that

$$\widetilde{\boldsymbol{r}}(t) = \boldsymbol{r}(t) + \alpha\,\boldsymbol{n}\times\boldsymbol{r}(t) = \boldsymbol{r}(t) + \boldsymbol{\alpha}\times\boldsymbol{r}(t) \ . \tag{10.14}$$

Lorentz transformation. $(ct,x,y,z) \mapsto (\gamma(ct+\beta x), \gamma(x+\beta ct), y, z)$, where c is the speed of light, whereas β and γ are dimensionless constants with $\gamma = 1/\sqrt{1-\beta^2}$. In vector notation we may write

$$(ct,\boldsymbol{r}) \mapsto (\gamma(ct + \beta\boldsymbol{n}\cdot\boldsymbol{r})\,,\, \boldsymbol{r} + (\gamma-1)(\boldsymbol{n}\cdot\boldsymbol{r})\boldsymbol{n} + \gamma\beta ct\boldsymbol{n}) \ , \tag{10.15}$$

where $\boldsymbol{n}$ is a unit vector, in our present example pointing along the x axis. This gives that

$$\widetilde{\boldsymbol{r}}\left(\gamma t + \frac{\gamma\beta}{c}\,\boldsymbol{n}\cdot\boldsymbol{r}(t)\right) = \boldsymbol{r}(t) + (\gamma-1)(\boldsymbol{n}\cdot\boldsymbol{r}(t))\,\boldsymbol{n} + \gamma\beta c t\boldsymbol{n}\ . \tag{10.16}$$

Here we find no explicit expression for $\widetilde{\boldsymbol{r}}(t)$, unless we limit ourselves to the case in which β is infinitesimal, so that we may compute only to first order in β. Then $\gamma = 1$, and

$$\widetilde{\boldsymbol{r}}\left(t + \frac{\boldsymbol{\beta}\cdot\boldsymbol{r}(t)}{c}\right) = \boldsymbol{r}(t) + ct\boldsymbol{\beta}\ , \tag{10.17}$$

with $\boldsymbol{\beta} = \beta\boldsymbol{n}$. Always to first order in β, we have

$$\widetilde{\boldsymbol{r}}(t) = \boldsymbol{r}(t) - \frac{\boldsymbol{\beta}\cdot\boldsymbol{r}(t)}{c}\,\dot{\boldsymbol{r}}(t) + ct\boldsymbol{\beta}\ . \tag{10.18}$$

10.2 Symmetries

By definition, the transformation $x(t) \mapsto \widetilde{x}(t)$ is a *symmetry* of the system if it preserves the equation of motion. That is, if the orbit $\widetilde{x}(t)$ is a solution of the equation of motion whenever the original orbit $x(t)$ is a solution, and if $x(t)$ is a solution whenever $\widetilde{x}(t)$ is a solution. Another reasonable condition on a symmetry transformation $x \mapsto \widetilde{x}$ is that it should be invertible.

One way to check whether the orbits $x(t)$ and $\widetilde{x}(t)$ are solutions of the equation of motion is to check whether the values of the action, $S[x]$ and $S[\widetilde{x}]$, are extremal. We may define a new action functional $\widetilde{S}$ such that

$$\widetilde{S}[x] = S[\widetilde{x}]\ . \tag{10.19}$$

The condition for the transformation $x \mapsto \widetilde{x}$ to be a symmetry is that the two action functionals $S[x]$ and $\widetilde{S}[x]$ are equivalent, giving the same Euler–Lagrange equation for x.

In particular, consider an infinitesimal transformation

$$x(t) \mapsto \widetilde{x}(t) = x(t) + \Delta x(t)\ , \tag{10.20}$$

transforming any given orbit $x(t)$, whether or not it is a solution of the equation of motion, into an orbit $\widetilde{x}(t) = x(t) + \Delta x(t)$, with Δx infinitesimal. We now define a new Lagrange function

$$\widetilde{L}(x, \dot{x}, t) = L(x + \Delta x, \dot{x} + \Delta\dot{x}, t)\ . \tag{10.21}$$

Thus we have $\widetilde{L} = L + \Delta L$, where ΔL is the infinitesimal variation in the Lagrange function due to the infinitesimal variation Δx in the orbit,

$$\Delta L = \frac{\partial L}{\partial x^i}\,\Delta x^i + \frac{\partial L}{\partial \dot{x}^i}\,\Delta\dot{x}^i. \tag{10.22}$$

The condition for Δx to define a symmetry is that the two Lagrange functions L and $\widetilde{L}$ give the same equation of motion for x. A sufficient, though not necessary, condition is that ΔL is an absolute time derivative, i.e. that there exists a function $\Delta M = \Delta M(x, \dot{x}, t)$ such that the equation

$$\Delta L = \frac{\mathrm{d}(\Delta M)}{\mathrm{d}t} \tag{10.23}$$

holds identically for any orbit x. What happens in practice is very often that the Lagrange function is invariant, i.e. that $\Delta L = 0$, and then we choose simply $\Delta M = 0$.

10.3 Noether's theorem

We say that a coordinate is *cyclic* if it does not occur in the Lagrange function. For example, x^1 is cyclic if

$$\frac{\partial L}{\partial x^1} = 0 \,. \tag{10.24}$$

In that case, the corresponding Euler–Lagrange equation is simply a conservation law for the momentum canonically conjugate to x^1,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^1}\right) = 0 \,. \tag{10.25}$$

That x^1 is cyclic means that the Lagrange function L, and hence the equation of motion, is invariant under any constant translation in the x^1 direction. Indeed, L might depend on the velocity $\dot{x}^1$, but a translation $x^1 \mapsto x^1 + d^1$, with d^1 constant, does not change $\dot{x}^1$.

The conservation law for a momentum component conjugate to a cyclic coordinate is a special case of *Noether's theorem*. The theorem states that a *continuous* symmetry of a certain type, in a physical system, implies a conservation law. The Noether type of symmetry, to which the theorem applies, is an infinitesimal transformation $x(t) \mapsto \widetilde{x}(t) = x(t) + \Delta x(t)$ having the property that Equation (10.23) holds for some $\Delta M = \Delta M(x, \dot{x}, t)$.

There exist continuous symmetries to which Noether's theorem does not apply, an example is the general type of scale invariance, see Problem 3. But the theorem encompasses the most important symmetries such as translations in space and time, rotations and Lorentz transformations.

Obviously, a conservation law depends on the equation of motion. A conserved quantity is constant along every orbit obeying the equation of motion, but is not in general constant along other orbits. Now if the orbit x is a solution of the Euler–Lagrange equation, then it is true for an *arbitrary* variation Δx that the variation of the Lagrange function is an absolute time derivative. The proof is easy,

$$\Delta L = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right)\Delta x^i + \frac{\partial L}{\partial \dot{x}^i}\,\Delta \dot{x}^i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\,\Delta x^i\right) = \frac{\mathrm{d}}{\mathrm{d}t}\left(p_i\,\Delta x^i\right). \tag{10.26}$$

Whenever we succeed in writing the variation ΔL as an absolute time derivative in two different ways, first by using that the transformation $x \mapsto x + \Delta x$ is a symmetry, and next by assuming that the original orbit x is a solution of the equation of motion, then we have a conservation law,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\,\Delta x^i - \Delta M\right) = 0 \,.} \tag{10.27}$$

This is Noether's theorem.

10.4 Examples

Important applications of Noether's theorem are the relation between translation invariance in time and space and conservation of energy and momentum, and between rotational invariance and conservation of angular momentum. Let us look at these examples.

Time translation. An infinitesimal time translation has the form $\Delta x^i(t) = -b\dot{x}^i(t)$, with b a constant infinitesimal parameter. It gives that $\Delta\dot{x}^i(t) = -b\ddot{x}^i(t)$, and if we assume that

$$\frac{\partial L}{\partial t} = 0 \,, \tag{10.28}$$

then we have

$$\Delta L = \frac{\partial L}{\partial x^i}\,\Delta x^i + \frac{\partial L}{\partial \dot{x}^i}\,\Delta\dot{x}^i = \frac{\mathrm{d}}{\mathrm{d}t}\,(-bL) \,. \tag{10.29}$$

The result that ΔL is an absolute time derivative, for an arbitrary orbit $x^i(t)$, means that the infinitesimal time translation is a symmetry. The only assumption needed is that the Lagrange function has no explicit time dependence. Noether's theorem then gives the following conservation law,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}^i}\,(-b\dot{x}^i) + bL\right) = -b\,\frac{\mathrm{d}}{\mathrm{d}t}\left(p_i\dot{x}^i - L\right) = -b\,\frac{\mathrm{d}H}{\mathrm{d}t} = 0 \,. \tag{10.30}$$

We see that time translation invariance, expressed by the condition $\partial L/\partial t = 0$, implies that the Hamilton function H is a constant of motion.

Space translation. As an example, let $L = L(\boldsymbol{r}, \boldsymbol{v}, t)$ be the Lagrange function of a particle in three dimensions, with position $\boldsymbol{r} = \boldsymbol{r}(t)$ and velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$. A constant translation

$$\boldsymbol{r}(t) \mapsto \widetilde{\boldsymbol{r}}(t) = \boldsymbol{r}(t) + \boldsymbol{d} \tag{10.31}$$

does not change the velocity, we have $\widetilde{\boldsymbol{v}}(t) = \boldsymbol{v}(t)$ when the displacement $\boldsymbol{d}$ is constant in space and time. When $\boldsymbol{d}$ is infinitesimal, then

$$\Delta L = \frac{\partial L}{\partial \boldsymbol{r}} \cdot \boldsymbol{d} \,. \tag{10.32}$$

Translation invariance in space usually means that the Lagrange function is invariant, i.e. that $\Delta L = 0$. By Noether's theorem, the component of the momentum $\boldsymbol{p} = \partial L/\partial \boldsymbol{v}$ along the displacement vector $\boldsymbol{d}$ is then conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\boldsymbol{p} \cdot \boldsymbol{d}) = 0 \,. \tag{10.33}$$

If there is translation invariance in all directions simultaneously, in the present case this means that all coordinates (x, y, z) are cyclic, then all three components of the vector $\boldsymbol{p}$ are conserved.

Rotation. Given $L = L(\boldsymbol{r}, \boldsymbol{v}, t)$, as above, and given the infinitesimal rotation

$$\boldsymbol{r}(t) \mapsto \widetilde{\boldsymbol{r}}(t) = \boldsymbol{r}(t) + \boldsymbol{\alpha} \times \boldsymbol{r}(t) \,, \tag{10.34}$$

where α is a constant and infinitesimal vector along the rotation axis. Differentiation gives that

$$\widetilde{\boldsymbol{v}}(t) = \boldsymbol{v}(t) + \boldsymbol{\alpha} \times \boldsymbol{v}(t) \,. \tag{10.35}$$

Now,

$$\Delta L = \frac{\partial L}{\partial \boldsymbol{r}} \cdot (\boldsymbol{\alpha} \times \boldsymbol{r}) + \frac{\partial L}{\partial \boldsymbol{v}} \cdot (\boldsymbol{\alpha} \times \boldsymbol{v}) = \boldsymbol{\alpha} \cdot \left(\boldsymbol{r} \times \frac{\partial L}{\partial \boldsymbol{r}} + \boldsymbol{v} \times \frac{\partial L}{\partial \boldsymbol{v}} \right) , \tag{10.36}$$

when we use that the triple product is invariant under cyclic permutations,

$$\boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}) . \tag{10.37}$$

Rotation invariance, like translation invariance, usually means that the Lagrange function is invariant, so that $\Delta L = 0$. In that case, Noether's theorem says that the component of the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ along the rotation axis $\boldsymbol{\alpha}$ is conserved,

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{p} \cdot (\boldsymbol{\alpha} \times \boldsymbol{r}) \right) = \frac{\mathrm{d}}{\mathrm{d}t} \left(\boldsymbol{\alpha} \cdot (\boldsymbol{r} \times \boldsymbol{p}) \right) = 0 . \tag{10.38}$$

If there is rotational invariance about an arbitrary axis, then all three components of the vector $\boldsymbol{L}$ are conserved.

10.5 Non-relativistic particles

A typical Lagrange function for a system of N non-relativistic point particles is of the form

$$L = \sum_{i=1}^{N} \frac{1}{2} m_i {\boldsymbol{v}_i}^2 - V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) . \tag{10.39}$$

m_i is the mass, $\boldsymbol{r}_i$ the position, and $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i$ the velocity of particle no. i, whereas V is the potential energy of the system as a whole. The canonical momentum of particle i is

$$\boldsymbol{p}_i = \frac{\partial L}{\partial \boldsymbol{v}_i} = m_i \boldsymbol{v}_i \qquad \text{(no sum over } i) . \tag{10.40}$$

The Hamilton function is

$$H = \sum_{i=1}^{N} \boldsymbol{p}_i \cdot \boldsymbol{v}_i - L = \sum_{i=1}^{N} \frac{1}{2} m_i {\boldsymbol{v}_i}^2 + V = \sum_{i=1}^{N} \frac{{\boldsymbol{p}_i}^2}{2m_i} + V . \tag{10.41}$$

It is not explicitly time dependent and therefore a constant of motion whenever the potential energy V is not explicitly time dependent, in fact we have then

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = \frac{\partial V}{\partial t} = 0 . \tag{10.42}$$

The Euler–Lagrange equation is Newton's second law,

$$0 = \frac{\partial L}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial L}{\partial \boldsymbol{v}_i} \right) = -\frac{\partial V}{\partial \boldsymbol{r}_i} - \frac{\mathrm{d}}{\mathrm{d}t} \left(m_i \boldsymbol{v}_i \right) . \tag{10.43}$$

Hamilton's equations are Newton's second law in another disguise,

$$\dot{\boldsymbol{r}}_i = \frac{\partial H}{\partial \boldsymbol{p}_i} = \frac{\boldsymbol{p}_i}{m_i} , \qquad \dot{\boldsymbol{p}}_i = -\frac{\partial H}{\partial \boldsymbol{r}_i} = -\frac{\partial V}{\partial \boldsymbol{r}_i} . \tag{10.44}$$

These equations describe interaction between the particles when the force on one particle depends on the positions of the other particles. Only if the potential energy is a sum of one particle contributions,

$$V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t) = \sum_{i=1}^{N} V_i(\boldsymbol{r}_i, t) \; , \tag{10.45}$$

is there no coupling between the equations of motion for different particles, so that the particles move independently of each other.

An infinitesimal rotation,

$$\boldsymbol{r}_i \mapsto \boldsymbol{r}_i + \Delta\boldsymbol{r}_i = \boldsymbol{r}_i + \boldsymbol{\alpha} \times \boldsymbol{r}_i \; , \tag{10.46}$$

with $\boldsymbol{\alpha}$ infinitesimal, leaves the kinetic energy invariant,

$$\Delta({\boldsymbol{v}_i}^2) = \boldsymbol{v}_i \cdot (\Delta\boldsymbol{v}_i) = \boldsymbol{v}_i \cdot (\boldsymbol{\alpha} \times \boldsymbol{v}_i) = -\boldsymbol{\alpha} \cdot (\boldsymbol{v}_i \times \boldsymbol{v}_i) = 0 \; . \tag{10.47}$$

Hence,

$$\Delta L = -\Delta V = -\sum_{i=1}^{N} \frac{\partial V}{\partial \boldsymbol{r}_i} \cdot (\boldsymbol{\alpha} \times \boldsymbol{r}_i) = -\boldsymbol{\alpha} \cdot \left(\sum_{i=1}^{N} \boldsymbol{r}_i \times \frac{\partial V}{\partial \boldsymbol{r}_i} \right) . \tag{10.48}$$

If the potential energy is invariant, then $\Delta L = 0$, and Noether's theorem gives the conservation law

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\sum_{i=1}^{N} \frac{\partial L}{\partial \boldsymbol{v}_i} \cdot \Delta\boldsymbol{r}_i \right) = \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{\alpha} \cdot \boldsymbol{L}) = 0 \; . \tag{10.49}$$

The conserved quantity $\boldsymbol{\alpha} \cdot \boldsymbol{L}$ is (porportional to) the component along the rotation axis of the total angular momentum

$$\boldsymbol{L} = \sum_{i=1}^{N} \boldsymbol{r}_i \times (m_i \boldsymbol{v}_i) \; . \tag{10.50}$$

If the potential V is invariant under rotations about any arbitrary rotation axis, then all three components of $\boldsymbol{L}$ are conserved. Invariance under rotations about three different axes is actually sufficient. One important case with full rotational symmetry is when the potential energy depends only on the distances between the particles,

$$V = \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} V_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) \; . \tag{10.51}$$

Charged particles in an external electromagnetic field

This example gives us the opportunity to get acquainted with a velocity dependent potential energy.

The electromagnetic field is described in the Lagrange formalism by a scalar potential $\Phi = \Phi(\boldsymbol{r}, t)$ and a three dimensional vector potential $\boldsymbol{A} = \boldsymbol{A}(\boldsymbol{r}, t)$. We call it an *external* field when we make the approximation that the field is not modified by the charged particles

moving around in the field. Thus, an external field does not mediate any interaction between the particles.

The Lagrange function of N non-relativistic particles in the external field is

$$L = \sum_{i=1}^{N} \left(\frac{1}{2} m_i \boldsymbol{v}_i{}^2 - q_i \Phi_i + q_i \boldsymbol{v}_i \cdot \boldsymbol{A}_i \right) . \tag{10.52}$$

Here q_i is the electrical charge of particle i, while $\Phi_i = \Phi(\boldsymbol{r}_i, t)$ and $\boldsymbol{A}_i = \boldsymbol{A}(\boldsymbol{r}_i, t)$. If L has an explicit time dependence, it will be because the potentials Φ and $\boldsymbol{A}$ vary with time. Note that the electromagnetic *fields*,

$$\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t} , \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} , \tag{10.53}$$

may well be constant in time, even when the *potentials* vary with time.

From this Lagrange function follows the Euler–Lagrange equation

$$\frac{\partial}{\partial \boldsymbol{r}_i} \left(-q_i \Phi_i + q_i \boldsymbol{v}_i \cdot \boldsymbol{A}_i \right) - \frac{\mathrm{d}}{\mathrm{d}t} \left(m_i \boldsymbol{v}_i + q_i \boldsymbol{A}_i \right) = 0 , \tag{10.54}$$

where there is still no summation convention for the particle index i. Every particle has its own equation of motion, independent of all the others. Therefore we may just as well drop the index i, there is in any case only one particle involved.

The equation of motion is, as always, Newton's second law,

$$m\boldsymbol{a} = q \left(\nabla \left(-\Phi + \boldsymbol{v} \cdot \boldsymbol{A} \right) - \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} \right) . \tag{10.55}$$

We may find a more explicit expression for the force. Let us look first at the x component of the equation,

$$m\ddot{x} = q \left(-\frac{\partial \Phi}{\partial x} + \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_y}{\partial x} + \dot{z} \frac{\partial A_z}{\partial x} - \frac{\mathrm{d}A_x}{\mathrm{d}t} \right) . \tag{10.56}$$

Since the coordinates (x, y, z) are all time dependent, $A_x = A_x(x, y, z, t)$ is an implicit function of t, in addition to its possible explicit time dependence. It follows that

$$\frac{\mathrm{d}A_x}{\mathrm{d}t} = \frac{\partial A_x}{\partial x} \dot{x} + \frac{\partial A_x}{\partial y} \dot{y} + \frac{\partial A_x}{\partial z} \dot{z} + \frac{\partial A_x}{\partial t} , \tag{10.57}$$

and hence

$$m\ddot{x} = q \left(-\frac{\partial \Phi}{\partial x} - \frac{\partial A_x}{\partial t} + \dot{y} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) + \dot{z} \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \right) . \tag{10.58}$$

Here we introduce the electrical field $\boldsymbol{E}$ and the magnetic flux density $\boldsymbol{B}$, given by the potentials Φ and $\boldsymbol{A}$ as in Equation (10.53). The resulting equation is

$$m\ddot{x} = q \left(E_x + \dot{y} B_z - \dot{z} B_y \right) . \tag{10.59}$$

We find the y and z components of the equation of motion in the same way, and it is easy enough to guess that the full vector equation is

$$m\boldsymbol{a} = q \left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right) . \tag{10.60}$$

The right hand side in this equation is called the *Lorentz force*.

Note that even though the potentials Φ and $\boldsymbol{A}$ represent the electromagnetic field in the Lagrange function, it is the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ that are present in the equation of motion. Since we measure the electromagnetic field by measuring the deflection of charged particles, the directly measurable quantities are the fields and not the potentials. In quantum mechanics on the other hand, e.g. in the Aharonov–Bohm effect, the potentials may influence the particles in more subtle ways, even when the fields vanish in the region where the particles move around.

The canonical momentum of particle i is now position dependent,

$$\boldsymbol{p}_i = \frac{\partial L}{\partial \boldsymbol{v}_i} = m_i \boldsymbol{v}_i + q_i \boldsymbol{A}_i \; . \tag{10.61}$$

The velocity $\boldsymbol{v}_i$ is directly observable, but the potential $\boldsymbol{A}_i$ is not, and so neither is the canonical momentum $\boldsymbol{p}_i$. The observable momentum is the *kinematic* momentum

$$\boldsymbol{\pi}_i = m_i \boldsymbol{v}_i = \boldsymbol{p}_i - q_i \boldsymbol{A}_i \; . \tag{10.62}$$

The Hamilton function is

$$H = \sum_{i=1}^{N} \boldsymbol{p}_i \cdot \boldsymbol{v}_i - L = \sum_{i=1}^{N} \left(\frac{1}{2} \, m_i {\boldsymbol{v}_i}^2 + q_i \Phi_i \right) = \sum_{i=1}^{N} \left(\frac{(\boldsymbol{p}_i - q_i \boldsymbol{A}_i)^2}{2 m_i} + q_i \Phi_i \right) . \tag{10.63}$$

Like the Lagrange function it contains the potentials. Hence, Hamilton's equations,

$$\dot{\boldsymbol{r}}_i = \frac{\partial H}{\partial \boldsymbol{p}_i} = \frac{\boldsymbol{p}_i - q_i \boldsymbol{A}_i}{m_i} \; , \qquad \dot{\boldsymbol{p}}_i = -\frac{\partial H}{\partial \boldsymbol{r}_i} \; , \tag{10.64}$$

also contain the potentials. But after eliminating the momentum $\boldsymbol{p}_i$ from these equations we get the same equation of motion as before, Equation (10.60), containing only the fields.

10.6 Relativistic particles

The Lagrange function L for a free relativistic particle of mass m should be chosen in such a way that it maximizes the proper time τ, as given by Equation (8.61). We define

$$L = -mc^2 \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} = -mc^2 \, \frac{\mathrm{d}\tau}{\mathrm{d}t} = -mc \frac{\mathrm{d}s}{\mathrm{d}t} = -mc \sqrt{g_{\mu\nu} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}t}} \; , \tag{10.65}$$

where $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$. Then the action is proportional to the proper time,

$$S = \int_{t_A}^{t_B} \mathrm{d}t \; L = -mc^2 \int_{\tau_A}^{\tau_B} \mathrm{d}\tau = -mc^2 (\tau_B - \tau_A) \; . \tag{10.66}$$

L has dimension of energy, and has the correct non-relativistic limit, up to a constant: when the velocity $\boldsymbol{v}$ is small compared to the speed of light, we have that

$$L \approx -mc^2 + \frac{1}{2} \, m \boldsymbol{v}^2 \; . \tag{10.67}$$

The canonical momentum of the particle is

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \frac{m \boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} \; . \tag{10.68}$$

Inversely, the velocity may be expressed in terms of the momentum as

$$\boldsymbol{v} = \frac{c\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2c^2}} \,. \tag{10.69}$$

The Hamilton function is

$$H = \boldsymbol{p}\cdot\boldsymbol{v} - L = \frac{mc^2}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} = c\sqrt{\boldsymbol{p}^2 + m^2c^2} \,. \tag{10.70}$$

The equation of motion for a free relativistic particle, whether we look at the Euler–Lagrange equation or Hamilton's equations, says that the momentum is constant,

$$\dot{\boldsymbol{p}} = \frac{\partial L}{\partial \boldsymbol{r}}\bigg|_{\dot{\boldsymbol{r}}} = -\frac{\partial H}{\partial \boldsymbol{r}}\bigg|_{\boldsymbol{p}} = 0 \,. \tag{10.71}$$

Relativistic particles in an external electromagnetic field

The Lagrange function for a relativistic particle in an external electromagnetic field is

$$L = -mc^2\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} \; - q\Phi + q\boldsymbol{v}\cdot\boldsymbol{A} \,. \tag{10.72}$$

The canonical momentum of the particle is

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} + q\boldsymbol{A} \,. \tag{10.73}$$

Compared to the non-relativistic case, the difference is the relativistic expression for the kinematic momentum,

$$\boldsymbol{\pi} = \boldsymbol{p} - q\boldsymbol{A} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} \,. \tag{10.74}$$

The relativistic Hamilton function is

$$H = \boldsymbol{p}\cdot\boldsymbol{v} - L = \frac{mc^2}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} + q\Phi = c\sqrt{(\boldsymbol{p} - q\boldsymbol{A})^2 + m^2c^2} + q\Phi \,. \tag{10.75}$$

The equation of motion, expressed in terms of the kinematic momentum, has the same form in the relativistic as in the non-relativistic case,

$$\dot{\boldsymbol{\pi}} = q\left(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}\right) \,. \tag{10.76}$$

The Lorentz force is the same in the two cases, and the only difference is the expression for the kinematic momentum $\boldsymbol{\pi}$.

Particles in external electromagnetic and gravitational fields

Above, we treated a relativistic particle by a non-relativistic formalism, regarding the position of the particle as a function of time, $\boldsymbol{r} = \boldsymbol{r}(t)$. This is the "$3+1$ dimensional" point of view. But it is also possible to adopt a "4 dimensional" point of view, treating space and time on an equal footing.

In the above Lagrange function for a particle in an external electromagnetic field, in the special theory of relativity, we may introduce the four dimensional vector potential, which we define as a covariant four vector,

$$A_\mu = \left(\frac{\Phi}{c}\,, -\boldsymbol{A}\right) = \left(\frac{\Phi}{c}\,, -A_x, -A_y, -A_z\right). \tag{10.77}$$

Then we may write

$$L = -mc\sqrt{g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu}\; - q\dot{x}^\mu A_\mu\,. \tag{10.78}$$

It is an assumption that the expression under the square root sign is non-negative.

In this expression for the Lagrange function we need no longer assume that the metric tensor $g_{\mu\nu}$ is constant, like it is in the special theory of relativity. Hence this is the Lagrange function of a particle with electric charge q moving in an external gravitational field described by a metric tensor $g_{\mu\nu}$ varying in space and time, and in an external electromagnetic field, given by the four vector potential A_μ.

The action integral along a given orbit $x^\mu = x^\mu(t)$ from one point A to another point B is

$$S = \int_{t_A}^{t_B} \mathrm{d}t\, L = \int_{t_A}^{t_B} \mathrm{d}t\, \left(-mc\sqrt{g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu}\; - q\dot{x}^\mu A_\mu\right)\,. \tag{10.79}$$

This integral has a purely geometric meaning, as we may see by writing it in the form

$$S = \int_A^B \left(-mc\sqrt{g_{\mu\nu}\;\mathrm{d}x^\mu\,\mathrm{d}x^\nu}\; - q\;\mathrm{d}x^\mu A_\mu\right)\,. \tag{10.80}$$

It is manifestly *reparametrization invariant*. That is, it depends on the curve $x^\mu = x^\mu(t)$ in the four dimensional spacetime, but not on how the curve is parametrized. We may introduce an arbitrary parameter u along the curve, a new "time", such that $t = t(u)$, or inversely $u = u(t)$, and if we generalize our definition of the time derivative, so that

$$\dot{x}^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}u} = \frac{\mathrm{d}x^\mu}{\mathrm{d}t}\,\frac{\mathrm{d}t}{\mathrm{d}u}\,, \tag{10.81}$$

then

$$S = \int_{u_A}^{u_B} \mathrm{d}u\, \left(-mc\sqrt{g_{\mu\nu}\,\dot{x}^\mu \dot{x}^\nu}\; - q\dot{x}^\mu A_\mu\right)\,. \tag{10.82}$$

It is understood that the parameters t and u define the same orientation of the curve, i.e. that $\dot{t} = \mathrm{d}t/\mathrm{d}u > 0$.

In the $3+1$ dimensional formalism we had three coordinates, $\boldsymbol{r} = (x, y, z)$, and three canonically conjugate momenta, $\boldsymbol{p} = (p_x, p_y, p_z)$. Here we have suddenly four coordinates, $x^\mu = (ct, x, y, z)$, and consequently four canonical momenta,

$$P_\mu = -p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -\frac{mc\, g_{\mu\nu}\dot{x}^\nu}{\sqrt{g_{\kappa\lambda}\dot{x}^\kappa \dot{x}^\lambda}} - qA_\mu\,. \tag{10.83}$$

Note that the canonical momentum, which we denote here by P_μ in an attempt to reduce the confusion, has the opposite sign as compared to the relativistic momentum p_μ (as the latter is usually defined).

Note also the strange circumstance that P_μ is reparametrization invariant, i.e. independent of how the parameter u is defined. The special form of the relation between the velocity $\dot{x}^\mu$ and the momentum P_μ implies a constraint on the momentum,

$$g^{\mu\nu}\,(P_\mu + qA_\mu)\,(P_\nu + qA_\nu) = m^2c^2\;, \tag{10.84}$$

which reduces the number of independent momentum components from four back to three.

Similarly to what we did earlier, we may define a kinematic momentum

$$\Pi_\mu = -\pi_\mu = P_\mu + qA_\mu = -\frac{mc\,g_{\mu\nu}\dot{x}^\nu}{\sqrt{g_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda}} = -mg_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\;. \tag{10.85}$$

Here τ is the proper time, defined by the relation

$$c\,\mathrm{d}\tau = \mathrm{d}u\,\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = \sqrt{g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu}\;. \tag{10.86}$$

The Euler–Lagrange equation, $\dot{P}_\mu = \partial L/\partial x^\mu$, then takes the following form,

$$\dot{\pi}_\mu + q\dot{A}_\mu = \frac{mc\,g_{\rho\sigma,\mu}\dot{x}^\rho\dot{x}^\sigma}{2\sqrt{g_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda}} + q\dot{x}^\rho A_{\rho,\mu}\;. \tag{10.87}$$

Here we use that $\dot{A}_\mu = A_{\mu,\rho}\,\dot{x}^\rho$, and introduce the electromagnetic *field tensor*,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}\;, \tag{10.88}$$

which is antisymmetric, $F_{\nu\mu} = -F_{\mu\nu}$. That gives the equation

$$\dot{\pi}_\mu = \frac{mc\,g_{\rho\sigma,\mu}\dot{x}^\rho\dot{x}^\sigma}{2\sqrt{g_{\kappa\lambda}\dot{x}^\kappa\dot{x}^\lambda}} + qF_{\mu\rho}\dot{x}^\rho\;. \tag{10.89}$$

Multiplying this equation by c and dividing by $\sqrt{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}$, we get that

$$m\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(g_{\mu\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\right) = \frac{1}{2}\,m\,g_{\rho\sigma,\mu}\,\frac{\mathrm{d}x^\rho}{\mathrm{d}\tau}\,\frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} + qF_{\mu\nu}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}\;. \tag{10.90}$$

This equation must hold as well in the special theory of relativity, where $g_{\mu\nu} = \eta_{\mu\nu}$, and it must then be identical to Equation (10.76). So it is indeed, when we identify the electromagnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ with the components of the field tensor $F_{\mu\nu}$. With $g_{\mu\nu} = \eta_{\mu\nu}$, we have

$$\pi_\mu = (\pi_0,\pi_1,\pi_2,\pi_3) = (\pi^0,-\boldsymbol{\pi}) = \left(\frac{mc}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}}\,,\,-\frac{m\boldsymbol{v}}{\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}}}\right), \tag{10.91}$$

and Equation (10.89) may be written as

$$\frac{\mathrm{d}\pi_\mu}{\mathrm{d}t} = qF_{\mu\rho}\frac{\mathrm{d}x^\rho}{\mathrm{d}t}\;. \tag{10.92}$$

Where $\mathrm{d}x^\rho/\mathrm{d}t = (c, \boldsymbol{v})$. We find the following relation,

$$F_{\mu\nu} = \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} . \tag{10.93}$$

A trick for computing the time derivative of π_0 is to use the constraint

$$\pi_\mu \pi^\mu = (\pi_0)^2 - \boldsymbol{\pi}^2 = m^2 c^2 \, . \tag{10.94}$$

Differentiation gives that

$$\frac{\mathrm{d}\pi_0}{\mathrm{d}t} = \frac{\boldsymbol{\pi}}{\pi_0} \cdot \frac{\mathrm{d}\boldsymbol{\pi}}{\mathrm{d}t} = \frac{q}{c} \, \boldsymbol{v} \cdot (\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) = \frac{q}{c} \, \boldsymbol{v} \cdot \boldsymbol{E} \, . \tag{10.95}$$

The four dimensional Hamilton formalism is rather more subtle than what we are used to. That has to do with the reparametrization invariance of the action integral, and with the resulting constraint on the canonical momentum. The reparametrization invariance is due to the identity

$$\mathrm{d}t \; L\biggl(x, \frac{\mathrm{d}x}{\mathrm{d}t}\biggr) = \mathrm{d}u \; L\biggl(x, \frac{\mathrm{d}x}{\mathrm{d}u}\biggr) \; , \tag{10.96}$$

or equivalently, with $\alpha = \mathrm{d}u/\mathrm{d}t > 0$ and $\mathrm{d}x/\mathrm{d}t = \alpha \, \mathrm{d}x/\mathrm{d}u$,

$$L\biggl(x, \alpha \, \frac{\mathrm{d}x}{\mathrm{d}u}\biggr) = \alpha \; L\biggl(x, \frac{\mathrm{d}x}{\mathrm{d}u}\biggr) \; , \tag{10.97}$$

which means, in the mathematical terminology, that the Lagrange function is homogeneous of degree one as a function of the velocity $\dot{x}$. Differentiating the last equation with respect to α and setting $\alpha = 1$ gives the relation

$$\frac{\partial L}{\partial \dot{x}^\mu} \, \dot{x}^\mu = L \, . \tag{10.98}$$

But this means that the Hamilton function vanishes identically, by the standard definition,

$$H = P_\mu \dot{x}^\mu - L = \frac{\partial L}{\partial \dot{x}^\mu} \, \dot{x}^\mu - L = 0 \, . \tag{10.99}$$

The root of the trouble is the freedom we have to choose the parameter u, but the same freedom offers a way out. The choice $u = \tau$ leads to the essential simplification that

$$\sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} = c \, . \tag{10.100}$$

Then we have $P_\mu = -m g_{\mu\nu} \dot{x}^\nu - q A_\mu$, and inversely,

$$\dot{x}^\mu = -\frac{1}{m} \, g^{\mu\nu} \, (P_\nu + q A_\nu) \, . \tag{10.101}$$

This is the equation of motion $\dot{x}^\mu = \partial H / \partial P_\mu$ resulting from the Hamilton function

$$H = -\frac{1}{2m} \, g^{\mu\nu} \, (P_\mu + q A_\mu) \, (P_\nu + q A_\nu) \, . \tag{10.102}$$

Therefore we postulate that this is the Hamilton function of the particle. It is a constant of motion, because it does not depend explicitly on the parameter $u = \tau$, and that is consistent with the constraint on the momentum which we know from before, that

$$H = -\frac{1}{2}\, mc^2 \,. \tag{10.103}$$

From the Hamilton function H we may go back to a Lagrange function $L' \neq L$. The relation between canonical momentum and velocity, $\dot{x}^\mu = \partial H/\partial P_\mu$, gives that

$$H = -\frac{1}{2}\, m g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \,. \tag{10.104}$$

This gives

$$L' = P_\mu \dot{x}^\mu - H = -\frac{1}{2}\, m g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu - q A_\mu \dot{x}^\mu \,, \tag{10.105}$$

a different Lagrange function from the one we started out with. It is equivalent, in as far as it gives the same equation of motion, and has the same non-relativistic limit. But it is not reparametrization invariant, which means that the equation of motion determines the parametrization of an orbit.

Problems

1. Given the Lagrange function

$$L = \frac{1}{2}\, m(\dot{y}^2 + \dot{y}^2) + a_1 x + a_2 y + \dot{x}(a_3 t + a_4 x + a_5 y + a_6 x y^2) + \dot{y}(a_7 t + a_8 x + a_9 y + a_{10} x^2 y) \ ,$$

where m and $a_1, a_2, \ldots, a_{10}$ are constants. It describes a particle of mass m moving in the (x, y) plane.

   a) Derive the Euler–Lagrange equations. Solve them, assuming that $a_6 = a_{10}$.

   b) Investigate for which values of the constants $a_1, a_2, \ldots, a_{10}$ the following transformations are symmetries.
   – translation in the x direction;
   – translation in the y direction;
   – time translation;
   – rotation.
   Use Noether's theorem to find the corresponding conservation laws.

2. Consider a non-relativistic system of N particles, with the Lagrange function

$$L = \sum_{i=1}^{N} \frac{1}{2}\, m_i \boldsymbol{v}_i{}^2 - \sum_{i<j} V_{ij}(|\boldsymbol{r}_i - \boldsymbol{r}_j|) \ .$$

m_i is the mass, $\boldsymbol{r}_i = \boldsymbol{r}_i(t)$ the position and $\boldsymbol{v}_i = \dot{\boldsymbol{r}}_i = \mathrm{d}\boldsymbol{r}_i/\mathrm{d}t$ the velocity of particle no. i, whereas V_{ij} is the potential energy due to the interaction of the two particles i and j. Show that the following transformations, where $\boldsymbol{r}_i(t)$ transforms into $\widetilde{\boldsymbol{r}}_i(t)$, are symmetries of the system, and find the corresponding conserved quantities.

   a) Time translation: $\widetilde{\boldsymbol{r}}_i(t) = \boldsymbol{r}_i(t - b)$, with b constant.

   b) Spatial translation in the x direction (d = constant):

$$\widetilde{x}_i(t) = x_i(t) + d \ , \quad \widetilde{y}_i(t) = y_i(t) \ , \quad \widetilde{z}_i(t) = z_i(t) \ .$$

   c) Rotation about the z axis (α = constant):

$$\begin{aligned} \widetilde{x}_i(t) &= x_i(t)\cos\alpha - y_i(t)\sin\alpha \ , \\ \widetilde{y}_i(t) &= x_i(t)\sin\alpha + y_i(t)\cos\alpha \ , \\ \widetilde{z}_i(t) &= z_i(t) \ . \end{aligned}$$

   d) Acceleration (also called boost) in the x direction (u = constant):

$$\widetilde{x}_i(t) = x_i(t) + ut \ , \quad \widetilde{y}_i(t) = y_i(t) \ , \quad \widetilde{z}_i(t) = z_i(t) \ .$$

   This is a *Galilei transformation*, the non-relativistic counterpart to a Lorentz transformation.

3. (Scale invariance I): Given the Lagrange function

$$L = \sum_{i=1}^{N} \frac{1}{2}\, m_i \boldsymbol{v}_i{}^2 - V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \ ,$$

where the potential energy V is a homogeneous function of the coordinates of degree k, i.e., for any $\alpha > 0$ we have

$$V(\alpha \boldsymbol{r}_1, \alpha \boldsymbol{r}_2, \ldots, \alpha \boldsymbol{r}_N) = \alpha^k\, V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N) \ .$$

Show that a scaling of space and time with constant factors α and β,

$$\boldsymbol{r} \mapsto \widehat{\boldsymbol{r}} = \alpha \boldsymbol{r} \ , \qquad t \mapsto \widehat{t} = \beta t \ ,$$

leads to the following transformation of a time dependent orbit,

$$\boldsymbol{r}_i \mapsto \widetilde{\boldsymbol{r}}_i(t) = \alpha \boldsymbol{r}_i\!\left(\frac{t}{\beta}\right) \ .$$

Show that α and β may be chosen such that this transformation is a symmetry of the system, i.e. preserves the equations of motion.
How is the Lagrange function transformed?
Does Noether's theorem apply?
Examples of scale invariance:
How does the period of a harmonic oscillator vary with the amplitude?
How does the time of free fall in a constant gravitational field vary with the height?
How does the period of a planet vary with its distance from the Sun?

4. (Scale invariance II): Given the following Lagrange function, with constants C_{ij},

$$L = \sum_{i=1}^{N} \frac{1}{2}\, m_i \boldsymbol{v}_i{}^2 - \sum_{i<j} \frac{C_{ij}}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^2} \ .$$

Show that the scaling transformation (also called dilatation, or conformal transformation)

$$\widetilde{\boldsymbol{r}}_i(t) = \mathrm{e}^{\lambda}\, \boldsymbol{r}_i(\mathrm{e}^{-2\lambda}\ t) \ ,$$

with λ constant, is a symmetry of Noether type, and find the conserved quantity.

Chapter 11

Field mechanics

A *field* in physics is a quantity varying in space and time. Examples of various types of physical fields are:

– scalar fields: mass density (non-relativistic), charge density, pressure, temperature;
– complex scalar fields: Schrödinger wave function, complex Klein–Gordon field;
– vector fields: electric and magnetic fields (non-relativistic);
– tensor fields: electromagnetic field (relativistic), gravitational field.

Mathematically speaking, a field ϕ is a function of space and time, with its function values in a "field space". Introducing coordinates $\phi^1, \phi^2, \ldots, \phi^n$ in the field space, we describe the field as a quantity having n components that are functions of space and time,

$$\phi = \phi^i(\boldsymbol{r}, t) = \phi^i(x^0, x^1, x^2, x^3) = \phi^i(x) \; . \tag{11.1}$$

In an analogy to particle mechanics, where the configuration at time t is denoted by $x = x^j(t)$, the field ϕ corresponds to the coordinate x. But there are two different ways to regard this correspondence, which we may call "$3+1$ dimensional" and "4 dimensional":

– $3+1$ dimensional: $j \leftrightarrow (i, \boldsymbol{r})$ and $t \leftrightarrow t$;
– 4 dimensional: $j \leftrightarrow i$ and $t \leftrightarrow (x^0, x^1, x^2, x^3)$.

The Lagrange formalism, with the Euler–Lagrange equation for the field, does not distinguish between space and time, and so invites the 4 dimensional interpretation. The Hamilton formalism, on the other hand, with Hamilton's equations for the field, is a $3+1$ dimensional point of view, in which space and time play different roles. The traditional way to quantize a field theory is $3+1$ dimensional, based on the Hamilton formalism, whereas the path integral quantization of Richard Feynman is 4 dimensional, based on the Lagrange formalism.

11.1 Hamilton's principle

The Lagrange function for a field ϕ is a three dimensional integral over a region Ω_3,

$$L = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \; \mathcal{L} \; , \tag{11.2}$$

where $\mathcal{L}$ is the *Lagrange density*, or *Lagrangian*,

$$\mathcal{L} = \mathcal{L}\left(\phi, \frac{\partial\phi}{\partial x}, x\right) = \mathcal{L}(\phi^1, \ldots, \phi^n, \phi^1_{,0}, \ldots, \phi^n_{,0}, \phi^1_{,1}, \ldots, \phi^n_{,3}, x^0, x^1, x^2, x^3) \; . \tag{11.3}$$

The action integral is four dimensional, and is a functional of the field,

$$S = S[\phi] = \int_{t_A}^{t_B} \mathrm{d}t\ L = \frac{1}{c}\int_\Omega \mathrm{d}^4x\ \mathcal{L}\ , \tag{11.4}$$

where $\Omega = [t_A, t_B] \times \Omega_3$. The four dimensional point of view is very natural here, and we may well allow Ω to be an arbitrary four dimensional region, not necessarily a Cartesian product of a time interval $[t_A, t_B]$ and a spatial region Ω_3.

In a similar way as we allowed the Lagrange function for a system of particles to have an explicit time dependence, we allow the Lagrange density for a field to have an explicit dependence on space and time. Physically, such an explicit dependence would in principle make it possible to measure position in space and time in an absolute way, hence it would violate the principle of relativity. Such a violation is physically meaningful if it is due to some external background field. For example, a weak electromagnetic field has little influence on the gravitational field, and it often makes sense to solve the electromagnetic field equations while treating the gravitational field as a fixed external field. However, $\mathcal{L}$ should not depend explicitly on space and time when we treat all fields as dynamical and have no fixed external field.

An infinitesimal variation $\delta\phi$ of the field ϕ gives the following infinitesimal variation of S,

$$\delta S = S[\phi + \delta\phi] - S[\phi] = \frac{1}{c}\int_\Omega \mathrm{d}^4x\ \delta\mathcal{L} = \frac{1}{c}\int_\Omega \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi^i}\,\delta\phi^i + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\delta\phi^i_{,\mu}\right). \tag{11.5}$$

Here the summation convention is used both for the field index i, to be summed over from 1 to n, and for the spacetime index μ, to be summed over from 0 to 3.

Again we use partial integration in the formula for δS. It follows from Gauss's theorem (the divergence theorem) that

$$\int_\Omega \mathrm{d}^4x\ \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\delta\phi^i_{,\mu} = \int_{\partial\Omega} \mathrm{d}^3S_\mu\ \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\delta\phi^i - \int_\Omega \mathrm{d}^4x\ \left(\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\right)\right)\delta\phi^i\ . \tag{11.6}$$

$\partial\Omega$ is the boundary of the region Ω, and d^3S_μ is the surface element on the boundary.

The notation $\mathrm{d}/\mathrm{d}x^\mu$ introduced here, emphasizes that we are dealing with an absolute differentiation, acting both on the explicit x^μ dependence and on the implicit dependence. Thus, if $F = F(\phi, \partial\phi/\partial x, x)$, and we interpret $\partial/\partial x^\mu$ as differentiation with respect to the explicit x^μ dependence, we have that

$$\frac{\mathrm{d}F}{\mathrm{d}x^\mu} = \frac{\partial F}{\partial\phi^i}\,\phi^i_{,\mu} + \frac{\partial F}{\partial\phi^i_{,\nu}}\,\phi^i_{,\nu\mu} + \frac{\partial F}{\partial x^\mu}\ . \tag{11.7}$$

We choose to restrict the variation $\delta\phi$ such that it vanishes on the boundary $\partial\Omega$. It follows that

$$\delta S = \frac{1}{c}\int_\Omega \mathrm{d}^4x \left(\frac{\partial\mathcal{L}}{\partial\phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\right)\right)\delta\phi^i\ . \tag{11.8}$$

Thus, the functional derivative of S with respect to the field component ϕ^i is

$$\frac{\delta S}{\delta\phi^i} = \frac{1}{c}\left(\frac{\partial\mathcal{L}}{\partial\phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\right)\right)\ . \tag{11.9}$$

The action integral determines the field equations, by Hamilton's principle for fields.

Hamilton's principle. A field configuration ϕ in a spacetime region Ω is dynamically allowed if it is an extremum of the action integral

$$S[\phi] = \frac{1}{c}\int_\Omega \mathrm{d}^4x\, \mathcal{L}\,. \tag{11.10}$$

That is, if the first order variation of $S[\phi]$ vanishes,

$$\delta S = 0\,, \tag{11.11}$$

for every infinitesimal variation $\delta\phi$ vanishing on the boundary $\partial\Omega$.

The field equations that follow from Hamilton's principle, are the Euler–Lagrange equations,

$$\boxed{c\,\frac{\delta S}{\delta\phi^i} = \frac{\partial\mathcal{L}}{\partial\phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\right) = 0\,.} \tag{11.12}$$

Non-relativistic form of the Euler–Lagrange equations

Equation (11.12) has a manifestly relativistic form, it makes no distinction between the time and space dependences of the field. That the coordinate $x^0 = ct$ represents time, has no consequence for the outward appearance of the equation. This does not mean that Hamilton's principle applies only to relativistic field theories, in fact we may write the same Euler–Lagrange equations in the non-relativistic form

$$\frac{\partial\mathcal{L}}{\partial\phi^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,t}}\right) - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,x}}\right) - \frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,y}}\right) - \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,z}}\right) = 0\,, \tag{11.13}$$

distinguishing explicitly between the time t and the space coordinates (x, y, z). We define

$$\phi_{,t} = c\phi_{,0} = \frac{\partial\phi}{\partial t}\,,\quad \phi_{,x} = \phi_{,1} = \frac{\partial\phi}{\partial x}\,,\quad \phi_{,y} = \phi_{,2} = \frac{\partial\phi}{\partial y}\,,\quad \phi_{,z} = \phi_{,3} = \frac{\partial\phi}{\partial z}\,. \tag{11.14}$$

Equation (11.13) may be written more compactly as

$$\frac{\partial\mathcal{L}}{\partial\phi^i} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,t}}\right) - \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{r}}\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\phi^i)}\right) = 0\,. \tag{11.15}$$

Second derivatives in the Lagrange density

Even though there is seldom any need, let us mention how to treat a Lagrange density containing second order derivatives of the field. Equation (11.5) will get an additional term, which we transform by integrating partially twice,

$$\begin{aligned}\int_\Omega \mathrm{d}^4x\,\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu\nu}}\,\delta\phi^i_{,\mu\nu} = {} & \int_{\partial\Omega}\mathrm{d}^3S_\rho\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu\rho}}\,\delta\phi^i_{,\mu} - \left(\frac{\mathrm{d}}{\mathrm{d}x^\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\rho\nu}}\right)\right)\delta\phi^i\right)\\ & + \int_\Omega \mathrm{d}^4x\,\left(\frac{\mathrm{d}^2}{\mathrm{d}x^\mu\,\mathrm{d}x^\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu\nu}}\right)\right)\delta\phi^i\,.\end{aligned} \tag{11.16}$$

A somewhat subtle point is how to interpret the derivative $\partial\mathcal{L}/\partial\phi^i_{,\mu\nu}$. Because of the symmetry $\phi^i_{,\mu\nu} = \phi^i_{,\nu\mu}$, there are 10 and not 16 independent second order derivatives of a field component ϕ^i. Nevertheles, we choose to differentiate the Lagrange density $\mathcal{L}$ with respect to $\phi^i_{,\mu\nu}$ as if all the 16 derivatives were independent variables. The symmetry implies then only that

$$\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu\nu}} = \frac{\partial\mathcal{L}}{\partial\phi^i_{,\nu\mu}} \,. \tag{11.17}$$

As usual, we disregard the boundary terms, with the justification that we choose the variation $\delta\phi$ of the field ϕ such that both $\delta\phi^i = 0$ and $\delta\phi^i_{,\mu} = 0$ on the boundary $\partial\Omega$. The resulting Euler–Lagrange equations are as follows,

$$\boxed{c\,\frac{\delta S}{\delta\phi^i} = \frac{\partial\mathcal{L}}{\partial\phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^\mu\,\mathrm{d}x^\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu\nu}}\right) = 0 \,.} \tag{11.18}$$

11.2 Complex field

Two real fields ϕ^1 and ϕ^2 may be combined into a complex field ϕ, defined such that

$$\phi = \frac{1}{\sqrt{2}}\left(\phi^1 + \mathrm{i}\phi^2\right), \qquad \phi^* = \frac{1}{\sqrt{2}}\left(\phi^1 - \mathrm{i}\phi^2\right). \tag{11.19}$$

The factor $1/\sqrt{2}$ is not essential, but is a much used convention. Obviously, ϕ^* is the complex conjugate of ϕ. Conversely, the real part ϕ^1 and the imaginary part ϕ^2 are expressed in terms of ϕ and ϕ^* as

$$\phi^1 = \frac{1}{\sqrt{2}}\left(\phi + \phi^*\right), \qquad \phi^2 = -\frac{\mathrm{i}}{\sqrt{2}}\left(\phi - \phi^*\right). \tag{11.20}$$

A Lagrange density which is a function of ϕ^1, ϕ^2 and their derivatives may be alternatively regard as a function of ϕ, ϕ^* and their derivatives.

A common and very useful practice is to simplify the notation by treating ϕ and ϕ^* formally as if they, and not ϕ^1 and ϕ^2, were the independent fields. We define

$$\frac{\partial}{\partial\phi} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\phi^1} - \mathrm{i}\,\frac{\partial}{\partial\phi^2}\right), \qquad \frac{\partial}{\partial\phi^*} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\phi^1} + \mathrm{i}\,\frac{\partial}{\partial\phi^2}\right). \tag{11.21}$$

These definitions are self explanatory, because they imply that

$$\frac{\partial\phi}{\partial\phi} = \frac{\partial\phi^*}{\partial\phi^*} = 1 \,, \qquad \frac{\partial\phi^*}{\partial\phi} = \frac{\partial\phi}{\partial\phi^*} = 0 \,. \tag{11.22}$$

The two Euler–Lagrange equations

$$\frac{\partial\mathcal{L}}{\partial\phi^1} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^1_{,\mu}}\right) = 0 \,, \qquad \frac{\partial\mathcal{L}}{\partial\phi^2} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^2_{,\mu}}\right) = 0 \,, \tag{11.23}$$

are equivalent to the two equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi_{,\mu}}\right) = 0 \,, \qquad \frac{\partial \mathcal{L}}{\partial \phi^*} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^*_{,\mu}}\right) = 0 \,. \tag{11.24}$$

We need actually only one of the last two equations. They are not independent, in fact they are complex conjugates of each other, since the Lagrange density $\mathcal{L}$ is always assumed to be real.

11.3 Hamilton's equations

Although we will have no use for it here, we should have a quick look at the Hamilton formalism for fields, which is often taken as the point of departure for quantization. As mentioned already, it is based on a 3+1 dimensional philosophy, where we treat the time dependence separate from the space dependence. The Hamilton formalism is more complicated in field theory than in particle mechanics, in spite of the close analogy.

We describe the field configuration at a given time t by giving the field components as functions of the position $\boldsymbol{r}$,

$$\phi^i = \phi^i(\boldsymbol{r}) = \phi^i(\boldsymbol{r}, t) \,. \tag{11.25}$$

These field components typically do not contain complete information about the *state* of the system at time t, in the sense that the time development is uniquely determined once we have specified the state at one given time. The additional information needed at the given time, in the typical case, is the first derivative of the field with respect to time. The time derivative is another space dependent field $\dot{\phi}$,

$$\dot{\phi}^i = \dot{\phi}^i(\boldsymbol{r}) = \frac{\partial}{\partial t}\,\phi^i(\boldsymbol{r}, t) \,. \tag{11.26}$$

The Lagrange function, as defined in Equation (11.2), we regard now as a functional of the space dependent fields $\phi(\boldsymbol{r})$ and $\dot{\phi}(\boldsymbol{r})$,

$$L = L[\phi(\boldsymbol{r}), \dot{\phi}(\boldsymbol{r})] = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\; \mathcal{L}(\phi(\boldsymbol{r}), \nabla\phi(\boldsymbol{r}), \dot{\phi}(\boldsymbol{r})) \,. \tag{11.27}$$

The Lagrange density $\mathcal{L}$ is typically an ordinary function of $\phi(\boldsymbol{r})$, its gradient $\nabla\phi(\boldsymbol{r})$, and of $\dot{\phi}(\boldsymbol{r})$.

The momentum canonically conjugate to the field ϕ is yet another field π, which is the functional derivative of the Lagrange function with respect to the velocity $\dot{\phi}$,

$$\pi_i = \frac{\delta L}{\delta \dot{\phi}^i} = \frac{\partial \mathcal{L}}{\partial \dot{\phi}^i} \,. \tag{11.28}$$

The Hamilton function is

$$H = \left(\int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\; \pi_i \dot{\phi}^i\right) - L = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}^i}\,\dot{\phi}^i - \mathcal{L}\right) \,. \tag{11.29}$$

Infinitesimal variations $\delta\phi$ and $\delta\dot{\phi}$ of ϕ and $\dot{\phi}$ give the following variation of H,

$$\begin{aligned}\delta H &= \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\delta\pi_i\,\dot{\phi}^i + \pi_i\,\delta\dot{\phi}^i\right) - \delta L = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\delta\pi_i\,\dot{\phi}^i + \pi_i\,\delta\dot{\phi}^i - \frac{\delta L}{\delta\phi^i}\,\phi^i - \frac{\delta L}{\delta\dot{\phi}^i}\,\dot{\phi}^i\right) \\ &= \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\delta\pi_i\,\dot{\phi}^i - \frac{\delta L}{\delta\phi^i}\,\phi^i\right) . \end{aligned} \tag{11.30}$$

This means that it is natural to regard H as a functional of the field ϕ and the conjugate momentum π,

$$H = H[\phi,\pi] = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r}\; \mathcal{H}(\phi(\boldsymbol{r}), \nabla\phi(\boldsymbol{r}), \pi(\boldsymbol{r})) . \tag{11.31}$$

The Hamilton density $\mathcal{H}$, similarly to the Lagrange density $\mathcal{L}$, is an ordinary function of $\phi(\boldsymbol{r})$, $\nabla\phi(\boldsymbol{r})$, and $\pi(\boldsymbol{r})$ at the position $\boldsymbol{r}$. By the definition of the variational derivatives $\delta H/\delta\phi_i$ and $\delta H/\delta\pi^i$ we have that

$$\delta H = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\frac{\delta H}{\delta\phi^i}\,\delta\phi^i + \frac{\delta H}{\delta\pi_i}\,\delta\pi_i\right) . \tag{11.32}$$

Comparison of the two expressions for δH gives the first Hamilton's equation,

$$\dot{\phi}^i = \frac{\delta H}{\delta\pi_i} , \tag{11.33}$$

as well as the following relation between variational derivatives,

$$\left.\frac{\delta H}{\delta\phi^i}\right|_{\pi} = -\left.\frac{\delta L}{\delta\phi^i}\right|_{\dot{\phi}} = -\frac{\partial\mathcal{L}}{\partial\phi^i} + \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\phi^i}\right) . \tag{11.34}$$

The variational derivatives with respect to ϕ^i of H and L, respectively, are taken at constant π, in the first case, and at constant $\dot{\phi}$, in the second case. It is easy to get lost among the various variational, total and partial derivatives!

The second Hamilton's equation,

$$\dot{\pi}_i = -\frac{\delta H}{\delta\phi^i} , \tag{11.35}$$

follows when we use also the Euler–Lagrange equation,

$$\dot{\pi}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}^i}\right) = \frac{\partial\mathcal{L}}{\partial\phi^i} - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial\nabla\phi_i}\right) . \tag{11.36}$$

A general observable of the field would be a functional $F = F[\phi,\pi,t]$, in the general case also with an explicit time dependence. The time development of the fields ϕ and π, according to Hamilton's equations, gives the following time derivative of F,

$$\dot{F} = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\frac{\delta F}{\delta\phi^i}\,\dot{\phi}^i + \frac{\delta F}{\delta\pi_i}\,\dot{\pi}_i\right) + \frac{\partial F}{\partial t} = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\frac{\delta F}{\delta\phi^i}\,\frac{\delta H}{\delta\pi_i} - \frac{\delta F}{\delta\pi_i}\,\frac{\delta H}{\delta\phi^i}\right) + \frac{\partial F}{\partial t} . \tag{11.37}$$

We write it as

$$\dot{F} = \{F,H\} + \frac{\partial F}{\partial t} , \tag{11.38}$$

after defining the Poisson bracket of two observables $F = F[\phi, \pi, t]$ and $G = G[\phi, \pi, t]$,

$$\{F, G\} = \int_{\Omega_3} \mathrm{d}^3\boldsymbol{r} \left(\frac{\delta F}{\delta \phi^i} \frac{\delta G}{\delta \pi_i} - \frac{\delta F}{\delta \pi_i} \frac{\delta G}{\delta \phi^i} \right) . \tag{11.39}$$

The fundamental Poisson brackets are

$$\{\phi^i(\boldsymbol{r}), \phi^j(\boldsymbol{s})\} = 0 \, , \qquad \{\pi_i(\boldsymbol{r}), \pi_j(\boldsymbol{s})\} = 0 \, , \qquad \{\phi^i(\boldsymbol{r}), \pi_j(\boldsymbol{s})\} = \delta^i_j \, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{s}) \, . \tag{11.40}$$

The schematic prescription for quantizing this classical field theory is now to let the field variables $\phi^i(\boldsymbol{r})$ and $\pi_i(\boldsymbol{r})$ become operators on some Hilbert space, and impose the following so called canonical commutation relations, corresponding to the classical Poisson brackets,

$$[\phi^i(\boldsymbol{r}), \phi^j(\boldsymbol{s})] = 0 \, , \qquad [\pi_i(\boldsymbol{r}), \pi_j(\boldsymbol{s})] = 0 \, , \qquad [\phi^i(\boldsymbol{r}), \pi_j(\boldsymbol{s})] = \mathrm{i}\hbar \delta^i_j \, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{s}) \, . \tag{11.41}$$

11.4 Equivalent Lagrange densities

We saw in Chapter 9 that two Lagrange functions L and $\widetilde{L}$ in particle mechanics are equivalent, in the sense that they give the same equation of motion, if the difference between them is an absolute time derivative,

$$\widetilde{L} - L = \frac{\mathrm{d}M}{\mathrm{d}t} \, . \tag{11.42}$$

In fact, a Lagrange function which is an absolute time derivative does not contribute to the equation of motion, simply because its action integral depends only on the positions and velocities at the start and end points of the orbit. Do not forget that this is a *sufficient* condition for equivalence, it is by no means a *necessary* condition!

The corresponding result in field theory is that two Lagrange densities $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are equivalent if the difference between them is a divergence, and the proof is completely analoguous. A Lagrange density which is an absolute divergence,

$$\widetilde{\mathcal{L}} - \mathcal{L} = \frac{\mathrm{d}\mathcal{M}^\mu}{\mathrm{d}x^\mu} \, , \tag{11.43}$$

with $\mathcal{M}^\mu = \mathcal{M}^\mu(\phi, \partial\phi/\partial x, x)$, has an action integral which depends only on the field and its derivatives on the boundary $\partial\Omega$ of the four dimensional region Ω,

$$\widetilde{(S)} - S = \int_\Omega \mathrm{d}^4x \, (\widetilde{\mathcal{L}} - \mathcal{L}) = \int_\Omega \mathrm{d}^4x \, \frac{\mathrm{d}\mathcal{M}^\mu}{\mathrm{d}x^\mu} = \int_{\partial\Omega} \mathrm{d}^3S_\mu \, \mathcal{M}^\mu \, . \tag{11.44}$$

As an alternative proof, direct insertion into the Euler–Lagrange equations (11.18) shows that an absolute divergence gives no contribution, i.e. that

$$\left(\frac{\partial}{\partial \phi^i} - \frac{\mathrm{d}}{\mathrm{d}x^\mu} \frac{\partial}{\partial \phi^i_{,\mu}} + \frac{\mathrm{d}^2}{\mathrm{d}x^\mu \, \mathrm{d}x^\nu} \frac{\partial}{\partial \phi^i_{,\mu\nu}} \right) \frac{\mathrm{d}\mathcal{M}^\rho}{\mathrm{d}x^\rho} = 0 \, , \tag{11.45}$$

identically, without restrictions on the field ϕ. In fact,

$$\frac{\mathrm{d}\mathcal{M}^\rho}{\mathrm{d}x^\rho} = \frac{\partial \mathcal{M}^\rho}{\partial \phi^i} \phi^i_{,\rho} + \frac{1}{2} \left(\frac{\partial \mathcal{M}^\rho}{\partial \phi^i_{,\mu}} + \frac{\partial \mathcal{M}^\mu}{\partial \phi^i_{,\rho}} \right) \phi^i_{,\mu\rho} + \frac{\partial \mathcal{M}^\rho}{\partial x^\rho} \, . \tag{11.46}$$

Here the middle term on the right hand side has been rewritten by means of the symmetry $\phi^i_{,\mu\rho} = \phi^i_{,\rho\mu}$. Equation (11.45) follows from the following three identities,

$$\begin{aligned}
\frac{\partial}{\partial\phi^i}\frac{\mathrm{d}\mathcal{M}^\rho}{\mathrm{d}x^\rho} &= \frac{\mathrm{d}}{\mathrm{d}x^\rho}\frac{\partial\mathcal{M}^\rho}{\partial\phi^i}\,, \\
\frac{\partial}{\partial\phi^i_{,\mu}}\frac{\mathrm{d}\mathcal{M}^\rho}{\mathrm{d}x^\rho} &= \frac{\partial\mathcal{M}^\mu}{\partial\phi^i} + \frac{\mathrm{d}}{\mathrm{d}x^\rho}\frac{\partial\mathcal{M}^\rho}{\partial\phi^i_{,\mu}}\,, \\
\frac{\partial}{\partial\phi^i_{,\mu\nu}}\frac{\mathrm{d}\mathcal{M}^\rho}{\mathrm{d}x^\rho} &= \frac{1}{2}\left(\frac{\partial\mathcal{M}^\nu}{\partial\phi^i_{,\mu}} + \frac{\partial\mathcal{M}^\mu}{\partial\phi^i_{,\nu}}\right)\,.
\end{aligned} \tag{11.47}$$

To prove these, we use that the partial derivatives $\partial/\partial\phi^i$, $\partial/\partial\phi^i_{,\mu}$ and $\partial/\partial x^\mu$ commute. To prove Equation (11.45), we use in addition the fact that the "absolute partial derivatives" $\mathrm{d}/\mathrm{d}x^\mu$ and $\mathrm{d}/\mathrm{d}x^\nu$ commute.

Problems

1. Check that all the terms in the Euler–Lagrange equation (11.12) have the same dimension.

2. Given a string with a constant mass density μ (mass per length).
   Choose the x axis along the string. The end points lie fixed at $x = 0$ and $x = A$, but for $0 < x < A$ the string is allowed to oscillate freely, with displacements $y = y(x,t)$ and $z = z(x,t)$ varying with the position x along the string and with the time t.
   Assume that the displacements are small, so that the length of the string, ℓ, varies little. Then the string tension σ is (approximately) constant, and the potential energy is $V = \sigma\ell$, up to an additive constant.

   a) Show that for small displacements the Lagrange function is (up to an additive constant)

   $$L = \int_0^A \mathrm{d}x \left(\frac{1}{2}\mu \left(\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2 \right) - \frac{1}{2}\sigma \left(\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2 \right) \right).$$

   b) Write down the Euler–Lagrange equations.
   Solve them (that is, find the most general solution).

   c) The functions $y = y(x,t)$ and $z = z(x,t)$ may be Fourier expanded as functions of x, with coefficients that are functions of t.
   Express the Lagrange function L in terms of the Fourier components of y and z.
   Write down the Euler–Lagrange equations for the Fourier components, and solve them. Compare to the general solution from point b) above.

Chapter 12

Symmetries and conservation laws for fields

In the present chapter we will see how to translate into the language of field theory the discussion in Chapter 10 of transformations, symmetries and conservation laws in particle mechanics. Since the concepts should be more or less familiar from Chapter 10, and since their adaptation to a new context is rather straightforward, we will concentrate mainly on the new features.

12.1 Field transformations

A field transformation

$$\phi \mapsto \widetilde{\phi} \tag{12.1}$$

is a rule which assigns to any given field configuration

$$\phi = \phi^i(x) = \phi^i(x^0, x^1, x^2, x^3) \tag{12.2}$$

another uniquely defined field configuration

$$\widetilde{\phi} = \widetilde{\phi}^i(x) = \widetilde{\phi}^i(x^0, x^1, x^2, x^3) \ . \tag{12.3}$$

We assume here most of the time that the transformation is invertible, so that there exists an inverse transformation $\widetilde{\phi} \mapsto \phi$.

In a similar way as we only considered a very restricted class of transformations of orbits in particle mechanics, the class of field transformations we consider here will be very restricted. In general, we may imagine that the field ϕ as a whole determines the transformed field $\widetilde{\phi}$ in some complicated way, but most of the time, if not always, the most general type of transformation we are going to meet with will be a linear transformation of the special form

$$\widetilde{\phi}^i(\widetilde{x}) = U^i{}_j(x)\, \phi^j(x) \ . \tag{12.4}$$

It may involve a transformation of space and time, $x \to \widetilde{x}$, which is a coordinate transformation if we interpret it as a passive transformation. In addition, it may involve a linear transformation $U^i{}_j(x)$ acting on the field index j.

This transformation is *local* in the sense that the value of the transformed field $\widetilde{\phi}$ at a given point $\widetilde{x}$ depends only on the value of the original field ϕ at one point x. It is called a *gauge transformation* in the case when $\widetilde{x} = x$, so that there is no spacetime transformation involved. By definition, a gauge transformation is *local* when the matrix $U^i{}_j(x)$ varies with x, as opposed to *global* when $U^i{}_j(x)$ is constant, independent of x. (Note that gauge transformations look different in the case of a gauge field such as the electromagnetic field.)

Familiar examples are the transformations of a scalar field ϕ,

$$\widetilde{\phi}(\widetilde{x}) = \phi(x) \ , \tag{12.5}$$

of a contravariant vector field A^μ,

$$\widetilde{A}^\mu(\widetilde{x}) = \frac{\partial \widetilde{x}^\mu}{\partial x^\nu} \, A^\nu(x) \ , \tag{12.6}$$

of a covariant vector field B_μ,

$$\widetilde{B}_\mu(\widetilde{x}) = \frac{\partial x^\nu}{\partial \widetilde{x}^\mu} \, B_\nu(x) \ , \tag{12.7}$$

and of a scalar density field D,

$$\widetilde{D}(\widetilde{x}) = \left| \det\left(\frac{\partial x}{\partial \widetilde{x}} \right) \right| \, D(x) \ . \tag{12.8}$$

In the special theory of relativity we are dealing with translations and Lorentz transformations, and then there is no distinction between a scalar and a scalar density, because the Jacobi determinant is always unity,

$$\det\left(\frac{\partial x}{\partial \widetilde{x}} \right) = 1 \ . \tag{12.9}$$

When discussing symmetries and conservation laws, we are particularly interested in infinitesimal transformations of a field ϕ, which in general will have the form

$$\phi(x) \mapsto \widetilde{\phi}(x) = \phi(x) + \Delta\phi(x) \ . \tag{12.10}$$

with $\Delta\phi$ infinitesimal. Note that in this formula we compare the field ϕ and the transformed field $\widetilde{\phi}$ at the same point x in space and time, whereas in Equation (12.4) we compare $\phi(x)$ and $\widetilde{\phi}(\widetilde{x})$.

Consider in particular the infinitesimal coordinate transformation $x \mapsto \widetilde{x} = x + \xi$, with $\xi = \xi^\mu(x)$ infinitesimal. To first order in ξ we have $\xi(\widetilde{x}) = \xi(x)$, thus we may write the inverse transformation as

$$\widetilde{x} \mapsto x = \widetilde{x} - \xi(x) = \widetilde{x} - \xi(\widetilde{x}) \ . \tag{12.11}$$

According to our general transformation rule, $\widetilde{\phi}(\widetilde{x})$ is given by $\phi(x) = \phi(\widetilde{x} - \xi(\widetilde{x}))$, which is the same as saying that $\widetilde{\phi}(x)$ is given by

$$\phi(x - \xi(x)) = \phi(x) - \xi^\mu(x) \, \phi_{,\mu}(x) \ . \tag{12.12}$$

For an infinitesimal coordinate transformation $\widetilde{x} = x + \xi$ also the matrix $U^i{}_j(x)$ must be infinitesimally different from the identity matrix, that is, we must have

$$U^i{}_j(x) = \delta^i_j + \sigma^i_j(x) \ , \tag{12.13}$$

with σ^i_j infinitesimal. Thus, the result of the infinitesimal coordinate transformation $\widetilde{x} = x+\xi$ will be a field transformation of the form

$$\widetilde{\phi}^i(x) = (\delta^i_j + \sigma^i_j(x))(\phi^j(x) - \xi^\mu(x)\,\phi^j_{,\mu}(x)) \ . \tag{12.14}$$

Or equivalently, $\widetilde{\phi} = \phi + \Delta\phi$, with

$$\Delta\phi^i = \sigma^i_j\,\phi^j - \xi^\mu\phi^j_{,\mu} \ . \tag{12.15}$$

When written in this form, the transformation involves not only the value $\phi(x)$ of the field at the point x, but also the derivative $\phi_{,\mu}(x)$.

We saw in Chapter 2 that we may define a special kind of differentiation, the Lie derivative, which describes how general tensor fields transform under infinitesimal coordinate transformations. In particular, we have for a scalar field ϕ,

$$\Delta\phi = -\xi^\mu\phi_{,\mu} \ , \tag{12.16}$$

for a contravariant vector field A^μ,

$$\Delta A^\mu = -\xi^\nu A^\mu_{,\nu} + \xi^\mu_{,\nu}A^\nu \ , \tag{12.17}$$

for a covariant vector field B_μ,

$$\Delta B_\mu = -\xi^\nu B_{\mu,\nu} - \xi^\nu_{,\mu}B_\nu \ , \tag{12.18}$$

and for a scalar density field D,

$$\Delta D = -\xi^\mu D_{,\mu} - \xi^\mu_{,\mu}D = -\frac{\partial(\xi^\mu D)}{\partial x^\mu} \ . \tag{12.19}$$

For a volume preserving infinitesimal coordinate transformation the divergence of the infinitesimal displacement ξ vanishes, that is, we have $\xi^\mu_{,\mu} = 0$. In that case, there is no distinction between a scalar and a scalar density.

12.2 Symmetries

We say that a field transformation $\phi \mapsto \widetilde{\phi}$ is a *symmetry* if it preserves the field equations, in the sense that the transformed field configuration $\widetilde{\phi}$ is a solution of the field equations if and only if the original field configuration ϕ is a solution. In addition we require that there must exist an inverse transformation $\widetilde{\phi} \mapsto \phi$.

According to Hamilton's principle, the field configurations ϕ and $\widetilde{\phi}$ are solutions of the field equations if and only if the action integrals $S[\phi]$ and $S[\widetilde{\phi}]$ are extremal. Let us define

$$\widetilde{S}[\phi] = S[\widetilde{\phi}] \ . \tag{12.20}$$

The condition for the transformation $\phi \mapsto \widetilde{\phi}$ to be a symmetry, is that the two action integrals $S[\phi]$ and $\widetilde{S}[\phi]$ are equivalent, i.e. that they give the same Euler–Lagrange equations for ϕ.

This may of course happen in many different ways. The simplest possibility is that the Lagrange density and hence the action integral is invariant, i.e. that $\widetilde{S} = S$. A more general possibility is that the transformed Lagrange density $\widetilde{\mathcal{L}}$, defined by the relation

$$\widetilde{\mathcal{L}}\left(\phi, \frac{\partial\phi}{\partial x}, x\right) = \mathcal{L}\left(\widetilde{\phi}, \frac{\partial\widetilde{\phi}}{\partial x}, x\right) \ , \tag{12.21}$$

differs from the original Lagrange density $\mathcal{L}$ by an absolute divergence.

As we will see next, an infinitesimal transformation of this type will give a conservation law. This is the field theory version of Noether's theorem. There exist continuous symmetries to which Noether's theorem does not apply, for example giving $\widetilde{S} = CS$ where C is a constant different from one.

In the case of an infinitesimal field transformation $\phi \mapsto \widetilde{\phi} = \phi + \Delta\phi$, we have that $\widetilde{\mathcal{L}} = \mathcal{L} + \Delta\mathcal{L}$, where $\Delta\mathcal{L}$ is the infinitesimal variation of the Lagrange density,

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^i}\,\Delta\phi^i + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\Delta\phi^i_{,\mu}\,. \tag{12.22}$$

The condition for this transformation to be a symmetry, is that the two Lagrange densities $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ give the same field equations for ϕ. A sufficient, but not necessary condition is that $\Delta\mathcal{L}$ is a divergence, i.e. that there exists a vector

$$\Delta\mathcal{M}^\mu = \Delta\mathcal{M}^\mu\left(\phi, \frac{\partial\phi}{\partial x}, x\right) \tag{12.23}$$

such that

$$\Delta\mathcal{L} = \frac{\mathrm{d}(\Delta\mathcal{M}^\mu)}{\mathrm{d}x^\mu} \tag{12.24}$$

for any field configuration ϕ. An infinitesimal transformation which is a symmetry because it satisfies Equation (12.24), is called a Noether symmetry in field theory.

The Lagrange density may be invariant in the sense that $\Delta\mathcal{L} = 0$, and then we choose simply $\Delta\mathcal{M}^\mu = 0$. But the Lagrange density may also be invariant in a slightly different sense. In fact, we often say that the Lagrange density is invariant under a coordinate transformation $x \to \widetilde{x}$, meaning that it transforms as a scalar,

$$\widetilde{\mathcal{L}} = \mathcal{L}\,, \tag{12.25}$$

or rather as a scalar density,

$$\widetilde{\mathcal{L}} = \left|\det\left(\frac{\partial x}{\partial\widetilde{x}}\right)\right|\mathcal{L}\,, \tag{12.26}$$

where $\widetilde{\mathcal{L}}$ is evaluated at $\widetilde{x}$ and $\mathcal{L}$ is evaluated at x. As we have seen, when $\mathcal{L}$ is a scalar density, or a scalar in the case of a volume preserving coordinate transformation, then we have not $\Delta\mathcal{L} = 0$, but instead

$$\Delta\mathcal{L} = -\frac{\partial(\xi^\mu\mathcal{L})}{\partial x^\mu}\,. \tag{12.27}$$

12.3 Noether's theorem

If the Lagrange density $\mathcal{L}$ depends on the field ϕ only through the first order derivatives $\partial\phi/\partial x$, i.e. if $\partial\mathcal{L}/\partial\phi = 0$, then $\mathcal{L}$ is invariant under the addition of a constant value to ϕ, and the Euler–Lagrange equation has the form of a conservation law,

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\mu}}\right) = 0\,. \tag{12.28}$$

Noether's theorem in field theory is a generalization of this simple observation. It applies to any infinitesimal field transformation $\phi \mapsto \widetilde{\phi} = \phi + \Delta\phi$ which is a symmetry because there exists a vector $\Delta\mathcal{M}^\mu = \Delta\mathcal{M}^\mu(\phi, \partial\phi/\partial x, x)$ such that Equation (12.24) holds.

Any conservation law is of course a consequence of the field equations, which we have so far not used. If the field configuration ϕ is a solution of the Euler–Lagrange equation, then the variation of the Lagrange density is a divergence, for an arbitrary variation $\Delta\phi$, since

$$\Delta\mathcal{L} = \left(\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\right)\right)\Delta\phi^i + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\Delta\phi^i_{,\mu} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\Delta\phi^i\right). \tag{12.29}$$

When we are able to write the variation $\Delta\mathcal{L}$ as a divergence in two different ways, first by using that the transformation $\phi \mapsto \phi + \Delta\phi$ is a symmetry, and next by assuming that the original field configuration ϕ is a solution of the field equations, then we have a conservation law,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\Delta\phi^i - \Delta\mathcal{M}^\mu\right) = 0\,.} \tag{12.30}$$

This relation is a continuity equation, written in the relativistic form

$$\frac{\partial j^\mu}{\partial x^\mu} = 0\,. \tag{12.31}$$

Like the Euler–Lagrange equation it has an equivalent non-relativistic form,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0\,, \tag{12.32}$$

where the density ρ and the three dimensional current density $\boldsymbol{j}$ are given by the relations

$$\rho = \frac{j^0}{c} = \frac{\partial\mathcal{L}}{\partial\phi^i_{,t}}\,\Delta\phi^i - \frac{\Delta\mathcal{M}^0}{c}\,, \qquad j_x = \frac{\partial\mathcal{L}}{\partial\phi^i_{,x}}\,\Delta\phi^i - \Delta\mathcal{M}^1\,,\ldots\,. \tag{12.33}$$

12.4 Gauge invariance

The simplest example of a symmetry giving a conservation law is an infinitesimal gauge transformation of the form

$$\widetilde{\phi}^{\,i}(\widetilde{x}) = \left(\delta^i_j + \tau^i{}_j\right)\phi^j(x)\,, \tag{12.34}$$

when it leaves the Lagrange density invariant. We take the infinitesimal matrix τ to be constant in space and time, so that this is a *global* transformation. Writing $\widetilde{\phi} = \phi + \Delta\phi$, we have that

$$\Delta\phi^i = \tau^i{}_j\,\phi^j\,. \tag{12.35}$$

The corresponding variation of the Lagrange density is

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi^i}\,\Delta\phi^i + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\Delta\phi^i_{,\mu} = \tau^i{}_j\left(\frac{\partial\mathcal{L}}{\partial\phi^i}\,\phi^j + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\phi^j_{,\mu}\right), \tag{12.36}$$

and gauge invariance means, in most cases, that $\Delta\mathcal{L} = 0$.

Noether's teorem gives the following conservation law, valid when the field fulfils the Euler–Lagrange equation,

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\Delta\phi^i\right) = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\tau^i{}_j\phi^j\right) = 0\,. \tag{12.37}$$

It is rather remarkable that a *global* symmetry gives rise to a *local* conservation law, we seem to get much more than we ask for. The explanation must be that locality is built into the Lagrange formalism, in that the Lagrange density $\mathcal{L}$ is a function of the field and its derivatives at one point in space and time. In the case of relativistic field theories we have the additional argument that only local conservation laws are consistent with Lorentz invariance.

12.5 Translation invariance

Let $d = (d^0, d^1, d^2, d^3)$ be constant, and consider the translation $x^\mu \mapsto x^\mu + d^\mu$. The translated field $\widetilde{\phi}$ is given by the relation

$$\widetilde{\phi}^{\,i}(x+d) = \phi^i(x), \tag{12.38}$$

or equivalently,

$$\widetilde{\phi}^{\,i}(x) = \phi^i(x-d) = \phi^i(x) - d^\mu\,\phi^i_{,\mu}(x)\,. \tag{12.39}$$

The last equality holds when d is infinitesimal, and then

$$\Delta\phi^i(x) = -d^\mu\,\phi^i_{,\mu}(x)\,, \tag{12.40}$$

in the notation used above. Equation (12.22) gives that

$$\Delta\mathcal{L} = -d^\mu\left(\frac{\partial\mathcal{L}}{\partial\phi^i}\,\phi^i_{,\mu} + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\nu}}\,\phi^i_{,\mu\nu}\right) = -d^\mu\left(\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}x^\mu} - \frac{\partial\mathcal{L}}{\partial x^\mu}\right). \tag{12.41}$$

The condition for translation invariance in the given direction d is that the Lagrange density $\mathcal{L}$ does not depend explicitly on x, in this direction, i.e. that

$$d^\mu\,\frac{\partial\mathcal{L}}{\partial x^\mu} = 0\,, \tag{12.42}$$

where the differentiation is with both ϕ and $\partial\phi/\partial x$ constant. In that case $\Delta\mathcal{L}$ is a divergence,

$$\Delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(-d^\mu\mathcal{L}\right). \tag{12.43}$$

As we have seen, this means that $\mathcal{L}$ transforms as a scalar density.

Using Noether's theorem, we get the following conservation law,

$$\frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\left(-d^\nu\,\phi^i_{,\nu}\right) + d^\mu\mathcal{L}\right) = 0\,. \tag{12.44}$$

This may be written as

$$\frac{\mathrm{d}\left(-d^\nu \widehat{T}_\nu{}^\mu\right)}{\mathrm{d}x^\mu} = 0 \,, \tag{12.45}$$

where $\widehat{T}_\nu{}^\mu$ is the *canonical energy momentum tensor*,

$$\widehat{T}_\nu{}^\mu = \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} \, \phi^i_{,\nu} - \delta^\mu_\nu \mathcal{L} \,. \tag{12.46}$$

The present derivation leads to the canonical energy momentum tensor with one upper and one lower index. Raising the lower index gives the following result,

$$\widehat{T}^{\mu\nu} = g^{\mu\lambda}\widehat{T}_\lambda{}^\nu = \frac{\partial \mathcal{L}}{\partial \phi^i_{,\nu}} \, \phi^{i,\mu} - g^{\mu\nu}\mathcal{L} \,. \tag{12.47}$$

Translation invariance in one direction d gives the conserved current $d^\nu \widehat{T}_\nu{}^\mu$. Translation invariance in all directions simultaneously gives a conserved tensor $\widehat{T}_\nu{}^\mu$, such that

$$\frac{\mathrm{d}\widehat{T}_\nu{}^\mu}{\mathrm{d}x^\mu} = 0 \,. \tag{12.48}$$

Here we see one more example of what we have seen before, that a global symmetry gives a local conservation law.

12.6 Lorentz invariance

An infinitesimal Lorentz transformation is of the form

$$x^\lambda \mapsto \widetilde{x}^\lambda = x^\lambda + \omega^\lambda{}_\nu \, x^\nu = x^\lambda + \frac{1}{2}\,\omega_{\mu\nu}\left(g^{\lambda\mu}x^\nu - g^{\lambda\nu}x^\mu\right), \tag{12.49}$$

where ω is infinitesimal, and $\omega_{\mu\nu} = g_{\mu\lambda}\omega^\lambda{}_\nu$ is antisymmetric,

$$\omega_{\nu\mu} = -\omega_{\mu\nu} \,. \tag{12.50}$$

This is an infinitesimal coordinate transformation $x \mapsto \widetilde{x} = x + \xi$, with

$$\xi^\lambda = \omega^\lambda{}_\nu \, x^\nu = \frac{1}{2}\,\omega_{\mu\nu}\left(g^{\lambda\mu}x^\nu - g^{\lambda\nu}x^\mu\right). \tag{12.51}$$

Taking a contravariant vector field as an example, Equation (12.17) shows how it transforms,

$$\Delta A^\lambda = -\xi^\sigma A^\lambda_{,\sigma} + \xi^\lambda_{,\rho} A^\rho = \frac{1}{2}\,\omega_{\mu\nu}\left(-\left(g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu\right)A^\lambda_{,\sigma} + \left(g^{\lambda\mu}\delta^\nu_\rho - g^{\lambda\nu}\delta^\mu_\rho\right)A^\rho\right). \tag{12.52}$$

Defining

$$S^\lambda{}_\rho{}^{\mu\nu} = -S^\lambda{}_\rho{}^{\nu\mu} = g^{\lambda\mu}\delta^\nu_\rho - g^{\lambda\nu}\delta^\mu_\rho \,, \tag{12.53}$$

we may regard this as a special case of the general transformation formula

$$\Delta\phi^i = \frac{1}{2}\,\omega_{\mu\nu}\left(-\left(g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu\right)\phi^i_{,\sigma} + S^i{}_j{}^{\mu\nu}\phi^j\right). \tag{12.54}$$

Here $S^i{}_j{}^{\mu\nu} = -S^i{}_j{}^{\nu\mu}$ represents the *spin* of the field. The spin is zero for scalar fields, but not for spinor, vector or general tensor fields. Note that $\Delta\phi$ is a sum of what we may call an *orbital* part,

$$(\Delta\phi^i)_{\text{orbital}} = -\frac{1}{2}\,\omega_{\mu\nu}\,(g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu)\,\phi^i_{,\sigma}\,, \tag{12.55}$$

which is present for fields with or without spin, and a *spin* part,

$$(\Delta\phi^i)_{\text{spin}} = \frac{1}{2}\,\omega_{\mu\nu}S^i{}_j{}^{\mu\nu}\phi^j\,. \tag{12.56}$$

The derivative of $\Delta\phi$ is

$$\Delta\phi^i_{,\rho} = \frac{1}{2}\,\omega_{\mu\nu}\left(-\left(g^{\sigma\mu}\delta^\nu_\rho - g^{\sigma\nu}\delta^\mu_\rho\right)\phi^i_{,\sigma} - (g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu)\,\phi^i_{,\sigma\rho} + S^i{}_j{}^{\mu\nu}\phi^j_{,\rho}\right). \tag{12.57}$$

Inserting this in Equation (12.22), we may write the result as a sum of three terms,

$$\Delta\mathcal{L} = \frac{1}{2}\,\omega_{\mu\nu}\,(\mathcal{A}^{\mu\nu} + \mathcal{B}^{\mu\nu} + \mathcal{C}^{\mu\nu})\,, \tag{12.58}$$

where

$$\begin{aligned}
\mathcal{A}^{\mu\nu} &= -\frac{\partial\mathcal{L}}{\partial\phi^i_{,\rho}}\left(g^{\sigma\mu}\delta^\nu_\rho - g^{\sigma\nu}\delta^\mu_\rho\right)\phi^i_{,\sigma}\,,\\
\mathcal{B}^{\mu\nu} &= S^i{}_j{}^{\mu\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i}\,\phi^j + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\rho}}\,\phi^j_{,\rho}\right)\,,\\
\mathcal{C}^{\mu\nu} &= -\,(g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu)\left(\frac{\partial\mathcal{L}}{\partial\phi^i}\,\phi^i_{,\sigma} + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\rho}}\,\phi^i_{,\sigma\rho}\right).
\end{aligned} \tag{12.59}$$

Note that $-\mathcal{A}^{\mu\nu}$ is the antisymmetric part of the canonical energy momentum tensor,

$$\widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu} = \frac{\partial\mathcal{L}}{\partial\phi^i_{,\nu}}\,\phi^{i,\mu} - \frac{\partial\mathcal{L}}{\partial\phi^i_{,\mu}}\,\phi^{i,\nu} = -\mathcal{A}^{\mu\nu}\,. \tag{12.60}$$

Lorentz invariance will follow as a consequence if the Lagrange density transforms as a scalar density, that is, if

$$\Delta\mathcal{L} = -\frac{\mathrm{d}(\xi^\sigma\mathcal{L})}{\mathrm{d}x^\sigma}\,. \tag{12.61}$$

We now impose two conditions that together will be sufficient. The first condition is that $\mathcal{A}^{\mu\nu} + \mathcal{B}^{\mu\nu} = 0$. It may be interpreteded as a condition on the antisymmetric part of the canonical energy momentum tensor, that

$$\widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu} = -\mathcal{A}^{\mu\nu} = \mathcal{B}^{\mu\nu} = S^i{}_j{}^{\mu\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i}\,\phi^j + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\rho}}\,\phi^j_{,\rho}\right)\,. \tag{12.62}$$

In some cases, for example with a scalar field, Lorentz invariance holds because both $\mathcal{A}^{\mu\nu} = 0$ and $\mathcal{B}^{\mu\nu} = 0$. Then the canonical energy momentum tensor is symmetric, $\widehat{T}^{\mu\nu} = \widehat{T}^{\nu\mu}$.

The second condition is that the Lagrange density $\mathcal{L}$ does not depend explicitly on space and time. It implies that

$$\mathcal{C}^{\mu\nu} = -\left(g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu\right)\frac{\mathrm{d}\mathcal{L}}{\mathrm{d}x^\sigma} = -\frac{\mathrm{d}}{\mathrm{d}x^\sigma}\left(\left(g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu\right)\mathcal{L}\right). \tag{12.63}$$

The last equality follows because

$$\frac{\partial}{\partial x^\sigma}\left(g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu\right) = g^{\sigma\mu}\delta^\nu_\sigma - g^{\sigma\nu}\delta^\mu_\sigma = g^{\nu\mu} - g^{\mu\nu} = 0\,, \tag{12.64}$$

which is just another proof that the Lorentz transformation preserves volume.

Under the above two conditions the infinitesimal Lorentz transformation is a symmetry, since

$$\Delta\mathcal{L} = \frac{1}{2}\,\omega_{\mu\nu}\mathcal{C}^{\mu\nu} = \frac{\mathrm{d}}{\mathrm{d}x^\lambda}\left(-\frac{1}{2}\,\omega_{\mu\nu}\left(g^{\lambda\mu}x^\nu - g^{\lambda\nu}x^\mu\right)\mathcal{L}\right). \tag{12.65}$$

As a consequence, Noether's theorem gives the following conservation law,

$$\frac{\mathrm{d}}{\mathrm{d}x^\rho}\left(\frac{1}{2}\,\omega_{\mu\nu}\left(\frac{\partial\mathcal{L}}{\partial\phi^i_{,\rho}}\left(-\left(g^{\sigma\mu}x^\nu - g^{\sigma\nu}x^\mu\right)\phi^i_{,\sigma} + S^i{}_j{}^{\mu\nu}\phi^j\right) + \left(g^{\rho\mu}x^\nu - g^{\rho\nu}x^\mu\right)\mathcal{L}\right)\right) = 0\,. \tag{12.66}$$

We may express this slightly simpler in terms of the canonical energy momentum tensor, as

$$\frac{\mathrm{d}}{\mathrm{d}x^\rho}\left(\frac{1}{2}\,\omega_{\mu\nu}\left(-x^\nu\widehat{T}^{\mu\rho} + x^\mu\widehat{T}^{\nu\rho} + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\rho}}\,S^i{}_j{}^{\mu\nu}\phi^j\right)\right) = 0\,. \tag{12.67}$$

We define the *canonical angular momentum tensor* $\widehat{M}^{\mu\nu\rho}$ as

$$\widehat{M}^{\mu\nu\rho} = x^\mu\widehat{T}^{\nu\rho} - x^\nu\widehat{T}^{\mu\rho} + \frac{\partial\mathcal{L}}{\partial\phi^i_{,\rho}}\,S^i{}_j{}^{\mu\nu}\phi^j\,. \tag{12.68}$$

For a single infinitesimal Lorentz transformation, given by the antisymmetric matrix $\omega_{\mu\nu}$, we have a conserved current,

$$\frac{\mathrm{d}}{\mathrm{d}x^\rho}\left(\frac{1}{2}\,\omega_{\mu\nu}\widehat{M}^{\mu\nu\rho}\right) = 0\,. \tag{12.69}$$

If all infinitesimal Lorentz transformations are symmetries, then the tensor $\widehat{M}^{\mu\nu\rho}$ is conserved, that is

$$\frac{\mathrm{d}\widehat{M}^{\mu\nu\rho}}{\mathrm{d}x^\rho} = 0\,. \tag{12.70}$$

A tensor of rank three has $4^3 = 64$ components, but not always that many independent components. Since $\widehat{M}^{\mu\nu\rho}$ is antisymmetric in the first two indices, and since an antisymmetric tensor of rank two has 6 independent components, the number of independent components of $\widehat{M}^{\mu\nu\rho}$ is $6 \times 4 = 24$.

12.7 Symmetrization of the energy momentum tensor

Combining Equation (12.62), which is one of the conditions for Lorentz invariance, with the Euler–Lagrange equations, we get that

$$\widehat{T}^{\mu\nu} - \widehat{T}^{\nu\mu} = \frac{\mathrm{d}}{\mathrm{d}x^\rho}\left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\rho}} S^i{}_j{}^{\mu\nu}\phi^j\right). \tag{12.71}$$

This equation, valid when the theory is Lorentz invariant and the field ϕ satisfies the Euler–Lagrange equations, may be used for symmetrizing $\widehat{T}^{\mu\nu}$. The symmetrized energy momentum tensor, named after Belinfante, is defined as

$$T^{\mu\nu} = \widehat{T}^{\mu\nu} + \frac{\mathrm{d}\psi^{\mu\nu\rho}}{\mathrm{d}x^\rho}\,. \tag{12.72}$$

Here $\psi^{\mu\nu\rho}$ has to be chosen such that, first, $\mathrm{d}(\psi^{\mu\nu\rho} - \psi^{\nu\mu\rho})/\mathrm{d}x^\rho$ equals minus the right hand side of Equation (12.71), and, second, that $\psi^{\mu\rho\nu} = -\psi^{\mu\nu\rho}$. Both these conditions are satisfied when we take

$$\psi^{\mu\nu\rho} = -\frac{1}{2}\left(\frac{\partial \mathcal{L}}{\partial \phi^i_{,\rho}} S^i{}_j{}^{\mu\nu} - \frac{\partial \mathcal{L}}{\partial \phi^i_{,\nu}} S^i{}_j{}^{\mu\rho} - \frac{\partial \mathcal{L}}{\partial \phi^i_{,\mu}} S^i{}_j{}^{\nu\rho}\right)\phi^j\,. \tag{12.73}$$

This choice implies that $T^{\mu\nu}$ is symmetric, $T^{\nu\mu} = T^{\mu\nu}$, maybe not for an arbitrary field configuration, but at least for a field configuration satisfying the Euler–Lagrange equations. Furthermore, $T^{\mu\nu}$ is conserved as a consequence of the Euler–Lagrange equations,

$$\frac{\mathrm{d}T^{\mu\nu}}{\mathrm{d}x^\nu} = \frac{\mathrm{d}\widehat{T}^{\mu\nu}}{\mathrm{d}x^\nu} + \frac{\mathrm{d}^2\psi^{\mu\nu\rho}}{\mathrm{d}x^\nu \mathrm{d}x^\rho} = \frac{\mathrm{d}\widehat{T}^{\mu\nu}}{\mathrm{d}x^\nu} = 0\,, \tag{12.74}$$

where we use that $\psi^{\mu\nu\rho}$ is antisymmetric in the last two indices.

The two tensors $\widehat{T}^{\mu\nu}$ and $T^{\mu\nu}$ give the same total four-momentum $P^\mu = \int \mathrm{d}^3\boldsymbol{r}\, T^{\mu 0}$, assuming that the field ϕ and its derivatives go to zero sufficiently fast when the spatial coordinates go to infinity. In fact, we have that $\psi^{\mu 00} = 0$, and

$$\int \mathrm{d}^3\boldsymbol{r}\; T^{\mu 0} = \int \mathrm{d}^3\boldsymbol{r}\left(\widehat{T}^{\mu 0} + \frac{\mathrm{d}\psi^{\mu 01}}{\mathrm{d}x^1} + \frac{\mathrm{d}\psi^{\mu 02}}{\mathrm{d}x^2} + \frac{\mathrm{d}\psi^{\mu 03}}{\mathrm{d}x^3}\right) = \int \mathrm{d}^3\boldsymbol{r}\; \widehat{T}^{\mu 0}\,, \tag{12.75}$$

since the three dimensional divergence may be integrated to a vanishing surface integral.

If we define

$$M^{\mu\nu\rho} = x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho}\,, \tag{12.76}$$

then we have that

$$\begin{aligned} M^{\mu\nu\rho} - \widehat{M}^{\mu\nu\rho} &= x^\mu \frac{\mathrm{d}\psi^{\nu\rho\sigma}}{\mathrm{d}x^\sigma} - x^\nu \frac{\mathrm{d}\psi^{\mu\rho\sigma}}{\mathrm{d}x^\sigma} - \frac{\partial \mathcal{L}}{\partial \phi^i_{,\rho}} S^i{}_j{}^{\mu\nu}\phi^j \\ &= \frac{\mathrm{d}}{\mathrm{d}x^\sigma}\left(x^\mu\psi^{\nu\rho\sigma} - x^\nu\psi^{\mu\rho\sigma}\right) - \psi^{\nu\rho\mu} + \psi^{\mu\rho\nu} - \frac{\partial \mathcal{L}}{\partial \phi^i_{,\rho}} S^i{}_j{}^{\mu\nu}\phi^j \\ &= \frac{\mathrm{d}}{\mathrm{d}x^\sigma}\left(x^\mu\psi^{\nu\rho\sigma} - x^\nu\psi^{\mu\rho\sigma}\right). \end{aligned} \tag{12.77}$$

The two tensors $M^{\mu\nu\rho}$ and $\widehat{M}^{\mu\nu\rho}$ give the same total angular momentum, in the same way as $T^{\mu\nu}$ and $\widehat{T}^{\mu\nu}$ give the same total momentum. $M^{\mu\nu\rho}$ is conserved, this follows e.g. from the fact that $T^{\mu\nu}$ is conserved and symmetric,

$$\frac{\mathrm{d}M^{\mu\nu\rho}}{\mathrm{d}x^{\rho}} = T^{\nu\mu} - T^{\mu\nu} + x^{\mu}\frac{\mathrm{d}T^{\nu\rho}}{\mathrm{d}x^{\rho}} - x^{\nu}\frac{\mathrm{d}T^{\mu\rho}}{\mathrm{d}x^{\rho}} = 0\,. \tag{12.78}$$

Problems

1. Show that an infinitesimal translation of a scalar field ϕ a distance ϵ in the x direction may be written as

$$\Delta\phi = -\epsilon \frac{\mathrm{i}}{\hbar} p_x \phi \,,$$

where p_x is the x component of the quantum mechanical momentum operator $\boldsymbol{p} = -\mathrm{i}\hbar\nabla$. For this reason, $\boldsymbol{p}$ is said to be the infinitesimal generator of translations.

2. The Lorentz transformations discussed in this chapter include rotations. Show that a rotation of the scalar field ϕ an infinitesimal angle α about the z axis may be written as

$$\Delta\phi = -\alpha \frac{\mathrm{i}}{\hbar} L_z \phi \,,$$

where L_z is the z component of the angular momentum operator $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$. Thus, $\boldsymbol{L}$ is the infinitesimal generator of rotations (for fields without spin).

3. The Lagrange density of a string with mass density μ and string tension σ is

$$\mathcal{L} = \frac{1}{2}\mu\left(\left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2\right) - \frac{1}{2}\sigma\left(\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right).$$

The end points of the string are fixed a distance A apart. See Problem 11 2.

   a) Find the energy momentum tensor.
   Are energy and momentum conserved? (Locally? Globally?)

   b) Show that for an arbitrary constant α the transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \widetilde{y} \\ \widetilde{z} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix}$$

   is a symmetry, and find the conservation law associated with the symmetry.
   What is the physical meaning of the symmetry, and of the conservation law?

4. Given the Lagrange density

$$\mathcal{L} = \frac{\mathrm{i}\hbar}{2}\left(\frac{\partial\psi}{\partial t}\psi^* - \psi\frac{\partial\psi^*}{\partial t}\right) - \frac{\hbar^2}{2m}(\nabla\psi)\cdot(\nabla\psi^*) - V\psi\psi^*$$

for the complex field $\psi = \psi(\boldsymbol{r}, t)$, where $V = V(\boldsymbol{r}, t)$ represents (in some sense) a potential energy.

   a) What do the field equations look like for ψ and for the complex conjugate field ψ^*?

   b) What is the condition for time translation invariance?
   Assume that this condition holds, and use it to derive a conservation law.

   c) What is the condition for space translation invariance?
   Assume that this condition holds, and derive a conservation law.

d) What is the total energy and the total momentum of the field?
How is this particular complex field theory related to non-relativistic quantum mechanics?
Assume that the field vanishes outside a finite region in space (just for simplicity, the condition may be relaxed).
The quantum mechanical operators for energy and momentum are

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V \ , \qquad \boldsymbol{p} = \frac{\hbar}{\mathrm{i}}\nabla \ .$$

e) A transformation of the form

$$\psi \mapsto \widetilde{\psi} = \mathrm{e}^{-\mathrm{i}\alpha}\psi \ , \qquad \psi^* \mapsto \widetilde{\psi}^* = \mathrm{e}^{\mathrm{i}\alpha}\psi^* \ ,$$

where α is a constant real phase, is called a *global gauge transformation*.
Show that it is a symmetry, for arbitrary α.
Which conservation law does this symmetry imply?

5. Let A_μ be a covariant vector field (like the electromagnetic four-vector potential), let the metric tensor $g_{\mu\nu}$ be constant, and consider the following Lagrange density, where a and b are constants,

$$\mathcal{L} = a g^{\mu\nu} g^{\rho\sigma} A_{\mu,\rho} A_{\nu,\sigma} + b g^{\mu\nu} g^{\rho\sigma} A_{\mu,\rho} A_{\sigma,\nu} \ . \tag{12.79}$$

a) Find the Euler–Lagrange equations.

b) Show that the partial derivative $A_{\mu,\nu}$ transforms as a tensor under translations and Lorentz transformations.
How does it transform under general coordinate transformations?
How does the antisymmetric derivative $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ transform under general coordinate transformations?

c) Which of the following transformations are symmetries of the theory. Note that the answers may depend on the values of a and b.
– A translation (infinitesimal or finite)?
– A Lorentz transformation (infinitesimal or finite)?
– The orbital part alone, or the spin part alone, of an infinitesimal Lorentz transformation?
– A general coordinate transformation?
– A gauge transformation of the form $\Delta A_\mu = -\chi_{,\mu}$, where χ is an arbitrary scalar field (infinitesimal or finite)?

d) Find the canonical energy momentum tensor. Is it symmetric?

Chapter 13

The Klein–Gordon field

The Klein–Gordon equation for a real or complex scalar field is the simplest of all relativistically invariant classical field equations. It was proposed first as a quantum mechanical wave equation, the most direct relativistic generalization of the non-relativistic Schrödinger equation. The relativistis equation now known as the Klein–Gordon equation was actually the equation that Schrödinger tried first, but since it did not give the correct fine structure of the energy levels of the hydrogen atom, he rejected it in favour of the non-relativistic wave equation.

Electrons have spin, and the correct relativistic wave equation for electrons is the Dirac equation. The Klein–Gordon equation is the quantum mechanical wave equation for spinless particles, like π-mesons. For example, it gives the correct energy levels for π mesic atoms, where a negative π meson replaces an electron. The nuclear forces binding protons and neutrons together in atomic nuclei have a very short range and are very complicated, but the component which has the longest range may be described as resulting from the exchange of π mesons. The Klein–Gordon equation is the classical field equation for the nuclear forces, in so far as it has any meaning to describe them by means of a classical field.

The relativistic relation between energy E, momentum $\boldsymbol{p}$ and rest mass M,

$$E^2 - \boldsymbol{p}^2 c^2 = M^2 c^4 \ , \tag{13.1}$$

motivates the Klein–Gordon equation. The formal substitution

$$E \to \mathrm{i}\hbar \frac{\partial}{\partial t} \ , \qquad \boldsymbol{p} \to -\mathrm{i}\hbar \nabla \ , \tag{13.2}$$

or simply $p_\mu = (E/c, -\boldsymbol{p}) \to \mathrm{i}\hbar\partial_\mu$, gives the equation

$$-\hbar^2 \left(\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi \right) = M^2 c^4 \phi \tag{13.3}$$

for the Klein–Gordon field ϕ. Another way to write the equation is

$$\Box \phi + \mu^2 \phi = 0 \ , \tag{13.4}$$

when we introduce the inverse Compton wave length $\mu = Mc/\hbar$ and the d'Alembert operator

$$\Box = \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 = g^{\mu\nu} \partial_\mu \partial_\nu \ . \tag{13.5}$$

An important problem for the interpretation of the Klein–Gordon equation as a quantum mechanical wave equation for free relativistic particles, like the Schrödinger equation for free non-relativistic particles, is that the energy levels have no lower bound. For every solution of the equation with quantized energy equal to E there is another solution, the complex conjugate, having energy $-E$. This means that this so called *first quantized* theory is not acceptable, and has to be *second quantized*. That is, the Klein–Gordon field must first be interpreted as a classical field, which is then quantized.

13.1 Lagrange formalism

Real Klein–Gordon field

The Lagrange density for the real Klein–Gordon field is

$$\mathcal{L} = \frac{1}{2}\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \mu^2\phi^2\right) = \frac{1}{2}\left(\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 - (\nabla\phi)^2 - \mu^2\phi^2\right). \tag{13.6}$$

The Euler–Lagrange equation following from it is the Klein–Gordon equation,

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}x^\rho}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\rho}}\right) = -\mu^2\phi - \frac{1}{2}\,\partial_\rho\left(g^{\rho\nu}\phi_{,\nu} + g^{\mu\rho}\phi_{,\mu}\right) = -\mu^2\phi - g^{\mu\rho}\phi_{,\mu\rho}\,. \tag{13.7}$$

The factor 1/2 in the Lagrange density may be regarded as a convention. The field equation is the same for the Lagrange densities $\mathcal{L}$ and $C\mathcal{L}$, where C is an arbitrary constant. Moreover, any positive proportionality constant C may be absorbed into the field by means of a redefinition $\sqrt{C}\phi \to \phi$.

From the above Lagrange density follows the canonical energy momentum tensor

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial\phi_{,\nu}}\,g^{\mu\rho}\phi_{,\rho} - g^{\mu\nu}\mathcal{L} = g^{\mu\rho}\phi_{,\rho}g^{\nu\sigma}\phi_{,\sigma} - g^{\mu\nu}\mathcal{L}\,. \tag{13.8}$$

It is symmetric, because the Klein–Gordon field has no spin. The energy density is

$$T^{00} = \phi_{,0}\phi_{,0} - \mathcal{L} = \frac{1}{2}\left(\frac{1}{c^2}\left(\frac{\partial\phi}{\partial t}\right)^2 + (\nabla\phi)^2 + \mu^2\phi^2\right). \tag{13.9}$$

It is non-negative, a property which is preserved when the classical Klein–Gordon field is second quantized. In this way the problem of negative energy levels in the first quantized theory is solved.

Complex Klein–Gordon field

Two real Klein–Gordon fields ϕ^1 and ϕ^2 of the same mass M may be combined into a complex field

$$\phi = \frac{1}{\sqrt{2}}\left(\phi^1 + \mathrm{i}\phi^2\right). \tag{13.10}$$

Again we include a factor $1/\sqrt{2}$ which is not essential, but is a frequently used convention. The Lagrange density will be a sum of contributions from the two real fields, and may also be expressed in terms of the complex field ϕ and its complex conjugate field ϕ^*,

$$\mathcal{L} = \frac{1}{2}\left(g^{\mu\nu}\left(\phi^1_{,\mu}\phi^1_{,\nu} + \phi^2_{,\mu}\phi^2_{,\nu}\right) - \mu^2\left((\phi^1)^2 + (\phi^2)^2\right)\right) = g^{\mu\nu}\phi^*_{,\mu}\phi_{,\nu} - \mu^2\phi^*\phi\,. \tag{13.11}$$

We get the Euler–Lagrange equation for ϕ by varying ϕ^* as if it were independent of ϕ,

$$\frac{\partial \mathcal{L}}{\partial \phi^*} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial \mathcal{L}}{\partial \phi^*_{,\mu}}\right) = -\mu^2\phi - g^{\mu\nu}\phi_{,\mu\nu} = 0 \,. \tag{13.12}$$

This is the Klein–Gordon equation for the complex field ϕ, and its real and imaginary parts are the Klein–Gordon equations for the real fields ϕ^1 and ϕ^2.

We get the Euler–Lagrange equation for ϕ^* by varying ϕ, but since the Lagrange density is real, this equation is simply the complex conjugate of the equation for ϕ.

The canonical energy momentum tensor for the complex Klein–Gordon field is

$$\begin{aligned} T^{\mu\nu} &= \frac{\partial \mathcal{L}}{\partial \phi_{,\nu}}\, g^{\mu\rho}\phi_{,\rho} + \frac{\partial \mathcal{L}}{\partial \phi^*_{,\nu}}\, g^{\mu\rho}\phi^*_{,\rho} - g^{\mu\nu}\mathcal{L} \\ &= g^{\sigma\nu}\phi^*_{,\sigma} g^{\mu\rho}\phi_{,\rho} + g^{\nu\sigma}\phi_{,\sigma} g^{\mu\rho}\phi^*_{,\rho} - g^{\mu\nu}\mathcal{L} \,. \end{aligned} \tag{13.13}$$

It is again symmetric, and the energy density is non-negative,

$$T^{00} = 2\phi^*_{,0}\phi_{,0} - \mathcal{L} = |\phi_{,0}|^2 + |\nabla\phi|^2 + \mu^2|\phi|^2 \,. \tag{13.14}$$

13.2 Plane wave solutions and quantization

We may verify easily that a plane wave of the form

$$\phi = \phi(\boldsymbol{r}, t) = \phi(x) = \chi \mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} = \chi \mathrm{e}^{-\mathrm{i}k_\mu x^\mu} \,, \tag{13.15}$$

with a constant amplitude χ, is a solution of the (complex) Klein–Gordon equation if

$$g^{\mu\nu}k_\mu k_\nu = \frac{\omega^2}{c^2} - \boldsymbol{k}^2 = \mu^2 \,. \tag{13.16}$$

Here $\boldsymbol{k}$ is the three dimensional and $k_\mu = (\omega/c, -\boldsymbol{k})$ the four dimensional wave number vector. This equation is the *dispersion relation*, giving the angular frequency ω as a function of $\boldsymbol{k}$,

$$\omega = \omega_{\boldsymbol{k}} = c\sqrt{\mu^2 + \boldsymbol{k}^2} \,. \tag{13.17}$$

The phase velocity

$$v_f = \frac{\omega_{\boldsymbol{k}}}{|\boldsymbol{k}|} \tag{13.18}$$

equals the velocity of light c if $\mu = 0$, but is larger than c if $\mu > 0$. This is no problem, however, since the the group velocity

$$\boldsymbol{v}_g = \frac{\partial \omega_{\boldsymbol{k}}}{\partial \boldsymbol{k}} = \frac{c\boldsymbol{k}}{\sqrt{\mu^2 + \boldsymbol{k}^2}} \tag{13.19}$$

never exceeds c in magnitude.

For such a plane wave solution we have $\phi_{,\mu} = -\mathrm{i}k_\mu\phi$. Hence the Lagrange density is $\mathcal{L} = 0$, and the energy momentum tensor is

$$T^{\mu\nu} = 2k^\mu k^\nu |\chi|^2 \,. \tag{13.20}$$

There is a factor of 2 because the field has two components, one real and one imaginary part.

The general solution of the Klein–Gordon equation is a superposition of plane wave solutions,

$$\phi(\boldsymbol{r},t) = \int \mathrm{d}^3\boldsymbol{k}\, \chi(\boldsymbol{k})\, \mathrm{e}^{-\mathrm{i}(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} \,. \tag{13.21}$$

We may transform the integral over $\boldsymbol{k}$ into a sum by means of a trick consisting in imposing periodic boundary conditions on ϕ, e.g. that

$$\phi(x+A,y,z,t) = \phi(x,y+A,z,t) = \phi(x,y,z+A,t) = \phi(x,y,z,t) \,. \tag{13.22}$$

The justification is that the period A is "invisible" if it is large enough. A consequence is that the wave number vector $\boldsymbol{k}$ becomes discrete,

$$k_i = n_i \frac{2\pi}{A} \,, \quad \text{where} \quad n_i = 0, \pm 1, \pm 2, \ldots \,, \tag{13.23}$$

so that we may write

$$\phi(\boldsymbol{r},t) = \sum_{\boldsymbol{k}} \chi_{\boldsymbol{k}}(t)\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}} \,. \tag{13.24}$$

The relation between integral and sum is that

$$\int \mathrm{d}^3\boldsymbol{k} = \left(\frac{2\pi}{A}\right)^3 \int \mathrm{d}^3\boldsymbol{n} \to \left(\frac{2\pi}{A}\right)^3 \sum_{\boldsymbol{k}} \,, \tag{13.25}$$

which gives that

$$\chi_{\boldsymbol{k}}(t) = \left(\frac{2\pi}{A}\right)^3 \chi(\boldsymbol{k})\, \mathrm{e}^{-\mathrm{i}\omega t} \,. \tag{13.26}$$

The time dependence in Equation (13.26) is such that Equation (13.24) is a solution of the Klein–Gordon equation. But every complex field $\phi(\boldsymbol{r},t)$ may be Fourier expanded like in Equation (13.24), if we allow the Fourier coefficients $\chi_{\boldsymbol{k}}$ to have an arbitrary time dependence. The Klein–Gordon equation is then derived from the Lagrange function L, which is the integral of the Lagrange density $\mathcal{L}$ over the volume $V = A^3$,

$$\begin{aligned} L &= \int_V \mathrm{d}^3\boldsymbol{r} \left(\frac{1}{c^2} \frac{\partial \phi^*}{\partial t} \frac{\partial \phi}{\partial t} - (\nabla\phi^*)\cdot(\nabla\phi) - \mu^2 \phi^* \phi \right) \\ &= \sum_{\boldsymbol{k}} \left(\frac{V}{c^2}\, \dot{\chi}^*_{\boldsymbol{k}} \dot{\chi}_{\boldsymbol{k}} - V\left(\boldsymbol{k}^2 + \mu^2\right) \chi^*_{\boldsymbol{k}} \chi_{\boldsymbol{k}} \right) . \end{aligned} \tag{13.27}$$

By varying $\chi_{\boldsymbol{k}}$ and $\chi^*_{\boldsymbol{k}}$ as if they were independent, we get the complex Klein–Gordon equation. For the real Klein–Gordon field there is the relation between the Fourier components that $\chi^*_{\boldsymbol{k}} = \chi_{-\boldsymbol{k}}$. A possible way to treat the real field is therefore to write

$$L = \sum_{\boldsymbol{k}} \left(\frac{V}{c^2}\, \dot{\chi}_{-\boldsymbol{k}} \dot{\chi}_{\boldsymbol{k}} - V\left(\boldsymbol{k}^2 + \mu^2\right) \chi_{-\boldsymbol{k}} \chi_{\boldsymbol{k}} \right) , \tag{13.28}$$

and then vary all the components $\chi_{\boldsymbol{k}}$ as if they were independent, ignoring the fact that $\chi_{\boldsymbol{k}}$ and $\chi_{-\boldsymbol{k}}$ have to be the complex conjugates of each other.

Here we have managed to write the Lagrange function for a *relativistic* field equation, the Klein–Gordon equation, in such a way that it looks exactly like the Lagrange function for infinitely many independent *non-relativistic* harmonic oscillators. To see the analogy even more explicitly, we may write $\mu = V/c^2$, and

$$\chi_{\boldsymbol{k}} = \frac{1}{\sqrt{2}} \left(X_{\boldsymbol{k}} + \mathrm{i}\, Y_{\boldsymbol{k}} \right) . \tag{13.29}$$

This gives, in the complex case, that

$$L = \sum_{\boldsymbol{k}} \left(\frac{1}{2}\, \mu \left((\dot{X}_{\boldsymbol{k}})^2 + (\dot{Y}_{\boldsymbol{k}})^2 \right) - \frac{1}{2}\, \mu\, {\omega_{\boldsymbol{k}}}^2 \left({X_{\boldsymbol{k}}}^2 + {Y_{\boldsymbol{k}}}^2 \right) \right) . \tag{13.30}$$

To every wave number vector $\boldsymbol{k}$ there correspond two oscillators of "mass" μ and angular frequency $\omega_{\boldsymbol{k}}$.

To quantize (second quantize) the Klein–Gordon field is the same as quantizing the corresponding set of harmonic oscillators. One energy quantum of one oscillator corresponds to one spinless particle of momentum $\boldsymbol{p} = \hbar \boldsymbol{k}$ and energy

$$\hbar \omega_{\boldsymbol{k}} = c \sqrt{M^2 c^2 + \boldsymbol{p}^2} \, . \tag{13.31}$$

We see that the mass M occurring as a parameter in the Klein–Gordon equation, is precisely the mass of the particles entering our theory as a result of the second quantization. The particles are bosons, because every oscillator may have an arbitrary number of energy quanta, corresponding to that number of particles in one single quantum mechanical state.

The fact that there correspond two oscillators to every wave number $\boldsymbol{k}$, is interpreted to mean that the quantization of the complex Klein–Gordon equation gives both particles and antiparticles. The real Klein–Gordon equation has half as many degrees of freedom, and its quantization gives particles that are their own antiparticles. Such particles are said to be *neutral*, they may not, for example, have any electric charge, since particle and antiparticle have opposite charges, by definition. The π^0 meson is an example of a neutral particle, being its own antiparticle.

13.3 The Yukawa potential

A simple generalization of the Klein–Gordon equation is to introduce another scalar field $\rho = \rho(\boldsymbol{r}, t)$ as a source, so that we obtain the inhomogeneous Klein–Gordon equation

$$\Box \phi + \mu^2 \phi = -\rho \, , \tag{13.32}$$

with the following Lagrange density, assuming the fields are real,

$$\mathcal{L} = \frac{1}{2} \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \mu^2 \phi^2 \right) - \rho \phi \, . \tag{13.33}$$

The sign convention for the source term used here corresponds to the convention used in electromagnetism. When the source field ρ is transformed as scalar field under Poincaré transformations, the Lagrange density is also transformed as a scalar field, and so the field equation is Poincaré invariant.

Field from point particles

If the source consists of N pointlike "charges" Q_i, with $i = 1, 2, \ldots, N$, at the positions $\boldsymbol{R}_i(t)$ at time t, and if we define $\boldsymbol{V}_i = \dot{\boldsymbol{R}}_i = \mathrm{d}\boldsymbol{R}_i/\mathrm{d}t$, then the density ρ is given by the formula

$$\rho(\boldsymbol{r}, t) = \sum_{i=1}^{N} Q_i \sqrt{1 - \frac{\boldsymbol{V}_i{}^2}{c^2}}\ \delta^{(3)}(\boldsymbol{r} - \boldsymbol{R}_i(t)) \ . \tag{13.34}$$

The square root in this formula is necessary to make ρ transform as a scalar under Lorentz transformations. One way to prove that ρ actually is a scalar, is to integrate over an arbitrary four dimensional region Ω, and check that the integral is invariant under Lorentz transformations. Let us assume that $(t, \boldsymbol{R}_i(t)) \in \Omega$ for $t_{iA} \le t \le t_{iB}$. As soon as we have integrated away the three dimensional δ-function, we see that the integral is invariant. In fact, the remaining time integral is just the proper time along the trajectory of each point charge,

$$\int_\Omega \mathrm{d}^4x\, \rho = \sum_{i=1}^{N} Q_i \int_{t_{iA}}^{t_{iB}} \mathrm{d}t \sqrt{1 - \frac{\boldsymbol{V}_i{}^2}{c^2}} \ . \tag{13.35}$$

In the special case of one single point charge Q at rest at the origin, the field ϕ should be time independent, and the field equation is then

$$-\nabla^2\phi + \mu^2\phi = -Q\, \delta^{(3)}(\boldsymbol{r}) \ . \tag{13.36}$$

The solution, found in Chapter 1 by Fourier transformation, is the Yukawa potential,

$$\phi = \phi(r) = -\frac{Q}{4\pi}\, \frac{\mathrm{e}^{-\mu r}}{r} \ , \tag{13.37}$$

with $r = |\boldsymbol{r}|$. If the Klein–Gordon field has mass $M = 0$, then the Yukawa potential has the same form as the Coulomb potential. However, if $M > 0$, then the Yukawa potential looks like the Coulomb potential in the limit $r \to 0$, but vanishes much faster when the distance r becomes larger than the Compton wave length $(1/\mu) = (\hbar/Mc)$. Thus, $1/\mu$ measures the range of the interaction mediated by the field ϕ. In the same way, the Coulomb potential would have been exponentially damped at large distances if the photon had a nonzero mass.

The gradient of the Yukawa potential is

$$\nabla\phi = \frac{\mathrm{d}\phi}{\mathrm{d}r}\, \nabla r = -\left(\mu + \frac{1}{r}\right) \phi\, \frac{\boldsymbol{r}}{r} \ , \tag{13.38}$$

and the energy density is

$$T^{00} = \frac{1}{2}\left((\nabla\phi)^2 + \mu^2\phi^2\right) = \frac{Q^2}{16\pi^2}\left(\mu^2 + \frac{\mu}{r} + \frac{1}{2r^2}\right) \frac{\mathrm{e}^{-2\mu r}}{r^2} \ . \tag{13.39}$$

In the limit $r \to 0$, the energy density diverges like $1/r^4$, hence the energy of the Yukawa field from a point charge is infinite,

$$E_F = \int \mathrm{d}^3\boldsymbol{r}\, T^{00} = 4\pi \int_0^\infty \mathrm{d}r\, r^2 T^{00} = \infty \ . \tag{13.40}$$

This shows that classical field theory with point particles as sources is inconsistent. To make sense of the theory we have to somehow ignore certain infinities that we meet with, such as the infinite field energy.

13.4 Interaction of particles and fields

The total Lagrange function for a system consisting of point particles interacting with a field is a sum of three terms, $L = L_F + L_P + L_I$, representing respectively the field alone, the particles alone, and the interaction between particles and field. We do not include here any direct interaction between the particles, in addition to the indirect interaction mediated by the field. For the real Klein–Gordon field ϕ with source ρ we have

$$L_F + L_I = \int \mathrm{d}^3\boldsymbol{r} \left(\frac{1}{2} \left(g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} - \mu^2 \phi^2 \right) - \rho\phi \right) . \tag{13.41}$$

The interaction term is, with point sources like in Equation (13.34), and with the notation $\phi_i = \phi(\boldsymbol{R}_i(t), t)$,

$$L_I = -\int \mathrm{d}^3\boldsymbol{r}\, \rho\phi = -\sum_{i=1}^{N} Q_i \phi_i \sqrt{1 - \frac{\boldsymbol{V}_i{}^2}{c^2}} \, . \tag{13.42}$$

We see here another example of how classical field theory with point particles as sources is inconsistent. We have seen, although only in one single example, that the field from a point sourc is infinite at the point where the source is located. In order to make the Lagrange function L_I for the interaction between particles and field finite, we therefore have to introduce an arbitrary rule saying that a particle interacts only with the field from all the other particles, and not with its own field.

For N point particles of masses m_i we have the particle Lagrange function

$$L_P = -\sum_{i=1}^{N} m_i\, c^2 \sqrt{1 - \frac{\boldsymbol{V}_i{}^2}{c^2}} \, . \tag{13.43}$$

We derive the equations of motion for the particles from the Lagrange function

$$L_P + L_I = -\sum_{i=1}^{N} (m_i\, c^2 + Q_i \phi_i) \sqrt{1 - \frac{\boldsymbol{V}_i{}^2}{c^2}} \, . \tag{13.44}$$

We may interpret this formula to mean that the Klein–Gordon field changes the mass of a particle, making it dependent on time and place.

Particle in external field

Let us consider the simple example of one particle of mass m, "Klein–Gordon charge" q, position $\boldsymbol{r} = \boldsymbol{r}(t)$, and velocity $\boldsymbol{v} = \dot{\boldsymbol{r}}$, moving in an external field $\phi = \phi(\boldsymbol{r}, t)$. The Lagrange function of the particle is then

$$L = -(mc^2 + q\phi) \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} \, . \tag{13.45}$$

The canonical momentum is

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \frac{(mc^2 + q\phi)\boldsymbol{v}}{c^2 \sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} \, , \tag{13.46}$$

and the Hamilton function is, under the assumption that $mc^2 + q\phi \geq 0$ everywhere,

$$H = \boldsymbol{p} \cdot \boldsymbol{v} - L = \frac{mc^2 + q\phi}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} = \sqrt{(mc^2 + q\phi)^2 + \boldsymbol{p}^2 c^2} \, . \tag{13.47}$$

Let us simplify even more and assume that ϕ is the Yukawa potential from a heavy particle lying at rest at the origin. Then we have

$$H = \sqrt{\left(mc^2 - \frac{qQ}{4\pi} \frac{\mathrm{e}^{-\mu r}}{r}\right)^2 + \boldsymbol{p}^2 c^2} \, . \tag{13.48}$$

The Hamiltonian H is conserved, since it is not explicitly time dependent. If now $qQ > 0$, then we may well have $H < mc^2$. In that case, the particle has not enough energy to escape to $r = \infty$, where $\phi = 0$ and

$$H = \sqrt{m^2 c^4 + \boldsymbol{p}^2 c^2} \; \geq mc^2 \, . \tag{13.49}$$

Reasoning in this way, we conclude, maybe somewhat rashly, that the force between two point charges Q and q is attractive if the charges are of the same sign. We also conclude that the force is repulsive between two charges of opposite signs.

An even more transparent way to reach the same conclusion may be to consider the non-relativistic limit, expanding the square roots and throwing away all terms with powers of c in the denominator. This gives

$$L = -(mc^2 + q\phi)\left(1 - \frac{\boldsymbol{v}^2}{2c^2} + \ldots\right) = -mc^2 + \frac{1}{2} m\boldsymbol{v}^2 - q\phi + \ldots \, , \tag{13.50}$$

and

$$H = mc^2 \sqrt{\left(1 + \frac{q\phi}{mc^2}\right)^2 + \frac{\boldsymbol{p}^2}{m^2 c^2}} = mc^2 + \frac{\boldsymbol{p}^2}{2m} + q\phi + \ldots \, . \tag{13.51}$$

The interaction mediated by the Klein–Gordon field resembles the electrostatic interaction between electric charges, in the non-relativistic limit. It also resembles gravitation, if the "Klein–Gordon charge" q of a particle is identified with the mass m. It has the property in common with gravitation that equal charges attrach each other, whereas equal electric charges repel each other.

It is an experimental fact that the nuclear forces are attractive between two nucleons, whether they are two protons, two neutrons, or one proton and a neutron. The attraction is sufficiently strong to produce stable atomic nuclei with over two hundred nucleons, in spite of the electrostatic repulsion between the protons. The size of a stable atomic nucleus is limited, because the nuclear forces have short range, whereas the electrostatic repulsion has long range. Both of these observations may be understood qualitatively if we assume that it is meaningful to regard the classical Klein–Gordon field of mass $M > 0$ as a model for the nuclear forces. If we identify the Klein–Gordon particles of the nuclear force as π mesons, then $M = 140\,\mathrm{MeV}/c^2 = 2,5 \times 10^{-28}\,\mathrm{kg}$, and the corresponding Compton wave length is

$$\frac{\hbar}{Mc} = 1.4 \times 10^{-15}\,\mathrm{m} = 1.4\,\mathrm{fm} \, . \tag{13.52}$$

This estimate of the range of the nuclear forces agrees well with the empirical formula for the radius of an atomic nucleus, which is $r_0 A^{1/3}$, where A is the mass number (the number of nucleons), and $r_0 \approx 1.2\,\mathrm{fm}$.

Elimination of the field

The extremalization of the action integral $S = S_F + S_P + S_V$ obtained from the Lagrange function $L = L_F + L_P + L_I$ gives the equations of motion for both the field and the particles. One way of treating these equations, possible at least in principle, is to eliminate the field by solving the field equations and expressing the field in terms of the particle positions. The solution for the field may be inserted either in the equations of motion for the particles, or else in the action integral. This procedure will give an action integral and equations of motion for the particles alone, containing direct interactions between the particles instead of the field.

In order to understand that it is permissible to eliminate variables by solving some of, but not all the Euler–Lagrange equations, and then insert the solutions back into the action integral, we may use an analogy. Remember that the point of departure for obtaining the Euler–Lagrange equations is to look for an extremal point of the action integral S. A simpler problem of the same type is to find the minimum of a function of two variables x og y, for example the not entirely trivial function

$$S = S(x, y) = x^4 + x^3 y + y^6 \, . \tag{13.53}$$

One possible approach is to first keep y fixed and minimize with respect to x, that is, to solve the equation

$$\frac{\partial S}{\partial x} = 4x^3 + 3x^2 y = 0 \, . \tag{13.54}$$

There are two solutions, $x = 0$ or $x = -3y/4$, for our purposes the second one is the most interesting. We may insert it into either the equation

$$\frac{\partial S}{\partial y} = x^3 + 6y^5 = 0 \, , \tag{13.55}$$

or the original function $S(x, y)$, this will give us a new function of the one variable y,

$$S = S(y) = -\frac{27}{256}\, y^4 + y^6 \, , \tag{13.56}$$

which is then to be minimized. Both methods give the same answer $y^2 = 9/128$, $x = -3y/4$, as they should.

Eliminating the field is easy in the non-relativistic limit, which we may also be justified in calling the static approximation. The basic assumption is then that all particle velocities are much smaller than the velocity of light and the velocity of propagation of disturbances in the field. Then we may use the instantaneous Yukawa potential

$$\phi(\boldsymbol{r}, t) = -\sum_{j=1}^{N} \frac{Q_j}{4\pi} \, \frac{\mathrm{e}^{-\mu|\boldsymbol{r} - \boldsymbol{R}_j(t)|}}{|\boldsymbol{r} - \boldsymbol{R}_j(t)|} \tag{13.57}$$

as an approximate solution of the field equations. This gives that

$$L_I = \sum_{i=1}^{N} \sum_{j \neq i} \frac{Q_i Q_j}{4\pi} \, \frac{\mathrm{e}^{-\mu|\boldsymbol{R}_i(t) - \boldsymbol{R}_j(t)|}}{|\boldsymbol{R}_i(t) - \boldsymbol{R}_j(t)|} \, . \tag{13.58}$$

Remember the convention that a particle does not interact with its own field.

This is not the end of the story, however. As we know, the total Lagrange function is $L_F + L_P + L_I$, and there remains an important contribution from L_F, the Lagrange function of the field, even after we have eliminated the field. To compute it, we write Equation (13.57) as

$$\phi(\boldsymbol{r}, t) = \sum_{j=1}^{N} \phi_{(j)}(\boldsymbol{r}, t) \; , \tag{13.59}$$

before insertion into L_F. In the non-relativistic limit the time derivative $\partial\phi/\partial t$ gives no contribution to L_F, and we have that

$$\begin{aligned} L_F \; &= \; -\frac{1}{2} \int \mathrm{d}^3\boldsymbol{r} \, (|\nabla\phi|^2 + \mu^2\phi^2) \\ &= \; -\frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{N} \int \mathrm{d}^3\boldsymbol{r} \, ((\nabla\phi_{(i)}) \cdot (\nabla\phi_{(j)}) + \mu^2 \phi_{(i)}\phi_{(j)}) \; . \end{aligned} \tag{13.60}$$

In this integral we must distinguish between two kinds of contributions, the *self energy terms*, with $i = j$, and the *interaction terms*, with $i \neq j$. The self energy is the field contribution to the energy of each particle, and it is infinite, because at least the integral of $|\nabla\phi_{(i)}|^2$ is divergent. For lack of a better alternative, we just disregard such pedantic worries concerning mathematical stringency, and choose simply to ignore the self energies. They are of no interest anyway in the present context, since they do not depend on the particle positions. Therefore we include only the interaction terms, and they are finite. There is the following integral with $i \neq j$ containing gradients,

$$\int \mathrm{d}^3\boldsymbol{r} \, (\nabla\phi_{(i)}) \cdot (\nabla\phi_{(j)}) = -\int \mathrm{d}^3\boldsymbol{r} \, \phi_{(i)} \, (\nabla^2\phi_{(j)}) \; . \tag{13.61}$$

The partial integration here introduces no boundary terms, since the fields vanish sufficiently fast at infinity. We may use the field equation,

$$-\nabla^2\phi_{(j)} = -\mu^2\phi_{(j)} - Q_j\delta^{(3)}(\boldsymbol{r} - \boldsymbol{R}_j(t)) \; , \tag{13.62}$$

to obtain the final result,

$$L_F = -(\text{infinite self energy}) - \frac{1}{2} L_I \; . \tag{13.63}$$

Ignoring the infinite self energy, we see that $L_F + L_I = L_I/2$. Thus, the correct interaction Lagrange function, after elimination of the fields, is not L_I but $L_I/2$. This reduction by a factor of 2 is of course essential, and also rather unexpected at first sight, therefore we may like to convince ourselves that it is indeed correct. We may justify it by the following intuitive argument.

In the expression for L_I in Equation (13.58) each particle position $\boldsymbol{R}_i$ plays in a sense a double role. On the one hand, it occurs as the argument $\boldsymbol{r}$ in the field function $\phi(\boldsymbol{r}, t)$, and this is the dependence that counts when we derive the Euler–Lagrange equations. In fact, we should derive the Euler–Lagrange equations *as if* the particles were moving in an external field $\phi(\boldsymbol{r}, t)$ independent of the particle positions. On the other hand, the particles are the sources of the field, and the field must obviously depend on the positions of the sources. In a correct derivation of the Euler–Lagrange equations for the particles, we should not vary with respect

to this source dependence of L_I. Nevertheless we do so, against better knowledge, because any distinction between the two kinds of position dependence would seem very artificial. This amounts to a double counting, which we have to compensate by using $L_I/2$ instead of L_I.

To repeat the conclusion, the correct interaction Lagrange function after elimination of the fields is

$$\frac{L_I}{2} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\frac{Q_iQ_j}{4\pi}\,\frac{\mathrm{e}^{-\mu|\boldsymbol{R}_i(t)-\boldsymbol{R}_j(t)|}}{|\boldsymbol{R}_i(t)-\boldsymbol{R}_j(t)|} = \sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\frac{Q_iQ_j}{4\pi}\,\frac{\mathrm{e}^{-\mu|\boldsymbol{R}_i(t)-\boldsymbol{R}_j(t)|}}{|\boldsymbol{R}_i(t)-\boldsymbol{R}_j(t)|}\,. \tag{13.64}$$

The correct prescription is to count each pair of particles exactly once.

13.5 Electrically charged Klein–Gordon field

The origin of the Klein–Gordon equation is the relativistic relation $g^{\mu\nu}p_\mu p_\nu = M^2c^2$, valid for a free particle. For a particle of electric charge q in an external electromagnetic field $A_\mu = (\Phi/c, -\boldsymbol{A})$ we have the modified constraint equation

$$g^{\mu\nu}(p_\mu - qA_\mu)(p_\nu - qA_\nu) = M^2c^2\,, \tag{13.65}$$

by Equation (10.84). Here again we substitute $p_\mu \to \mathrm{i}\hbar\partial_\mu$, in order to obtain a differential equation,

$$g^{\mu\nu}\left(\partial_\mu + \mathrm{i}a_\mu\right)\left(\partial_\nu + \mathrm{i}a_\nu\right)\phi + \mu^2\phi = 0\,. \tag{13.66}$$

We simplify our typography by defining

$$a_\mu = \frac{q}{\hbar}\,A_\mu\,. \tag{13.67}$$

This recipe for coupling the complex Klein–Gordon field to the electromagnetic field is called *minimal coupling*. All that is needed, is to replace the partial derivative ∂_μ by something that we may call a *covariant* derivative,

$$D_\mu = \partial_\mu + \mathrm{i}a_\mu\,. \tag{13.68}$$

The field has to be complex, because the covariant derivative contains the factor $\mathrm{i} = \sqrt{-1}$. However, there is nothing fundamental in the complex formalism as such. In fact, the Lagrange density is real, and we might just as well formulate the theory as a real theory throughout, with two real fields ϕ^1 and ϕ^2 that are the real and imaginary parts of the complex field ϕ, such that

$$\sqrt{2}\,\phi = \phi^1 + \mathrm{i}\phi^2 \leftrightarrow \begin{pmatrix}\phi^1\\ \phi^2\end{pmatrix}\,. \tag{13.69}$$

The factor $\mathrm{i} = \sqrt{-1}$ then corresponds to the real 2×2 matrix

$$J = \begin{pmatrix}J^1{}_1 & J^1{}_2\\ J^2{}_1 & J^2{}_2\end{pmatrix} = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}\,, \tag{13.70}$$

since

$$\mathrm{i}\left(\phi^1 + \mathrm{i}\phi^2\right) = -\phi^2 + \mathrm{i}\phi^1 \leftrightarrow \begin{pmatrix} -\phi^2 \\ \phi^1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi^1 \\ \phi^2 \end{pmatrix} . \tag{13.71}$$

The real version of Equation (13.66) is then the following,

$$g^{\mu\nu}\left(\delta^i_j\,\partial_\mu + J^i{}_j\,a_\mu\right)\left(\delta^j_k\,\partial_\nu + J^j{}_k\,a_\nu\right)\phi^k + \mu^2\phi^i = 0 . \tag{13.72}$$

The Lagrange density for the Klein–Gordon field with electric charge is

$$\begin{aligned} \mathcal{L} &= g^{\mu\nu}(D_\mu\,\phi)^*(D_\nu\,\phi) - \mu^2\phi^*\phi \\ &= g^{\mu\nu}(\phi^*_{,\mu} - \mathrm{i}a_\mu\,\phi^*)(\phi_{,\nu} + \mathrm{i}a_\nu\,\phi) - \mu^2\phi^*\phi \\ &= \frac{1}{2}\,\delta_{ik}\left(g^{\mu\nu}\left(\phi^i_{,\mu} + J^i{}_j\,a_\mu\,\phi^j\right)\left(\phi^k_{,\nu} + J^k{}_l\,a_\nu\,\phi^l\right) - \mu^2\phi^i\phi^k\right) . \end{aligned} \tag{13.73}$$

The four dimensional electric current density, which is the source of the electromagnetic field, is derived from the Lagrange density by variation of the vector potential A_μ,

$$\begin{aligned} j^\mu &= \frac{\partial\mathcal{L}}{\partial A_\mu} = \frac{q}{\hbar}\,\frac{\partial\mathcal{L}}{\partial a_\mu} = -\mathrm{i}\,\frac{q}{\hbar}\,g^{\mu\nu}\left(\phi^*(\phi_{,\nu} + \mathrm{i}a_\nu\,\phi) - (\phi^*_{,\nu} - \mathrm{i}a_\nu\,\phi^*)\phi\right) \\ &= \frac{q}{\hbar}\,g^{\mu\nu}\ \mathrm{Im}(\phi^*(\phi_{,\nu} + \mathrm{i}a_\nu\,\phi)) = \frac{q}{\hbar}\,g^{\mu\nu}\,\delta_{ik}\,J^i{}_j\,\phi^j\left(\phi^k_{,\nu} + J^k{}_l\,a_\nu\,\phi^l\right) . \end{aligned} \tag{13.74}$$

Geometric interpretation

Minimal coupling is a natural recipe because it gives a geometric meaning to the electromagnetic field, in the following way. We imagine that there exists at every point x in spacetime a local field space $\mathcal{F}(x)$ for the Klein–Gordon field. This field space we regard either as a one dimensional complex vector space, with one single basis vector $\mathbf{f}(x)$, or as a two dimensional real vector space, with two basis vectors $\mathbf{e}_1(x)$ and $\mathbf{e}_2(x)$. The value of the field at the point x is a vector $\Phi(x) \in \mathcal{F}(x)$. The complex field $\phi(x)$, or equivalently, the real fields $\phi^1(x)$ and $\phi^2(x)$, we regard as the coordinates of $\Phi(x)$ relative to the local basis in every field space. That is, we write either

$$\Phi(x) = \phi(x)\,\mathbf{f}(x) , \tag{13.75}$$

or

$$\Phi(x) = \phi^i(x)\,\mathbf{e}_i(x) = \phi^1(x)\,\mathbf{e}_1(x) + \phi^2(x)\,\mathbf{e}_2(x) . \tag{13.76}$$

In this geometric picture there will in general be nothing to single out one particular local basis $\mathbf{e}_1(x), \mathbf{e}_2(x)$ in favour of any other basis

$$\begin{aligned} \widetilde{\mathbf{e}}_1(x) &= \ \ \cos\alpha(x)\ \mathbf{e}_1(x) + \sin\alpha(x)\ \mathbf{e}_2(x) , \\ \widetilde{\mathbf{e}}_2(x) &= -\sin\alpha(x)\ \mathbf{e}_1(x) + \cos\alpha(x)\ \mathbf{e}_2(x) , \end{aligned} \tag{13.77}$$

rotated by an arbitrary angle $\alpha(x)$ relative to the first one. The field $\Phi(x)$ is of course independent of the local basis, but the field components must depend on our choice of basis, in such a way that

$$\Phi(x) = \phi^i(x)\,\mathbf{e}_i(x) = \widetilde{\phi}^i(x)\,\widetilde{\mathbf{e}}_i(x) . \tag{13.78}$$

Thus, the rotation of the basis vectors gives the following transformation of the field components,

$$\begin{aligned} \widetilde{\phi}^1(x) &= \cos\alpha(x)\ \phi^1(x) + \sin\alpha(x)\ \phi^2(x)\ , \\ \widetilde{\phi}^2(x) &= -\sin\alpha(x)\ \phi^1(x) + \cos\alpha(x)\ \phi^2(x)\ . \end{aligned} \tag{13.79}$$

In the complex formulation the corresponding formula is

$$\widetilde{\phi}(x) = \mathrm{e}^{-\mathrm{i}\alpha(x)}\,\phi(x)\ . \tag{13.80}$$

We say that we *choose a gauge* when we choose a local basis in the field space $\mathcal{F}(x)$ at every point x, and a rotation of the basis by an angle $\alpha(x)$ depending on space and time, we call a *local gauge transformation*. A rotation of the basis by a constant angle α we call, naturally, a *global* gauge transformation.

A local gauge transformation $\widetilde{\phi} = \mathrm{e}^{-\mathrm{i}\alpha}\phi$ gives for the partial derivatives of the field that

$$\widetilde{\phi}_{,\mu} = \mathrm{e}^{-\mathrm{i}\alpha}(\phi_{,\mu} - \mathrm{i}\,\alpha_{,\mu}\,\phi)\ . \tag{13.81}$$

Thus, the partial derivatives $\phi_{,\mu}$ transform differently from ϕ under a local gauge transformation. Our motivation for introducing the covariant derivative $\phi_{,\mu} + \mathrm{i}a_\mu\phi$, is that we want it to get transformed in the same way as ϕ under a local gauge transformation,

$$\widetilde{\phi}_{,\mu} + \mathrm{i}\,\widetilde{a}_\mu\,\widetilde{\phi} = \mathrm{e}^{-\mathrm{i}\alpha}(\phi_{,\mu} + \mathrm{i}\,a_\mu\,\phi)\ . \tag{13.82}$$

The solution is the following transformation formula for the vector potential a_μ,

$$\widetilde{a}_\mu = a_\mu + \alpha_{,\mu}\ . \tag{13.83}$$

Which means for the electromagnetic vector potential that

$$\widetilde{A}_\mu = A_\mu + \frac{\hbar}{q}\,\alpha_{,\mu}\ . \tag{13.84}$$

13.6 Klein–Gordon field in an external gravitational field

So far in this chapter we have assumed that the metric is constant, and in particular that $g_{\mu\nu} = \eta_{\mu\nu}$. It is very easy to generalize to an external gravitational field, where the metric depends on space and time. We want a theory which is generally covariant, in the sense that the field equations have the same form in every coordinate system. All we need is an action integral

$$S = \frac{1}{c}\int_\Omega \mathrm{d}^4x\,\mathcal{L} \tag{13.85}$$

which is invariant under arbitrary coordinate transformations. The simplest possibility is to choose a Lagrange density $\mathcal{L}$ which is a scalar density, and again the simplest choice is of the form

$$\mathcal{L} = \sqrt{|g|}\,\mathcal{L}_s\ , \tag{13.86}$$

where $\mathcal{L}_s$ is a scalar function, the Lagrange scalar. See Equation (7.25). The following Lagrange density for the Klein–Gordon field suggests itself,

$$\mathcal{L} = \frac{1}{2}\sqrt{|g|}\,\left(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - \mu^2\phi^2\right)\,. \tag{13.87}$$

It gives the following Euler–Lagrange equation,

$$0 = \frac{\partial\mathcal{L}}{\partial\phi} - \frac{\mathrm{d}}{\mathrm{d}x^\rho}\left(\frac{\partial\mathcal{L}}{\partial\phi_{,\rho}}\right) = -\mu^2\sqrt{|g|}\,\phi - \partial_\rho\left(\sqrt{|g|}\,g^{\rho\nu}\phi_{,\nu}\right)\,. \tag{13.88}$$

This field equation is invariant under arbitrary coordinate transformations, because it is derived from an invariant action integral. It follows that the two terms on the right hand side get transformed in the same way under a coordinate transformation. We know from Chapter 3 that the first term, $\sqrt{|g|}\,\phi$, transforms as a scalar density, since ϕ transforms as a scalar. Consequently, the second term is also a scalar density, so that

$$\Delta\phi = \frac{1}{\sqrt{|g|}}\,\partial_\rho\left(\sqrt{|g|}\,g^{\rho\nu}\,\partial_\nu\phi\right) \tag{13.89}$$

is a scalar. In this way we actually derive the Laplace–Beltrami operator Δ from our invariant variational principle. See Equation (5.56).

A new derivation of the energy momentum tensor

A surprising use of the above Lagrange density, including the factor $\sqrt{|g|}$, is in an alternative derivation of the energy momentum tensor $T^{\mu\nu}$. The idea is simply to vary the metric tensor $g_{\mu\nu}$. In Einstein's gravitational theory the metric tensor describes the gravitational field, and the energy momentum tensor plays the role of the source of the gravitational field. This is so precisely because the energy momentum tensor is what we get when we vary $g_{\mu\nu}$ in the Lagrange density.

Thus, let $\delta g_{\mu\nu}$ be an infinitesimal variation of $g_{\mu\nu}$, respecting the symmetry, that is, with $\delta g_{\mu\nu} = \delta g_{\nu\mu}$. From Equation (2.81) we have that

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}\,g_{\nu\mu}\,\delta g^{\mu\nu}\,. \tag{13.90}$$

Using this relation, we get that

$$\delta\mathcal{L} = \delta g^{\mu\nu}\left(-\frac{1}{2}\,g_{\nu\mu}\mathcal{L} + \frac{1}{2}\sqrt{|g|}\,\phi_{,\mu}\phi_{,\nu}\right) = \frac{1}{2}\,\delta g^{\mu\nu}\sqrt{|g|}\,T_{\nu\mu} = -\frac{1}{2}\,\delta g_{\mu\nu}\sqrt{|g|}\,T^{\nu\mu}\,. \tag{13.91}$$

Note that $\sqrt{|g|}\,T^{\mu\nu}$ is a symmetric tensor density, since $T^{\mu\nu}$ is a symmetric tensor.

Problems

1. Study the Klein–Gordon equation on the surface of a sphere of radius R.
   Try to solve it, e.g. by separating variables.
   What about the limit $R \to \infty$?
   What if the radius R is time dependent?
   (See also Problem 5.4.)

Chapter 14

The Maxwell field

The *electric field* $\boldsymbol{E} = \boldsymbol{E}(\boldsymbol{r}, t)$ and the *magnetic flux density* $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{r}, t)$ together constitute the *electromagnetic* field. The field equations for the electromagnetic field are Maxwell's equations, in which the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are coupled, so that they clearly have to be regarded as components of one and the same field. Maxwell discovered that his equations unifying electricity and magnetism have wave solutions, and that the waves propagate with the speed of light. This led to the important insight that light is electromagnetic waves.

The electromagnetic field pulls on electrically charged particles, and we measure the field by measuring the forces. Ideally, we should measure the force on a *test charge* which is so small that it does not change the field by influencing the sources. In classical field theory we assume that such ideal measuremants are always possible, in principle if not in practice.

The basis for the field measurement is the equation of motion for a point particle of charge q. Assuming that the metric tensor is constant, $g_{\mu\nu} = \eta_{\mu\nu}$, Equation (10.76) holds,

$$\dot{\boldsymbol{\pi}} = q\left(\boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{v} \times \boldsymbol{B}(\boldsymbol{r}, t)\right) \ . \tag{14.1}$$

We measure $\dot{\boldsymbol{\pi}}$ by measuring the position of the particle as a function of time, $\boldsymbol{r} = \boldsymbol{r}(t)$, and that determines the right hand side of the equation, which is called the Lorentz force. Recall that $\boldsymbol{\pi}$ is the mechanical momentum, given by the velocity $\boldsymbol{v} = \dot{\boldsymbol{r}} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, as distinguished from the canonical momentumen $\boldsymbol{p}$, containing in addition the three dimensional vector potential $\boldsymbol{A}$,

$$\boldsymbol{\pi} = \frac{m\boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} \ , \qquad \boldsymbol{p} = \boldsymbol{\pi} + q\boldsymbol{A}(\boldsymbol{r}, t) \ . \tag{14.2}$$

The above equation of motion is relativistically invariant, and we may make the invariance manifest by writing it in the form

$$\dot{\pi}_\mu = qF_{\mu\nu}\dot{x}^\nu \ . \tag{14.3}$$

This is Equation (10.89) with a constant metric. Here we have

$$\pi_\mu = (\pi^0, -\boldsymbol{\pi}) = \left(\frac{mc}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}} \ , -\frac{m\boldsymbol{v}}{\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}}}\right) , \tag{14.4}$$

and the field tensor $F_{\mu\nu}$ is an antisymmetric tensor, a 2-form, with $\boldsymbol{E}$ and $\boldsymbol{B}$ as components,

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{14.5}$$

14.1 Maxwell's equations

In vacuum

Maxwell's equations in vacuum are four homogeneous equations, that is, equations without source terms,

$$\nabla \cdot \boldsymbol{B} = 0\,, \tag{14.6}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0\,, \tag{14.7}$$

and four inhomogeneous equations, with source terms,

$$\nabla \cdot (\epsilon_0 \boldsymbol{E}) = \rho\,, \tag{14.8}$$

$$\nabla \times \left(\frac{\boldsymbol{B}}{\mu_0}\right) - \frac{\partial(\epsilon_0 \boldsymbol{E})}{\partial t} = \boldsymbol{j}\,, \tag{14.9}$$

which may also be written in the form

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0} = \mu_0 c^2 \rho\,, \tag{14.10}$$

$$\nabla \times \boldsymbol{B} - \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \boldsymbol{j}\,. \tag{14.11}$$

Here ϵ_0 and μ_0 are constants of nature, ϵ_0 is called the *permittivity* and μ_0 the *permeability* of vacuum. They determine the speed of light in vacuum to be $c = 1/\sqrt{\epsilon_0 \mu_0}$.

The density ρ and current density $\boldsymbol{j}$ of electric charge act as sources of the field. That the corresponding density and current density of magnetic charge vanish, is an experimental fact: magnetic monopoles have not yet been observed. Note that the field quantities having ρ and $\boldsymbol{j}$ as sources, are the *electric flux density* $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$ and the *magnetic field* $\boldsymbol{H} = \boldsymbol{B}/\mu_0$.

In a medium

For many purposes, a metal may be treated as a perfect conductor, and inside such an idealized medium there can be no electric field, because the electric charge distribution would readjust itself instantly to neutralize a nonzero field. Inside an insulating medium there may be electric and magnetic fields, and even when the atoms and molecules of the medium have no net charge, they may have electric and magnetic dipole moments, quadrupole moments, and so on, interacting with the fields.

The electromagnetic field inside such a medium may be described phenomenologically by modified Maxwell's equations, of the same form as Equations (14.8) and (14.9), but with ϵ_0 and μ_0 replaced by $\epsilon_r\epsilon_0$ and $\mu_r\mu_0$. The dimensionless proportionality constants ϵ_r and μ_r, called relative permittivity and relative permeability, are then phenomenological quantities characteristic of the medium. It is an important point that ϵ_r and μ_r occur inside the derivative operators ∇ and $\partial/\partial t$, if for some reason they vary in space and time. On the boundary between two different media they may even vary discontinuously.

Describing a medium by means of two constants ϵ_r and μ_r looks like a crude simplification. However, when describing the response of a medium to an electromagnetic field varying in time, we may take ϵ_r and μ_r to depend on the frequency. Such a frequency dependence is a major modification, since it means, strictly speaking, that Maxwell's equations become non-local in time.

Constraints and equations of motion

For Maxwell's equations to have any solution at all, it is necessary that the electric charge is conserved. In fact, it follows from the four equations with sources that

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = \frac{\partial(\nabla\cdot\boldsymbol{D})}{\partial t} + \nabla\cdot\left(\nabla\times\boldsymbol{H} - \frac{\partial\boldsymbol{D}}{\partial t}\right) = 0\,. \tag{14.12}$$

In a similar way, it is necessary that the magnetic charge is conserved, but of course this condition is trivially fulfilled since the magnetic charge is identically zero in Maxwell's equations.

The two consistency conditions, conservation of electric and magnetic charge, arise because we have altogether eight field equations for six field components $\boldsymbol{E}$ and $\boldsymbol{B}$. Of the eight equations there are six, (14.7) and (14.11), that contain time derivatives of $\boldsymbol{E}$ and $\boldsymbol{B}$, they are the *equations of motion* for the Maxwell field. The two remaining equations, (14.6) and (14.10), are *constraints*, that is, they are restrictions imposed on the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ at any given time. One of the constraints, Equation (14.6), is always consistent with the corresponding equation of motion (14.7), which gives that

$$\frac{\partial(\nabla\cdot\boldsymbol{B})}{\partial t} = \nabla\cdot\frac{\partial\boldsymbol{B}}{\partial t} = -\nabla\cdot(\nabla\times\boldsymbol{E}) = 0\,. \tag{14.13}$$

Equation (14.12) proves that the second constraint, Equation (14.10), is consistent with its corresponding equation of motion (14.11), assuming that the electric charge is conserved. Let us repeat the argument, to make it clear. Equation (14.11) together with conservation of electric charge imply that

$$\frac{\partial}{\partial t}\left(\nabla\cdot\boldsymbol{E} - \mu_0c^2\rho\right) = \nabla\cdot\left(\frac{\partial\boldsymbol{E}}{\partial t}\right) + \mu_0c^2\,\nabla\cdot\boldsymbol{j} = c^2\nabla\cdot(\nabla\times\boldsymbol{B}) = 0\,. \tag{14.14}$$

From this result follows that if Equation (14.10) holds at one given time, it holds at all times.

To repeat, if the constraints on $\boldsymbol{E}$ and $\boldsymbol{B}$ hold at one time, they hold at all times, by the equations of motion. Always under the condition that electric charge is conserved.

Maxwell's equations in relativistic form

The Maxwell field has six components $\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{B} = (B_x, B_y, B_z)$, just as many as an antisymmetric tensor of rank two. In fact, if $F_{\nu\mu} = -F_{\mu\nu}$, then

$$F_{\mu\nu} = \begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix} = \begin{pmatrix} 0 & F_{01} & F_{02} & F_{03} \\ -F_{01} & 0 & F_{12} & F_{13} \\ -F_{02} & -F_{12} & 0 & F_{23} \\ -F_{03} & -F_{13} & -F_{23} & 0 \end{pmatrix} . \tag{14.15}$$

The independent components are e.g. those above the diagonal.

As we have seen, the electromagnetic field tensor is indeed an antisymmetric tensor of rank two,

$$\boxed{F_{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix} .} \tag{14.16}$$

In other words, it is a 2-form,

$$\begin{aligned} \mathbf{F} = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \; = \; & F_{01}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 + F_{01}\, \mathrm{d}x^0 \wedge \mathrm{d}x^2 + F_{01}\, \mathrm{d}x^0 \wedge \mathrm{d}x^3 \\ & + F_{32}\, \mathrm{d}x^3 \wedge \mathrm{d}x^2 + F_{13}\, \mathrm{d}x^1 \wedge \mathrm{d}x^3 + F_{21}\, \mathrm{d}x^2 \wedge \mathrm{d}x^1 \\ = \; & E_x\, \mathrm{d}t \wedge \mathrm{d}x + E_y\, \mathrm{d}t \wedge \mathrm{d}y + E_z\, \mathrm{d}t \wedge \mathrm{d}z \\ & + B_x\, \mathrm{d}z \wedge \mathrm{d}y + B_y\, \mathrm{d}x \wedge \mathrm{d}z + B_z\, \mathrm{d}y \wedge \mathrm{d}x \; . \end{aligned} \tag{14.17}$$

We may use the metric tensor to raise both indices, defining as usual $F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}$. The Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$ gives that $F^{0j} = -F_{0j}$ and $F^{jk} = F_{jk}$, with $j, k = 1, 2, 3$, hence,

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} . \tag{14.18}$$

The four source free Maxwell's equations say that $F_{\mu\nu}$ is a closed 2-form, that is, its external derivative vanishes, $\mathrm{d}\mathbf{F} = 0$, or in component notation,

$$\boxed{(\mathrm{d}F)_{\rho\mu\nu} = F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu} = 0 \; .} \tag{14.19}$$

This is a tensor equation, since the external derivative of a 2-form is a 3-form. Hence the equation is invariant, not only under Poincaré transformations, but under arbitrary coordinate transformations.

The conservation law for electric charge, Equation (14.12), may be written in the special theory of relativity on the form $j^{\mu}{}_{,\mu} = 0$, where

$$j^{\mu} = (j^0, j^1, j^2, j^3) = (\rho c, \boldsymbol{j}) \; . \tag{14.20}$$

Hence, when we want to write the four equations with source terms on relativistic form, j^{μ} should appear on the right hand side of the equation, as the source. We see that the four equations may be joined in the one tensor equation

$$\boxed{F^{\mu\nu}{}_{,\nu} = -\mu_0 j^{\mu} \; .} \tag{14.21}$$

This equation is manifestly invariant under Poincaré transformations. We may make it generally covariant, that is, invariant under arbitrary coordinate transformations, simply by writing the partial derivative as a covariant derivative,

$$\boxed{F^{\mu\nu}{}_{;\nu} = -\mu_0 j^{\mu} \; .} \tag{14.22}$$

In the general theory of relativity the covariant derivative is uniquely given by the metric tensor, since the connection is assumed to be metric and symmetric. Then this last equation may be written in yet another form, which we will derive below from an invariant variational principle.

14.2 Lagrange formalism

Potentials

The four source free equations hold automatically if the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are given by a scalar potential Φ and a vector potential $\boldsymbol{A}$, as follows,

$$\boxed{\boldsymbol{E} = -\nabla\Phi - \frac{\partial \boldsymbol{A}}{\partial t} \; , \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \; .} \tag{14.23}$$

The potentials are not uniquely given by the fields, since a gauge transformation

$$\Phi \mapsto \widetilde{\Phi} = \Phi + \frac{\partial \chi}{\partial t} \; , \qquad \boldsymbol{A} \mapsto \widetilde{\boldsymbol{A}} = \boldsymbol{A} - \nabla\chi \; , \tag{14.24}$$

in which $\chi = \chi(\boldsymbol{r}, t)$ is an arbitrary function of space and time, does not change the fields $\boldsymbol{E}$ and $\boldsymbol{B}$. Since $\boldsymbol{E}$ and $\boldsymbol{B}$ are all that can be measured, at least according to the classical theory, this means that the potentials Φ and $\boldsymbol{A}$ are only partially measurable.

In a four dimensional notation we combine the scalar potential Φ and the vector potential $\boldsymbol{A}$ into a *four vector potential* with contravariant components

$$A^{\mu} = \left(\frac{\Phi}{c} \, , \boldsymbol{A}\right) = \left(\frac{\Phi}{c} \, , A_x \, , A_y \, , A_z\right) , \tag{14.25}$$

and covariant components

$$\boxed{A_\mu = \left(\frac{\Phi}{c}\,, -\boldsymbol{A}\right) = \left(\frac{\Phi}{c}\,, -A_x\,, -A_y\,, -A_z\right).} \tag{14.26}$$

Since there is seldom any risk of confusion, the four vector potential is also called a vector potential. Then we may express the field tensor as the external derivative of the vector potential,

$$\boxed{F_{\mu\nu} = (\mathrm{d}A)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = A_{\nu,\mu} - A_{\mu,\nu}\,.} \tag{14.27}$$

This is another tensor equation, which has the same form in any coordinate system. It shows immediately that the field tensor $F_{\mu\nu}$ is gauge invariant, since the gauge transformation of Equation (14.24) may be written as

$$A_\mu \mapsto \widetilde{A}_\mu = A_\mu - \partial_\mu \chi\,. \tag{14.28}$$

By a Lorentz transformation $x \mapsto \widetilde{x} = \Lambda x$ the vector potential A^μ is transformed into $\widetilde{A}^\mu$, given by the relation $\widetilde{A}^\rho(\widetilde{x}) = \Lambda^\rho{}_\sigma A^\sigma(x)$, or equivalently,

$$\widetilde{A}^\rho(x) = \Lambda^\rho{}_\sigma A^\sigma(\Lambda^{-1}x)\,. \tag{14.29}$$

In particular, we have for an infinitesimal Lorentz transformation

$$\Lambda^\rho{}_\sigma = \delta^\rho_\sigma + \omega^\rho{}_\sigma = \delta^\rho_\sigma + \frac{1}{2}\,\omega_{\mu\nu}\,(g^{\rho\mu}\delta^\nu_\sigma - g^{\rho\nu}\delta^\mu_\sigma) \tag{14.30}$$

that

$$\widetilde{A}_\kappa(x) = g_{\kappa\rho}\Lambda^\rho{}_\sigma g^{\sigma\lambda} A_\lambda(\Lambda^{-1}x) = \left(\delta^\lambda_\kappa + \frac{1}{2}\,\omega_{\mu\nu}\left(\delta^\mu_\kappa g^{\nu\lambda} - \delta^\nu_\kappa g^{\mu\lambda}\right)\right) A_\lambda(x - \omega x)\,. \tag{14.31}$$

This corresponds to Equation (12.54) when we substitute

$$\phi^i \to A_\kappa\,, \qquad S^i{}_j{}^{\mu\nu} \to S_\kappa{}^{\lambda\mu\nu} = \delta^\mu_\kappa g^{\nu\lambda} - \delta^\nu_\kappa g^{\mu\lambda}\,. \tag{14.32}$$

Lagrange density

Maxwell's equations for the free electromagnetic field, with zero charge density and current density, may be derived from the Lagrange density

$$\mathcal{L}_0 = \frac{1}{2}\left(\epsilon_0 \boldsymbol{E}^2 - \frac{\boldsymbol{B}^2}{\mu_0}\right) = \frac{1}{2\mu_0}\left(\frac{\boldsymbol{E}^2}{c^2} - \boldsymbol{B}^2\right), \tag{14.33}$$

by variation, not of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, but of the potentials Φ and $\boldsymbol{A}$. Thus, we write

$$\boxed{\mathcal{L}_0 = \frac{1}{2\mu_0}\left(\frac{1}{c^2}\left(\frac{\partial \boldsymbol{A}}{\partial t} + \nabla\Phi\right)^2 - (\nabla \times \boldsymbol{A})^2\right).} \tag{14.34}$$

The Lagrange density for the field with an external source is

$$\boxed{\mathcal{L} = \mathcal{L}_0 - \rho\Phi + \boldsymbol{j}\cdot\boldsymbol{A}\ .} \tag{14.35}$$

The Lagrange density $\mathcal{L}$ contains no time derivative of the scalar potential Φ. This means that Φ is a kind of Lagrange multiplier, and that the Euler–Lagrange equationen we get by varying Φ is a constraint rather than an equation of motion,

$$\frac{\partial\mathcal{L}}{\partial\Phi} - \nabla\cdot\left(\frac{\partial\mathcal{L}}{\partial(\nabla\Phi)}\right) = -\rho + \nabla\cdot\left(\frac{\boldsymbol{E}}{\mu_0 c^2}\right) = 0\ . \tag{14.36}$$

This is Equation (14.10).

By variation of $\boldsymbol{A}$ we get Equation (14.11), but we do not carry out that computation here, because it is much simpler to use the relativistic notation. Note that the electric field contains the time derivative of $\boldsymbol{A}$, and hence $\boldsymbol{E}^2$ in the Lagrange density may be regarded as a density of kinetic energy. Similarly, $\boldsymbol{B}^2$ may be regarded as a density of potential energy.

The expressions for $\mathcal{L}_0$ and $\mathcal{L}$ written in relativistic form are,

$$\boxed{\mathcal{L}_0 = -\frac{1}{4\mu_0}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4\mu_0}g^{\mu\rho}g^{\nu\sigma}\left(A_{\nu,\mu} - A_{\mu,\nu}\right)\left(A_{\sigma,\rho} - A_{\rho,\sigma}\right)\ ,} \tag{14.37}$$

and

$$\boxed{\mathcal{L} = \mathcal{L}_0 - j^\mu A_\mu\ .} \tag{14.38}$$

Thus we get that

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial A_{\kappa,\lambda}} &= -\frac{1}{4\mu_0}\left(\left(g^{\lambda\rho}g^{\kappa\sigma} - g^{\kappa\rho}g^{\lambda\sigma}\right)\left(A_{\sigma,\rho} - A_{\rho,\sigma}\right) + \left(g^{\mu\lambda}g^{\nu\kappa} - g^{\mu\kappa}g^{\nu\lambda}\right)\left(A_{\nu,\mu} - A_{\mu,\nu}\right)\right)\\ &= \frac{1}{\mu_0}g^{\kappa\rho}g^{\lambda\sigma}\left(A_{\sigma,\rho} - A_{\rho,\sigma}\right) = \frac{1}{\mu_0}F^{\kappa\lambda}\ .\end{aligned} \tag{14.39}$$

Which leads to the following Euler–Lagrange equations,

$$\frac{\partial\mathcal{L}}{\partial A_\mu} - \frac{\mathrm{d}}{\mathrm{d}x^\nu}\left(\frac{\partial\mathcal{L}}{\partial A_{\mu,\nu}}\right) = -j^\mu - \frac{1}{\mu_0}\partial_\nu F^{\mu\nu} = 0\ . \tag{14.40}$$

Lagrange density in a medium

Maxwell's equations in a medium may be derived from the Lagrange density

$$\mathcal{L}' = \frac{1}{2}\left(\epsilon_r\epsilon_0\boldsymbol{E}^2 - \frac{\boldsymbol{B}^2}{\mu_r\mu_0}\right) - \rho\Phi + \boldsymbol{j}\cdot\boldsymbol{A}\ . \tag{14.41}$$

The only difference compared to the Lagrange density in vacuum is that it contains the relative permittivity and permeability. Like before, $\boldsymbol{E}$ and $\boldsymbol{B}$ are expressed in terms of the potentials Φ and $\boldsymbol{A}$, and by varying the potentials, we get Maxwell's equations with ϵ_r and μ_r within the differentiation operators, as is correct.

This Lagrange density is no longer Lorentz-invariant, for the good reason that the medium is at rest and hence breaks the Lorentz invariance.

14.3 Energy momentum tensor

The canonical energy momentum tensor for the electromagnetic field, without sources, is

$$\widehat{T}^{\mu\nu} = g^{\mu\sigma} \frac{\partial \mathcal{L}_0}{\partial A_{\rho,\nu}} A_{\rho,\sigma} - g^{\mu\nu} \mathcal{L}_0 = \frac{1}{\mu_0} g^{\mu\sigma} F^{\rho\nu} A_{\rho,\sigma} + \frac{1}{4\mu_0} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \,. \tag{14.42}$$

It is not symmetric, because the electromagnetic field has spin. Even more important, it is not gauge invariant, even though the Lagrange density $\mathcal{L}_0$ is gauge invariant.

In order to symmetriz it, we add a term $\psi^{\mu\nu\rho}{}_{,\rho}$, where $\psi^{\mu\nu\rho}$ is given in terms of the spin, Equation (14.32), like in Equation (12.73),

$$\begin{aligned}
\psi^{\mu\nu\rho} &= -\frac{1}{2} \left(\frac{\partial \mathcal{L}}{\partial A_{\kappa,\rho}} S_\kappa{}^{\lambda\mu\nu} - \frac{\partial \mathcal{L}}{\partial A_{\kappa,\nu}} S_\kappa{}^{\lambda\mu\rho} - \frac{\partial \mathcal{L}}{\partial A_{\kappa,\mu}} S_\kappa{}^{\lambda\nu\rho} \right) A_\lambda \\
&= -\frac{1}{2\mu_0} \Big(F^{\kappa\rho} \left(\delta^\mu_\kappa g^{\nu\lambda} - \delta^\nu_\kappa g^{\mu\lambda} \right) - F^{\kappa\nu} \left(\delta^\mu_\kappa g^{\rho\lambda} - \delta^\rho_\kappa g^{\mu\lambda} \right) \\
&\qquad - F^{\kappa\mu} \left(\delta^\nu_\kappa g^{\rho\lambda} - \delta^\rho_\kappa g^{\nu\lambda} \right) \Big) A_\lambda \\
&= -\frac{1}{2\mu_0} \left(F^{\mu\rho} A^\nu - F^{\nu\rho} A^\mu - F^{\mu\nu} A^\rho + F^{\rho\nu} A^\mu - F^{\nu\mu} A^\rho + F^{\rho\mu} A^\nu \right) \\
&= \frac{1}{\mu_0} F^{\nu\rho} A^\mu \,.
\end{aligned} \tag{14.43}$$

Under the assumption that the field satisfies Maxwell's equations without source, $F^{\nu\rho}{}_{,\rho} = 0$, it follows that Belinfante's symmetric energy momentum tensor is

$$\boxed{\begin{aligned}
T^{\mu\nu} &= \widehat{T}^{\mu\nu} + \psi^{\mu\nu\rho}{}_{,\rho} \\
&= \frac{1}{\mu_0} \left(g^{\mu\sigma} F^{\rho\nu} A_{\rho,\sigma} + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} + F^{\nu\rho} A^\mu{}_{,\rho} \right) \\
&= \frac{1}{\mu_0} \left(F^{\rho\nu} g^{\mu\sigma} \left(A_{\rho,\sigma} - A_{\sigma,\rho} \right) + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right) \\
&= \frac{1}{\mu_0} \left(-F^{\rho\nu} F_\rho{}^\mu + \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right) .
\end{aligned}} \tag{14.44}$$

This energy momentum tensor is actually symmetric independent of whether the field equation holds. Moreover, it is gauge-invariant. Note that its trace vanishes,

$$T_\mu{}^\mu = \frac{1}{\mu_0} \left(-F^{\rho\mu} F_{\rho\mu} + \frac{1}{4} \delta^\mu_\mu F^{\rho\sigma} F_{\rho\sigma} \right) = 0 \,. \tag{14.45}$$

It may be of interest to express the components of the tensor

$$T_\mu{}^\nu = \frac{1}{\mu_0}\left(-F^{\rho\nu}F_{\rho\mu} + \frac{1}{4}\,\delta^\nu_\mu F^{\rho\sigma}F_{\rho\sigma}\right) \tag{14.46}$$

more explicitly in terms of the fields $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$ and $\boldsymbol{H}$. First, we have that

$$\begin{aligned} T_0{}^0 &= \frac{1}{\mu_0}\left(-F^{\rho 0}F_{\rho 0} + \frac{1}{4}\,F^{\rho\sigma}F_{\rho\sigma}\right) = \frac{1}{2\mu_0}\left(\frac{\boldsymbol{E}^2}{c^2} + \boldsymbol{B}^2\right) \\ &= \frac{1}{2}\left(\epsilon_0\boldsymbol{E}^2 + \frac{\boldsymbol{B}^2}{\mu_0}\right) = \frac{1}{2}\,\left(\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{H}\cdot\boldsymbol{B}\right). \end{aligned} \tag{14.47}$$

This is the energy density, let us call it U, which is non-negative. The advantage of introducing the quantities $\boldsymbol{D}$ and $\boldsymbol{H}$ here is that we may take over the formula directly in a medium with $\epsilon_r \neq 1$ and/or $\mu_r \neq 1$.

Second, we have that

$$T_0{}^1 = -\frac{1}{\mu_0}\,F^{\rho 1}F_{\rho 0} = \frac{1}{\mu_0 c}\,\left(B_zE_y - B_yE_z\right) = \frac{1}{c}\,\left(H_zE_y - H_yE_z\right) = \frac{S_x}{c}\,, \tag{14.48}$$

where S_x is the x component of the Poynting vector

$$\boldsymbol{S} = \boldsymbol{E}\times\boldsymbol{H}\,, \tag{14.49}$$

which is the energy current density. In the same way, $T_0{}^2 = S_y/c$ and $T_0{}^3 = S_z/c$.

Energy conservation in the free electromagnetic field is expressed in the continuity equation

$$T_0{}^\mu{}_{,\mu} = \frac{1}{c}\left(\frac{\partial U}{\partial t} + \nabla\cdot\boldsymbol{S}\right) = 0\,, \tag{14.50}$$

which follows from Maxwell's equations for the free field, i.e. with $\rho = 0$ and $\boldsymbol{j} = 0$.

Third, we have that

$$T_1{}^0 = -\frac{1}{\mu_0}\,F^{\rho 0}F_{\rho 1} = -\frac{1}{\mu_0 c}\,\left(E_yB_z - E_zB_y\right) = -c\,\left(D_yB_z - D_zB_y\right) = -cg_x\,. \tag{14.51}$$

Here, g_x is the x component of the momentum density,

$$\boldsymbol{g} = \boldsymbol{D}\times\boldsymbol{B}\,. \tag{14.52}$$

Similarly, $T_2{}^0 = -g_y$ and $T_3{}^0 = -g_z$. In the vacuum we have that

$$\boldsymbol{g} = \epsilon_0\boldsymbol{E}\times\boldsymbol{B} = \frac{1}{c^2}\,\boldsymbol{E}\times\boldsymbol{H} = \frac{1}{c^2}\,\boldsymbol{S}\,, \tag{14.53}$$

which means that

$$T^{10} = cg_x = \frac{S_x}{c} = T^{01}\,. \tag{14.54}$$

In a medium with $\epsilon_r \neq 1$ or $\mu_r \neq 1$ this symmetry relation no longer holds, because Lorentz invariance no longer holds.

Fourth, we have that

$$T_1{}^1 = \frac{1}{\mu_0}\left(-F^{\rho 1}F_{\rho 1} + \frac{1}{4}F^{\rho\sigma}F_{\rho\sigma}\right) = \frac{1}{2\mu_0}\left(\frac{{E_x}^2 - {E_y}^2 - {E_z}^2}{c^2} + {B_x}^2 - {B_y}^2 - {B_z}^2\right)$$
$$= D_xE_x + H_xB_x - \frac{1}{2}\left(\boldsymbol{D}\cdot\boldsymbol{E} + \boldsymbol{H}\cdot\boldsymbol{B}\right). \tag{14.55}$$

And fifth, we have that

$$T_1{}^2 = -\frac{1}{\mu_0}F^{\rho 2}F_{\rho 1} = \frac{1}{\mu_0}\left(\frac{E_xE_y}{c^2} + B_xB_y\right) = D_xE_y + H_xB_y\ . \tag{14.56}$$

We get corresponding formulae for all the spatial components $T_j{}^k$. This 3×3 tensor is the stress tensor of the electromagnetic field.

For example, conservation of the x component of the momentum of the free electromagnetic field is expressed in the continuity equation

$$T_1{}^\mu{}_{,\mu} = -\left(\frac{\partial g_x}{\partial t} + \frac{\partial(-T_1{}^1)}{\partial x} + \frac{\partial(-T_1{}^2)}{\partial y} + \frac{\partial(-T_1{}^3)}{\partial z}\right) = 0\ . \tag{14.57}$$

Which again follows from Maxwell's equations for the free field.

14.4 Plane wave solutions

A plane wave solution of Maxwell's equations will have the form

$$A_\mu(x) = a_\mu\, f(k_\rho x^\rho)\ . \tag{14.58}$$

Here a_μ is a constant *polarization vector*, and k_ρ is a constant four dimensional *wave number vector*, whereas f is an arbitrary function of one variable. This vector potential gives the field tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = (k_\mu a_\nu - k_\nu a_\mu)f'(k_\rho x^\rho)\ , \tag{14.59}$$

or,

$$F^{\mu\nu} = (k^\mu a^\nu - k^\nu a^\mu)f'(k_\rho x^\rho)\ . \tag{14.60}$$

Maxwell's equations in vacuum say that

$$F^{\mu\nu}{}_{,\nu} = k_\nu(k^\mu a^\nu - k^\nu a^\mu)f''(k_\rho x^\rho) = 0\ , \tag{14.61}$$

and excluding the trivial case where $f'' = 0$ identically, this means that

$$k^\mu(k_\nu a^\nu) - (k_\nu k^\nu)a^\mu = 0\ . \tag{14.62}$$

There are two ways to solve this equation. Either we have $k_\nu k^\nu \neq 0$, and consequently $a^\mu = \alpha k^\mu$ for some constant α, but this gives nothing but the trivial solution $F^{\mu\nu} = 0$. Or else we have that

$$k_\nu k^\nu = 0 \qquad \text{and} \qquad k_\nu a^\nu = 0\ . \tag{14.63}$$

The last two conditions give the only non-trivial plane wave solutions. Since the four dimensional polarisation vector a^μ is subject to the single condition $k_\nu a^\nu = 0$, it apparently has three degrees of freedom, but in reality it has only two. Indeed, if it is longitudinal, that is if $a^\mu = \alpha k^\mu$, then it is true that $k_\nu a^\nu = 0$, but it is also true that $F^{\mu\nu} = 0$.

If, for example, the wave moves in the positive z direction, then

$$k^\mu = (k^0, 0, 0, k^0) \ , \qquad k_\mu = (k^0, 0, 0, -k^0) \ . \tag{14.64}$$

With $a^\mu = (a^0, a^1, a^2, a^3)$ this means that

$$k_\nu a^\nu = k^0(a^0 - a^3) = 0 \ . \tag{14.65}$$

This condition alone has the three linearly independent solutions

$$a^\mu = \begin{cases} (0,1,0,0) \ , \\ (0,0,1,0) \ , \\ (1,0,0,1) \ . \end{cases} \tag{14.66}$$

But of these the third one gives $F^{\mu\nu} = 0$, because $a^\mu = \alpha k^\mu$. For the two remaining cases we have $a_\mu a^\mu = -1$. Thus, a plane wave propagating in the z direction has a polarization vector lying in the (x, y) plane.

The plane wave solution found above has

$$F^{\rho\sigma}F_{\rho\sigma} = (k^\rho a^\sigma - k^\sigma a^\rho)(k_\rho a_\sigma - k_\sigma a_\rho)\left(f'(k_\lambda x^\lambda)\right)^2 = 0 \ , \tag{14.67}$$

and its energy momentum tensor is

$$\begin{aligned} T^{\mu\nu} \ &= \ -\frac{1}{\mu_0}F^{\rho\nu}F_\rho{}^\mu = -\frac{1}{\mu_0}(k^\rho a^\nu - k^\nu a^\rho)(k_\rho a^\mu - k^\mu a_\rho)\left(f'(k_\lambda x^\lambda)\right)^2 \\ &= \ -\frac{1}{\mu_0}\,(a_\rho a^\rho)k^\mu k^\nu \left(f'(k_\lambda x^\lambda)\right)^2 \ . \end{aligned} \tag{14.68}$$

The energy density T^{00} is non-negative, since $a_\rho a^\rho < 0$.

Note that the plane wave moves without changing its shape, which is given by the arbitrary function f. In other words, there is no dispersion. In order to examine this concept more closely, let us Fourier transform the function f,

$$f(\xi) = \int_{-\infty}^{\infty} \mathrm{d}\kappa\, \widetilde{f}(\kappa)\, \mathrm{e}^{-\mathrm{i}\kappa\xi} \ . \tag{14.69}$$

This gives that

$$A_\mu(x) = a_\mu f(k_\rho x^\rho) = a_\mu \int_{-\infty}^{\infty} \mathrm{d}\kappa\, \widetilde{f}(\kappa)\, \mathrm{e}^{-\mathrm{i}\kappa k_\rho x^\rho} \ . \tag{14.70}$$

Writing here $\kappa k_\rho x^\rho = \omega' t - \boldsymbol{k}' \cdot \boldsymbol{r}$, we have that

$$\omega' = c\kappa k^0 \ , \qquad \boldsymbol{k}' = \kappa \boldsymbol{k} \ . \tag{14.71}$$

Since $k_\nu k^\nu = (k^0)^2 - \boldsymbol{k}^2 = 0$, we get the dispersion relation

$$\omega' = c|\boldsymbol{k}'| \ , \tag{14.72}$$

which implies that both the phase velocity v_f and the group velocity v_g equal c,

$$v_f = \frac{\omega'}{|\boldsymbol{k}'|} = c \ , \qquad \boldsymbol{v}_g = \frac{\partial \omega'}{\partial \boldsymbol{k}'} = c\,\frac{\boldsymbol{k}'}{|\boldsymbol{k}'|} \ . \tag{14.73}$$

14.5 Interaction of particles and fields

Field from point particles

If the sources of the Maxwell field are N pointlike electric charges q_i, with $i = 1, 2, \ldots, N$, at the positions $\boldsymbol{R}_i(t)$ at time t, then the charge density is

$$\rho = \rho(\boldsymbol{r}, t) = \sum_{i=1}^{N} q_i \, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{R}_i(t)) \; . \tag{14.74}$$

The current density contains the velocities $\boldsymbol{V}_i = \dot{\boldsymbol{R}}_i = \mathrm{d}\boldsymbol{R}_i/\mathrm{d}t$, and it equals

$$\boldsymbol{j} = \boldsymbol{j}(\boldsymbol{r}, t) = \sum_{i=1}^{N} q_i \boldsymbol{V}_i \, \delta^{(3)}(\boldsymbol{r} - \boldsymbol{R}_i(t)) \; . \tag{14.75}$$

We may derive these expressions by starting with a point charge at rest, and giving it a constant velocity by means of an active Lorentz transformation. Remember that we should transform $j^\mu = (\rho c, \boldsymbol{j})$ as a contravariant vector field. This particular example was discussed in Chapter 1, although in only one spatial dimension. From that simple example it is easy to generalize to N point charges with variable velocities, in three dimensions. Compare with Equation (13.34), which gives a scalar charge density, as opposed to the electromagnetic charge and current densities, which are the components of a four vector.

Maxwell's equations with point sources can be solved explicitly, and the solution is the Liénard–Wiechert potential.

In the special case of one single point charge Q at rest at the origin, we assume that the fields Φ and $\boldsymbol{A}$ are time independent, and we may even assume that $\boldsymbol{A} = 0$. The field equation is then

$$\nabla \cdot \boldsymbol{E} = -\nabla^2 \Phi = \frac{\rho}{\epsilon_0} = \frac{Q}{\epsilon_0} \, \delta^{(3)}(\boldsymbol{r}) \; . \tag{14.76}$$

The solution is the Coulomb potential, which is a function of $r = |\boldsymbol{r}|$,

$$\Phi = \Phi(r) = \frac{Q}{4\pi\epsilon_0 r} \; . \tag{14.77}$$

Note that the sign here is the opposite of the sign of the Yukawa potential in Equation (13.37). The sign is obviously important, because it determines whether the force between equal charges is repulsive or attractive. The difference between the Klein–Gordon field and the Maxwell field may be traced back to the Lagrange density. In fact, the gradient $\nabla\phi$ of the Klein–Gordon field contributes to what we may call the potential energy of the field, hence the negative sign of that term in the Lagrange density. The gradient $\nabla\Phi$ of Maxwell's scalar potential, on the other hand, contributes to what is more properly called the kinetic energy, which must have a positive sign in the Lagrange density.

The energy density of the Coulomb potential is

$$T^{00} = \frac{1}{2} \, \epsilon_0 \boldsymbol{E}^2 = \frac{Q^2}{16\pi^2 \epsilon_0 r^4} \; , \tag{14.78}$$

implying that the field energy is infinite, in the same way as for the Yukawa potential. Again we have to admit that classical field theory with point particles as sources is inconsistent.

Maxwell's equations determine uniquely the time development of an electromagnetic field in vacuum, with vanishing density and current density of electric charge. However, this is no longer true if the field has singularities representing (for example) point charges. Maxwell's equations demand that electric charge is conserved but otherwise gives no equations of motion for the singularities. Thus, the orbits $\boldsymbol{R}_i(t)$ above are arbitrary, and the singularities in the field give rise to a kind of indeterminism in Maxwell's equations, which has to be repaired by means of separate equations of motion for the singularities.

Interaction between point particles

The Lagrange function of a system consisting of point particles in interaction with the electromagnetic field is the sum of three terms, $L = L_F + L_P + L_I$, representing field, particles and interaction between field and particles, like in the case of the Klein–Gordon field. With point sources as in Equation (14.74), the interaction term is

$$L_I = \int \mathrm{d}^3\boldsymbol{r}\,(-\rho\Phi + \boldsymbol{j}\cdot\boldsymbol{A}) = \sum_{i=1}^{N}(-q_i\Phi_i + q_i\boldsymbol{V}_i\cdot\boldsymbol{A}_i)\;, \tag{14.79}$$

where $\Phi_i = \Phi(\boldsymbol{R}_i(t),t)$ and $\boldsymbol{A}_i = \boldsymbol{A}(\boldsymbol{R}_i(t),t)$. In order to make the Lagrange function L_I finite, we again have to introduce the arbitrary rule that a point particle does not interact with its own field.

The equations of motion for N point particles of masses m_i in an external field may be derived from the following Lagrange function, which we already know,

$$L_P + L_I = \sum_{i=1}^{N}\left(-m_i\,c^2\sqrt{1-\frac{\boldsymbol{V}_i{}^2}{c^2}} - q_i\Phi_i + q_i\boldsymbol{V}_i\cdot\boldsymbol{A}_i\right). \tag{14.80}$$

We want now to consider the interaction of point particles with the electromagnetic field when they themselves are the sources of the field. That is, we want to consider the interaction between point particles mediated by the electromagnetic field. In order to illustrate the principles, we will use not the complete Liénard–Wiechert potential, but only the static Coulomb potential, which is the non-relativistic approximation when particle velocities are small compared to the speed of light. This gives the following interaction Lagrange function, excluding self interactions,

$$L_I = -\sum_{i=1}^{N}\sum_{j\neq i}\frac{q_iq_j}{4\pi\epsilon_0\,|\boldsymbol{r}_i-\boldsymbol{r}_j|} \tag{14.81}$$

Just as with the Klein–Gordon field, L_I alone is too large by a factor of two, and again the way to obtain the correct interaction term for the particles is to include the Lagrange function of the field,

$$L_F = \frac{1}{2\mu_0c^2}\int \mathrm{d}^3\boldsymbol{r}\;(\nabla\Phi)^2 = \frac{1}{2\mu_0c^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\int \mathrm{d}^3\boldsymbol{r}\;(\nabla\Phi_{(i)})\cdot(\nabla\Phi_{(j)})\;. \tag{14.82}$$

In the static approximation we neglect the vector potential $\boldsymbol{A}$, and we write here $\Phi_{(i)}$ for the Coulomb potential from particle i,

$$\Phi_{(i)} = \Phi_{(i)}(\boldsymbol{r}) = \frac{q_i}{4\pi\epsilon_0\,|\boldsymbol{r}-\boldsymbol{r}_i|}\;. \tag{14.83}$$

As always in such cases, we ignore the self energy terms in Equation (14.82), with $i = j$, because they are infinite, and because they are not relevant for the interaction between the particles. The interaction terms, with $i \neq j$, are not very difficult to calculate directly, but the simplest calculation is by means of a partial integration,

$$\int \mathrm{d}^3\boldsymbol{r}\ (\nabla\Phi_{(i)}) \cdot (\nabla\Phi_{(j)}) = -\int \mathrm{d}^3\boldsymbol{r}\ \Phi_{(i)}(\nabla^2\Phi_{(j)})\ , \tag{14.84}$$

after which we use Maxwell's equation

$$\nabla^2\Phi_{(j)} = -\frac{q_j}{\epsilon_0}\,\delta^{(3)}(\boldsymbol{r} - \boldsymbol{R}_j)\ . \tag{14.85}$$

The result in the end is that $L_F = -L_I/2$, and hence

$$L_F + L_I = \frac{L_I}{2} = -\sum_{i=1}^{N}\sum_{j<i}\frac{q_i q_j}{4\pi\epsilon_0\,|\boldsymbol{r}_i - \boldsymbol{r}_j|}\ . \tag{14.86}$$

Again we see that we get the correct result by counting each pair of particles once, and not twice as in Equation (14.81).

14.6 Maxwell's equations in an external gravitational field

So far in this chapter we have assumed that the metric is constant, that $g_{\mu\nu} = \eta_{\mu\nu}$. The generalization to an external gravitational field is just as simple for the Maxwell field as for the Klein–Gordon field. All we have to do is to introduce an explicit factor $\sqrt{|g|}$ in the Lagrange density, such that

$$\mathcal{L} = \sqrt{|g|}\,\mathcal{L}_s\ , \tag{14.87}$$

where $\mathcal{L}_s$ is a Lagrange scalar, the same as above.

Thus, the Lagrange density for the Maxwell field, with source term, is

$$\mathcal{L} = -\frac{1}{4\mu_0}\sqrt{|g|}\,g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} - \sqrt{|g|}\,j^\mu A_\mu\ . \tag{14.88}$$

It is transformed as a scalar density when j^μ is transformed as a contravariant and A_μ as a covariant vector. When j^μ is a vector, it means that $\sqrt{|g|}\,j^\mu$ is a vector density. Logically speaking we ought to call j^μ a current *vector*, and reserve the name current *density* for the quantity $\sqrt{|g|}\,j^\mu$.

The Euler–Lagrange equation resulting from this Lagrange density is Equation (14.21) once more, except that there appears an extra factor $\sqrt{|g|}$,

$$\partial_\nu\left(\sqrt{|g|}\,F^{\mu\nu}\right) = -\mu_0\sqrt{|g|}\,j^\mu\ . \tag{14.89}$$

We know that this equation is invariant under arbitrary coordinate transformations, because it is derived from an invariant action integral. This is in fact one of the cases where covariant differentiation with a metric and symmetric connection reduces to ordinary partial differentiation. Another case, relevant for the Klein–Gordon equation, as we have seen, is the

divergence of a vector density. Thus, the derivative in Equation (14.89) is actually a covariant derivative, and the equation is nothing but Equation (14.22) multiplied by $\sqrt{|g|}$.

That the equation is invariant, means that the left and right hand sides transform in the same way under coordinate transformations. We know that the right hand side $\sqrt{|g|}\, j^\mu$ transforms as a vector density, hence the left hand side is also a vector density. The left hand side contains the quantity $\sqrt{|g|}\, F^{\mu\nu}$, which is an antisymmetric contravariant tensor density of rank 2, or simply a density of rank 2. Thus we have actually proved in general that the divergence of a density of rank 2, with ordinary partial differentiation, is a density av rank 1, that is, a vector density. These concepts are treated in more detail in the Chapters 3 and 5.

From Equation (14.89) follows directly the conservation law for electric charge,

$$\partial_\mu \left(\sqrt{|g|}\, j^\mu \right) = -\frac{1}{\mu_0}\, \partial_\mu \partial_\nu \left(\sqrt{|g|}\, F^{\mu\nu} \right) = 0 \, . \tag{14.90}$$

The conservation law in this form is invariant under arbitrary coordinate transformations, since the divergence $\partial_\mu \left(\sqrt{|g|}\, j^\mu \right)$ of a vector density $\sqrt{|g|}\, j^\mu$ is a scalar density. In order to prove that the divergence vanishes, we use that $F^{\mu\nu}$ is antisymmetric, and that partial derivatives commute,

$$\partial_\mu \partial_\nu \left(\sqrt{|g|}\, F^{\mu\nu} \right) = \partial_\nu \partial_\mu \left(\sqrt{|g|}\, F^{\mu\nu} \right) = -\partial_\nu \partial_\mu \left(\sqrt{|g|}\, F^{\nu\mu} \right) = -\partial_\mu \partial_\nu \left(\sqrt{|g|}\, F^{\mu\nu} \right) . \tag{14.91}$$

The symmetric energy momentum tensor once more

It is a remarkable fact that we may derive the symmetric energy momentum tensor of a field by varying the action integral with respect to the metric tensor. We saw that this was true for the Klein–Gordon field, and the derivation for the Maxwell field is just as simple.

The Lagrange density of the Maxwell field without sources, but in an external gravitational field, is

$$\mathcal{L} = -\frac{1}{4\mu_0} \sqrt{|g|}\, g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} \, . \tag{14.92}$$

We vary the metric tensor $g_{\mu\nu}$ and use that

$$\delta \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|}\, g_{\lambda\kappa}\, \delta g^{\kappa\lambda} \, , \tag{14.93}$$

according to Equation (2.81). This gives that

$$\begin{aligned} \delta\mathcal{L} \;&=\; \delta g^{\kappa\lambda} \left(-\frac{1}{2}\, g_{\lambda\kappa} \mathcal{L} - \frac{1}{4\mu_0} \sqrt{|g|}\, (g^{\nu\sigma} F_{\kappa\nu} F_{\lambda\sigma} + g^{\mu\rho} F_{\mu\kappa} F_{\rho\lambda}) \right) \\ &=\; \frac{1}{2}\, \delta g^{\kappa\lambda} \sqrt{|g|}\, T_{\lambda\kappa} = -\frac{1}{2}\, \delta g_{\kappa\lambda} \sqrt{|g|}\, T^{\lambda\kappa} \, . \end{aligned} \tag{14.94}$$

14.7 Duality between electric and magnetic fields

The dual field tensor

Using both the metric and the Levi–Civita symbol we may transform the electromagnetic field tensor $F_{\mu\nu}$ into a dual tensor, the Hodge dual, defined as

$${}^*F_{\alpha\beta} = \frac{1}{2}\, \epsilon_{\kappa\lambda\alpha\beta} \sqrt{|g|}\, g^{\kappa\mu} g^{\lambda\nu} F_{\mu\nu} \, . \tag{14.95}$$

The dual of ${}^*F_{\mu\nu}$ again is ${}^{**}F_{\mu\nu} = -F_{\mu\nu}$. See Chapter 3.

Maxwell's equations do not look essentially different when written in terms of the dual field tensor. All that happens is that the source free equations and the equations with source terms exchange their roles. The four source free equations may be written in the form

$$\partial_\nu \left(\sqrt{|g|}\ {}^*F^{\mu\nu} \right) = 0 \ , \tag{14.96}$$

whereas the four source equations may be written as

$$(\mathrm{d}\,{}^*F)_{\rho\mu\nu} = {}^*F_{\mu\nu,\rho} + {}^*F_{\rho\mu,\nu} + {}^*F_{\nu\rho,\mu} = -\mu_0\, \epsilon_{\lambda\mu\nu\rho} \sqrt{|g|}\ j^\lambda \ . \tag{14.97}$$

This equation equates two 3-forms.

That the dual field tensor is to such high degree equivalent to the original one, depends on the fact that spacetime is four dimensional. Only in four dimensions is it true that the dual of a 2-form is again a 2-form, in general it is a $(d-2)$-form in d dimensions.

The effect of the duality transformation is to exchange the electric and magnetic fields. Spesializing to the Minkowski metric $g_{\mu\nu} = \eta_{\mu\nu}$, we have that $\sqrt{|g|} = 1$, and for example,

$$\begin{aligned} {}^*F_{01} &= \frac{1}{2}\, \epsilon_{\kappa\lambda 01}\, F^{\kappa\lambda} = F^{23} = -B_x \ , \\ {}^*F_{23} &= \frac{1}{2}\, \epsilon_{\kappa\lambda 23}\, F^{\kappa\lambda} = F^{01} = -\frac{E_x}{c} \ . \end{aligned} \tag{14.98}$$

If we introduce the complex quantity

$$\boldsymbol{G} = \frac{\boldsymbol{E}}{c} + \mathrm{i}\boldsymbol{B} \ , \tag{14.99}$$

we see that the dual of $\boldsymbol{G}$ is

$${}^*\boldsymbol{G} = -\boldsymbol{B} + \mathrm{i}\,\frac{\boldsymbol{E}}{c} = \mathrm{i}\boldsymbol{G} \ . \tag{14.100}$$

Alternative Lagrange densities?

An interesting question about any physical theory is to what degree it is uniquely determined by some set of general principles. We may ask for example whether it is possible to modify Maxwell's equations without destroying their gauge invariance and general covariance. Another phrasing of the same question is whether there exists a scalar density that we might add to the Lagrange density, containing the four vector potential A_μ in such a way that it is still gauge invariant.

We might try to add to the Lagrange density a mass term proportional to $A_\mu A^\mu$, in analogy to the mass term ϕ^2 of the Klein–Gordon field. However, such a mass term would not be gauge-invariant. Thus, we may relate the fact that the photon is massless to the gauge invariance of Maxwell's equations.

Another scalar density which is gauge invariant and hence a candidate for inclusion in the Lagrange density, is

$$\mathcal{L}' = a\, \epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda} F_{\mu\nu} = 4a\, \epsilon^{\kappa\lambda\mu\nu} (\partial_\kappa A_\lambda)(\partial_\mu A_\nu) \ , \tag{14.101}$$

with a constant. To be more precise, $\mathcal{L}'$ is a pseudoscalar density. Note that this particular density owes its existence to the fact that the Levi–Civita symbol $\epsilon^{\kappa\lambda\mu\nu}$ has four indices in four dimensional spacetime. In terms of the dual field tensor ${}^*F_{\mu\nu}$ we have that

$$\mathcal{L}' = -2a\sqrt{|g|}\, g^{\mu\rho} g^{\nu\sigma}\, {}^*F_{\mu\nu} F_{\rho\sigma} \ . \tag{14.102}$$

In the Minkowski case, where the relation between the field tensor $F_{\mu\nu}$ and the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ is given by Equation (14.5), we have that

$$\epsilon^{\kappa\lambda\mu\nu} F_{\kappa\lambda} F_{\mu\nu} = -\frac{8}{c}\, \boldsymbol{E}\cdot\boldsymbol{B} \ . \tag{14.103}$$

That the three dimensional scalar product $\boldsymbol{E}\cdot\boldsymbol{B}$ is rotationally invariant, is evident, but that it is also Lorentz invariant is less evident.

What is wrong with the pseudoscalar density $\mathcal{L}'$ is that it is the divergence of a pseudovector density,

$$\mathcal{L}' = \partial_\kappa \left(4a\, \epsilon^{\kappa\lambda\mu\nu} A_\lambda (\partial_\mu A_\nu) \right) \ . \tag{14.104}$$

Therefore the action integral of $\mathcal{L}'$ may be transformed into a surface integral, which can not give any contribution to the classical Euler–Lagrange equations.

A term like this in the Lagrange density would have broken explicitly both the parity invariance and the time reversal invariance of Maxwell's equations, if it had contributed to the field equations. In fact, a Lagrange density of the form $\mathcal{L}+\mathcal{L}'$ gets transformed into $\mathcal{L}-\mathcal{L}'$ under a coordinate transformation with negative Jacobi determinant, such as a time or space inversion. And the two Lagrange densities $\mathcal{L}\pm\mathcal{L}'$ would have given different Euler–Lagrange equations.

Another interesting property of the action integral of $\mathcal{L}'$ is that even though it does not contain the metric, it is nevertheless invariant under arbitrary coordinate transformations, except for a change of sign under coordinate transformations with negative Jacobi determinant. A field theory not involving any metric and yet invariant under arbitrary coordinate transformations, is called a *topological* theory. $\mathcal{L}'$ is an example of a conceivable topological contribution to the Lagrange density. That it is a trivial contribution in the classical theory, because $\mathcal{L}'$ is a divergence and so contributes only a boundary term to the action integral, is just too bad if one is dreaming of exotic theories.

In quantum field theory, on the other hand, all contributions to the action integral are potentially important from the point of view of the path integral formulation. It may happen that boundary terms are essential, even though they do not change the local field equations. For example, it is an open problem to explain why topological terms corresponding to $\mathcal{L}'$ do not contribute in "quantum chromodynamics", QCD, which is the theory of the strong interactions. The presence of such a term would violate parity invariance and time reversal invariance. Such violations are not observed in strong interactions, at least they are very small.

Problems

1. Under a Lorentz transformation $x^\mu \mapsto \widetilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu$ the electromagnetic field tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

   is transformed into $\widetilde{F}^{\mu\nu}$, given by the relation $\widetilde{F}^{\mu\nu}(\widetilde{x}) = \Lambda^\mu{}_\rho \Lambda^\mu{}_\sigma F^{\rho\sigma}(x)$.
   Show that the electric field $\boldsymbol{E}$ transforms as a vector, and the magnetic flux density $\boldsymbol{B}$ as a pseudovector, under rotations and space inversion.
   (Hint: investigate three different rotations, about the x axis as given by Equation (8.41), the y axis and the z axis, in addition to space inversion, given by Equation (8.27).)
   How do $\boldsymbol{E}$ and $\boldsymbol{B}$ transform under a pure Lorentz transformation along the x axis, Equation (8.52)? Along the y axis? Along the z axis?
   Check that $\boldsymbol{E} \cdot \boldsymbol{B}$ is always invariant.

2. Use that the field tensor is a 2-form,

$$\begin{aligned} \mathbf{F} = \frac{1}{2} F_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \; = \; & E_x \mathrm{d}t \wedge \mathrm{d}x + E_y \mathrm{d}t \wedge \mathrm{d}y + E_z \mathrm{d}t \wedge \mathrm{d}z \\ & + B_x \mathrm{d}z \wedge \mathrm{d}y + B_y \mathrm{d}x \wedge \mathrm{d}z + B_z \mathrm{d}y \wedge \mathrm{d}x \; . \end{aligned}$$

   Compute explicitly the exterior derivative $\mathrm{d}\mathbf{F}$ to show that the equation $\mathrm{d}\mathbf{F} = 0$ is identical to Maxwell's four source free equations, (14.6) and (14.7).

3. Derive Maxwell's equation in a medium from the Lagrange density given in Equation (14.41).
   Consider the possibility that ϵ_r and μ_r may vary in space. Consider especially the limiting case where ϵ_r or μ_r changes discontinuously across a surface perpendicular to the z axis.
   Which components of the fields $\boldsymbol{E}, \boldsymbol{B}, \boldsymbol{D}, \boldsymbol{H}$ have to be continuous at the surface?

4. A plane, linearly polarized light wave of intensity 1 MW/cm^2 falls vertically down onto a plane water surface.
   The refraction index of water is 1.33, that is, the velocity of light in water is $c/1.33$. We neglect dispersion. Assume that $\epsilon_r = \mu_r = 1$ in air, and that $\mu_r = 1$ in water.
   Compute the radiation pressure against the surface, by using

   a) the stress tensor (the spatial part of the energy momentum tensor $T^{\mu\nu}$);

   b) momentum conservation.

   By how much will the water surface be pushed down, or lifted?

5. Derive the dual equations (14.96) and (14.97) from Maxwell's equations.

Chapter 15

Gravitation and geometry

In Newton's theory of gravitation the potential energy of a collection of N point masses is

$$V = -G \sum_{i=2}^{N} \sum_{j=1}^{i-1} \frac{m_i\, m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} = -G \sum_{i=2}^{N} \sum_{j<i} \frac{m_i\, m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} = -\frac{G}{2} \sum_{i=1}^{N} \sum_{j\neq i} \frac{m_i\, m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \ . \tag{15.1}$$

Here $G = 6.6726 \times 10^{-11}$ m^3 kg^{-1} s^{-2} is Newton's gravitational constant. Particle i has a mass m_i and position $\boldsymbol{r}_i$, and an acceleration given by Newton's second law,

$$m_i \, \frac{\mathrm{d}^2 \boldsymbol{r}_i}{\mathrm{d}t^2} = -\frac{\partial V}{\partial \boldsymbol{r}_i} = -G \sum_{j\neq i} \frac{m_i\, m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} \, (\boldsymbol{r}_i - \boldsymbol{r}_j) \ . \tag{15.2}$$

One way to interpret this equation is that particle i moves in a *gravitational field* from the other particles, given by a scalar potential

$$\phi_i(\boldsymbol{r}) = -G \sum_{j\neq i} \frac{m_j}{|\boldsymbol{r} - \boldsymbol{r}_j|} \ , \tag{15.3}$$

and has a potential energy in this gravitational field which is

$$V_i(\boldsymbol{r}_i) = m_i \, \phi_i(\boldsymbol{r}_i) \ . \tag{15.4}$$

The scalar potential ϕ_i satisfies the second order partial differential equation

$$\nabla^2 \phi_i = 4\pi G \rho_i \ , \tag{15.5}$$

where ρ_i is the *mass density*, excluding particle i,

$$\rho_i(\boldsymbol{r}) = \sum_{j\neq i} m_j \, \delta(\boldsymbol{r} - \boldsymbol{r}_j) \ . \tag{15.6}$$

This field equation for the gravitational potential ϕ_i contains no time derivative. Hence, ϕ_i has no dynamics of its own, it is purely an auxiliary quantitity that we may use when we find it useful, and throw away when we no longer need it.

The electric and magnetic forces between electric charges are mediated *with a finite speed* through an electromagnetic field described by Maxwell's equations. Newton's gravitational theory is no field theory in the same sense as electromagnetism, since it describes a force mediated *with an infinite speed*. A famous quotation indicates that Newton regarded this as an essential weakness of the theory, two hundred years before Faraday and Maxwell introduced the field concept in electromagnetism:

> "That one body may act upon another at a distance through a vacuum, without the mediation of any thing else, by and through which their action and force may be conveyed from one to another, is to me so great an absurdity, that I believe no man, who has in philosophical matters a competent faculty of thinking, can ever fall into it."

The need for a genuine field theory of gravitation became even more obvious when Einstein formulated his special theory of relativity, which forbids velocities greater than the speed of light. Einstein and other physicists therefore set out to find a relativistic theory of gravitation.

Such a theory can not have exactly the same form as Maxwell's theory, even though Newton's gravitational potential between point masses has nearly the same form as the Coulomb potential between point charges. One important difference between the two theories is that all masses are positive, whereas there exist both positive and negative electric charges. Another main difference is the sign of the potential energy: the gravitational energy is always negative, that is, the force is always attractive, whereas the Coulomb energy is positive for charges of the same sign, that is, the force is then repulsive. A third difference is that there is nothing in Newton's gravitational theory corresponding to the three dimensional electromagnetic vector potential, and hence the magnetic force, which is proportional to the velocity of the charged particle, has no obvious counterpart in the gravitational theory.

15.1 The principle of equivalence

Since the *inertial mass* m_i, on the left hand side of Newton's equation of motion, Equation (15.2), equals the *gravitational mass* m_i, on the right hand side, it cancels from the equation. The motion of a particle in a gravitational field is *independent of the mass of the particle*, according to Newton's theory.

In general, a mass distribution with mass density $\rho = \rho(\boldsymbol{r}, t)$ gives rise to a Newtonian gravitational potential ϕ satisfying the equation

$$\nabla^2 \phi = 4\pi G \rho \, . \tag{15.7}$$

In this gravitational field a pointlike "test particle" with a mass m so small that it does not influence the mass density ρ, will have a potential energy $V = m\phi$, and will be subject to a gravitational force $\boldsymbol{F} = -\nabla V = -m\nabla\phi$. Since the force is proportional to the mass m, it is natural to define the *gravitational field* as the force divided by the mass,

$$\boldsymbol{g} = \frac{\boldsymbol{F}}{m} = -\nabla\phi \, . \tag{15.8}$$

The equation of motion of the test particle is independent of its mass,

$$\frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = \boldsymbol{g} \, . \tag{15.9}$$

Thus, the gravitational field at the position $\boldsymbol{r}$ at the time t is the same as the *acceleration of gravity* there: it is the acceleration that any test particle will have when placed there. Newton's field equation of gravitation relates the divergence of the gravitational field to the mass density,

$$\nabla \cdot \boldsymbol{g} = -4\pi G \rho \, . \tag{15.10}$$

Recall that Maxwell's equations relate the electric field to the charge density in the same way.

The equivalence between "heavy" mass, which is the mass you feel when holding a hammer in your hand, and "inertial" mass, which you feel when setting the hammer in motion to hit a nail, was the point of departure for Einstein's gravitational theory, the general theory of relativity. Inertial and heavy mass are just two aspects of the same reality, according to Einstein. He made this *principle of equivalence*, as he called it, an axiom of the new theory:

> *There is no observable difference between gravitational forces and the inertial forces that are observed in an accelerated coordinate system.*

Einstein took as an example an observer in an elevator without a window. There is no possible experiment that this enclosed observer could do inside the elevator in order to learn whether the elevator is accelerated, or whether it is at rest in a gravitational field. In other words, a gravitational field may always be transformed away *locally*, that is, inside the elevator at a given time, by means of a suitable transformation of space and time.

The principle of equivalence as stated here is actually not quite true, because the gravitational field from stars and planets will always be inhomogeneous. When we try to transform away an inhomogeneous gravitational field by a coordinate transformation, there will always remain a residual field which can not be transformed away. The residual gravitational force is called a *tidal force*, because it gives rise to the ocean tides when the Earth is falling freely in the gravitational field from the Sun and the Moon. We will return to the subject of tidal forces at the end of this chapter.

The basis of Einstein's principle of equivalence was not only Newton's theory, which is of course well supported by experiments, but also a very precise experiment performed by the Hungarian physicist Baron Lorand von Eötvös. The idea behind the experiment was that the direction of the force of gravity on an object in the laboratory depends on the ratio between the inertial and heavy mass of the object. In fact, the observed force of gravity is the sum of two components having not only different physical origins but also different directions. The object is attracted towards the centre of the Earth, and the attractive force is proportional to the heavy mass. But there appears also a centrifugal force due to the acceleration of the laboratory as it follows the rotation of the Earth, this force is directed away from the rotation axis, and is proportional to the inertial mass. Eötvös compared different materials, and found that they all had the same ratio between inertial and heavy mass, to a precision of 5×10^{-9} (published in 1922). The experimental precision has later been improved to 10^{-12}.

Since the equation of motion of a particle in a gravitational field is independent of its mass, it is possible to imagine that the particle follows a *geodesic*, that is, a curve in spacetime of extremal length. The relevant "length", in the special as well as in the general theory of relativity, is $s = c\tau$, where τ is the proper time of the particle. We saw in Chapter 8 that the "law of the cosmic laziness" holds in the special theory of relativity, that a free particle follows an orbit such that the proper time is maximal. It is natural to assume that the same law holds in a gravitational field. Then the Lagrange function will be

$$\boxed{L = -mc\sqrt{g_{\mu\nu}(x)\,\frac{\mathrm{d}x^\mu}{\mathrm{d}t}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}t}}\,.} \tag{15.11}$$

The only new aspect as compared to the special theory of relativity, Equation (10.65), is that

the metric tensor is now taken to depend on time and place, $g_{\mu\nu} = g_{\mu\nu}(x)$, and in general it will be impossible to find a coordinate system in which it is constant.

The general theory of relativity is a gravitational theory in which the metric tensor $g_{\mu\nu} = g_{\mu\nu}(x)$ describes the gravitational field.

15.2 Non-relativistic limit

In order to see that Equation (15.11) makes sense, we should check that it is possible to choose the metric in such a way that we get the correct non-relativistic limit

$$L \to \text{constant} + \frac{1}{2}\, m\boldsymbol{v}^2 - m\phi \;, \tag{15.12}$$

when the velocity $\boldsymbol{v} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ is small compared to the speed of light c, and when the gravitational potential ϕ is also small compared to c^2. The last condition is well satisfied anywhere in our solar system. The gravitational potential in the solar system has its maximum (in absolute value) at the centre of the Sun, but is nearly as large at the surface of the Sun, where it is

$$\phi_S = -G\,\frac{M_\odot}{R_\odot} = -1.9 \times 10^{11}\ \mathrm{m}^2/\mathrm{s}^2 = -2.1 \times 10^{-6}\, c^2 \;. \tag{15.13}$$

Here $M_\odot = 2.0 \times 10^{30}$ kg is the solar mass, and $R_\odot = 7.0 \times 10^8$ m is the solar radius.

One possible solution is to take $x^0 = ct$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and

$$g_{00} = 1 + \frac{2\phi}{c^2} \;, \qquad g_{11} = g_{22} = g_{33} = -1 \;, \qquad \text{all other } g_{\mu\nu} = 0 \;. \tag{15.14}$$

From Equation (15.11) we then get the wanted result that

$$L = -mc^2 \sqrt{1 + \frac{2\phi}{c^2} - \frac{\boldsymbol{v}^2}{c^2}} \approx -mc^2 + \frac{1}{2}\, m\boldsymbol{v}^2 - m\phi \;. \tag{15.15}$$

It is worth noting that Newton's theory holds only for velocities that are small relative to the speed of light, and therefore the non-relativistic limit gives limited information about the metric $g_{\mu\nu}$. Separating the time coordinate $x^0 = ct$ from the spatial coordinates x^j, with $j = 1, 2, 3$, in Equation (15.11), we get that

$$L = -mc \sqrt{g_{00}c^2 + 2g_{0j}c\,\frac{\mathrm{d}x^j}{\mathrm{d}t} + g_{jk}\,\frac{\mathrm{d}x^j}{\mathrm{d}t}\,\frac{\mathrm{d}x^k}{\mathrm{d}t}} \;. \tag{15.16}$$

We get the correct non-relativistic limit by substituting $g_{0j} = g_{j0} = 0$ and $g_{jk} = \delta_{jk}$, but in this limit the velocity $|\boldsymbol{v}|$ is by definition small relative to the speed of light, hence these components of the metric are less precisely determined than g_{00}. In order to determine possible corrections to g_{jk} of order ϕ/c^2, we would have to know the equation of motion for a relativistic particle in a gravitational potential ϕ, when ϕ/c^2 is small. But this is not known, at least not from Newton's theory. These corrections to g_{jk} can be found, however, from Einstein's field equations, which will be the subject of the next chapter.

15.3 Gravitational redshift

According to the metric of Equation (15.14), the proper time of a particle at rest is

$$\mathrm{d}\tau = \mathrm{d}t \sqrt{1 + \frac{2\phi}{c^2}} \approx \mathrm{d}t \left(1 + \frac{\phi}{c^2}\right) . \tag{15.17}$$

Since the gravitational potential is negative, this means that a clock in a gravitational field will be retarded compared to a clock outside the field. The ratio between two proper times $\mathrm{d}\tau_1$ in a gravitational potential ϕ_1, and $\mathrm{d}\tau_2$ in a potential ϕ_2, is

$$\frac{\mathrm{d}\tau_1}{\mathrm{d}\tau_2} \approx \frac{1 + \frac{\phi_1}{c^2}}{1 + \frac{\phi_2}{c^2}} \approx 1 + \frac{\phi_1 - \phi_2}{c^2} . \tag{15.18}$$

One way to measure this effect is to compare the frequencies ν_1 and ν_2 of two identical light sources at the two points. We assume that the sources are monochromatic, that their light is of one single frequency. The frequency is the number of oscillations, n, per time interval Δt, that is,

$$\nu_1 = \frac{n_1}{\Delta t} , \qquad \nu_2 = \frac{n_2}{\Delta t} . \tag{15.19}$$

If the coordinate time interval Δt is the same for the two sources, then

$$\frac{\nu_1}{\nu_2} = \frac{n_1}{n_2} = \frac{\Delta\tau_1}{\Delta\tau_2} \approx 1 + \frac{\phi_1 - \phi_2}{c^2} . \tag{15.20}$$

This *gravitational redshift*, as it is called, was first observed in the spectra of so called white dwarf stars. But it is also observed directly in terrestrial laboratories. Since the acceleration of gravity on the surface of the Earth is

$$g = 9.8\,\mathrm{m/s^2} , \tag{15.21}$$

a height difference of $h = 1\,\mathrm{m}$ makes a difference in gravitational potential of

$$\Delta\phi = gh = 9.8\,\mathrm{m^2/s^2} \approx 10^{-16}\, c^2 . \tag{15.22}$$

Pound and Rebka measured this effect in 1960 (Physical Review Letters 4, 337 (1960)), in an experiment where the height difference was 20 m. Thus, the measured effect was of order 2×10^{-15}.

15.4 The equation of motion for a point mass

The action along an orbit from one point A to another point B is, according to Equation (15.11),

$$S = \int_A^B \mathrm{d}t\, L = -mc \int_A^B \mathrm{d}t \sqrt{g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \frac{\mathrm{d}x^\nu}{\mathrm{d}t}} . \tag{15.23}$$

Here t denotes time, but the action is unchanged if we replace t by an arbitrary parameter $u = u(t)$, such that the spacetime coordinates along the orbit followed by the particle are given as functions of u,

$$x^\mu = x^\mu(u) \; . \tag{15.24}$$

In fact, a change of integration variable gives that

$$S = -mc \int_A^B \mathrm{d}t \, \sqrt{g_{\mu\nu}(x) \, \frac{\mathrm{d}x^\mu}{\mathrm{d}t} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}t}} = -mc \int_A^B \mathrm{d}u \, \sqrt{g_{\mu\nu}(x) \, \frac{\mathrm{d}x^\mu}{\mathrm{d}u} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}u}} \; . \tag{15.25}$$

The reparametrization invariance of the action integral S becomes obvious when we write

$$S = -m \int_A^B \mathrm{d}s = -mc \int_A^B \mathrm{d}\tau \; . \tag{15.26}$$

Let us define the velocity as

$$w = \frac{\mathrm{d}s}{\mathrm{d}u} = \sqrt{g_{\mu\nu}(x) \, \frac{\mathrm{d}x^\mu}{\mathrm{d}u} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}u}} \; , \tag{15.27}$$

so that the action integral is $S = -mc \int_A^B \mathrm{d}u \, w$. Hamilton's principle applied to this action integral gives the Euler–Lagrange equations

$$0 = \frac{\partial w}{\partial x^\rho} - \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{\partial w}{\partial \left(\frac{\mathrm{d}x^\rho}{\mathrm{d}u} \right)} \right) = \frac{1}{2w} \, g_{\mu\nu,\rho} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}u} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}u} - \frac{\mathrm{d}}{\mathrm{d}u} \left(\frac{1}{2w} \left(g_{\rho\nu} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}u} + g_{\mu\rho} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}u} \right) \right) . \tag{15.28}$$

The reparametrization invariance of S means that the Euler–Lagrange equations do not determine how the parameter u varies along the orbit. We take advantage of this arbitrariness to choose $u = s$, this implies that $w = 1$, and hence the Euler–Lagrange equations get simplified as follows,

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(g_{\rho\nu} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \right) - \frac{1}{2} \, g_{\mu\nu,\rho} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \; . \tag{15.29}$$

This may indeed be a convenient form of the equation of motion, but it is customary to rewrite further by expanding the derivative,

$$\begin{aligned} 0 \; &= \; g_{\rho\nu} \, \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}s^2} + g_{\rho\nu,\sigma} \, \frac{\mathrm{d}x^\sigma}{\mathrm{d}s} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}s} - \frac{1}{2} \, g_{\mu\nu,\rho} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \\ &= \; g_{\rho\nu} \, \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}s^2} + \frac{1}{2} \left(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho} \right) \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \; . \end{aligned} \tag{15.30}$$

Finally, multiplication by $g^{\lambda\rho}$ gives the *geodesic equation*,

$$\boxed{\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}s^2} + \Gamma^\lambda_{\mu\nu} \, \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \, \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0 \; .} \tag{15.31}$$

We recognize $\Gamma^\lambda_{\mu\nu}$ as the metric and symmetric connection, also known as the Christoffel connection. The Christoffel connection coefficients are symmetric in their lower indices, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$, and are given by the metric tensor and its first order derivatives as

$$\boxed{\Gamma^\lambda_{\mu\nu} = \frac{1}{2}\, g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})\ .} \tag{15.32}$$

We have seen in Chapter 5 that the connection Γ is not a tensor, but transforms in a different way under coordinate transformations.

The symmetry $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ is the mathematical way of expressing Einstein's principle of equivalence, because it means that the connection may be "transformed away" *at one arbitrary point* by a suitable choice of coordinates. In other words, around any given point in spacetime it is possible to introduce a local coordinate system such that all connection coefficients vanish at this single point.

From Equation (15.32) follows that the covariant derivative of the metric tensor vanishes,

$$g_{\mu\nu;\rho} = g_{\mu\nu,\rho} - g_{\sigma\nu}\Gamma^\sigma_{\mu\rho} - g_{\mu\sigma}\Gamma^\sigma_{\nu\rho} = 0\ . \tag{15.33}$$

This equation together with the geodesic equation implies that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\left(g_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\right) &= g_{\mu\nu,\rho}\,\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s} - \left(g_{\mu\nu}\Gamma^\mu_{\rho\sigma}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s} + g_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\Gamma^\nu_{\rho\sigma}\right)\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\,\frac{\mathrm{d}x^\sigma}{\mathrm{d}s} \\ &= \left(g_{\mu\nu,\rho} - g_{\sigma\nu}\Gamma^\sigma_{\mu\rho} - g_{\mu\sigma}\Gamma^\sigma_{\nu\rho}\right)\frac{\mathrm{d}x^\rho}{\mathrm{d}s}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0\ .\end{aligned} \tag{15.34}$$

This result is important, because it demonstrates that the geodesic equation is compatible with the assumption that $w = 1$, which helped us to simplify the equation.

It is worth noting that the geodesic equation may be derived from a different Lagrange function, Equation (10.105),

$$L' = -\frac{1}{2}\, mc^2 g_{\mu\nu}(x)\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s}\ . \tag{15.35}$$

This Lagrange function fixes the parametrization of the orbit, since the action integral is not reparametrization invariant. It has the same non-relativistic limit, up to an additive constant.

15.5 Tidal forces due to geodesic deviation

Physical bodies are not point masses, and it is natural to ask what will happen to an extended body moving in a gravitational field. The answer, as we know from Newton's theory, is that it will feel the effect of an inhomogeneous field as *tidal forces* squeezing in some directions and tearing in other directions. According to the geometric point of view we want to take here, the tidal forces result from the relative accelerations of neighbouring geodesics, since each part of the body tries to follow its own geodesic. We want now to calculate the relative acceleration of two point particles.

Tidal forces according to Newton

The relative acceleration of two point particles moving in a gravitational field $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{r}, t)$, one along the orbit $\boldsymbol{r}_1 = \boldsymbol{r}_1(t)$ and the other one along the nearby orbit $\boldsymbol{r}_2 = \boldsymbol{r}_2(t) = \boldsymbol{r}_1(t) + \Delta\boldsymbol{r}(t)$, is

$$\Delta\boldsymbol{a} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\,\Delta\boldsymbol{r} = \Delta\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{r}_2(t), t) - \boldsymbol{g}(\boldsymbol{r}_1(t), t)\ . \tag{15.36}$$

If we regard the first particle as an observer observing the second particle, then the observed position is $\Delta\boldsymbol{r}$, and the observed acceleration $\Delta\boldsymbol{a}$ will be interpreted as being due to the residual gravitational field $\Delta\boldsymbol{g}$. The three spatial components of $\Delta\boldsymbol{g}$ are, to first order in $\Delta\boldsymbol{r}$,

$$\Delta g^i(\Delta x^1, \Delta x^2, \Delta x^3) = g^i_{,j}\,\Delta x^j\ , \tag{15.37}$$

where the partial derivatives $g^i_{,j}$ are computed at the position of the first particle. The linearity in $\Delta\boldsymbol{r}$ implies that the tidal forces at opposite points $\Delta\boldsymbol{r}$ and $-\Delta\boldsymbol{r}$ balance.

Note that the field equation in vacuum,

$$\nabla\cdot\boldsymbol{g} = g^1_{,1} + g^2_{,2} + g^3_{,3} = g^i_{,i} = 0\ , \tag{15.38}$$

means that the tidal forces will tend to preserve volumes. If they pull in one direction, they push in another direction, and vice versa. The field equation does not say anything about whether the tidal forces exert a torque on an extended body.

Tidal forces from geodesic motion

Now we consider one point particle moving along a geodesic $x^\mu = x^\mu(s)$ which is a solution of Equation (15.31), and a second point particle moving along a nearby geodesic

$$y^\mu = y^\mu(s) = x^\mu(s) + \Delta x^\mu(s)\ . \tag{15.39}$$

We choose to parametrize this second geodesic by the same parameter s which is the geodesic parameter of the first geodesic, but we should remember that the geodesic parameter of the second geodesic, let us call it $\widetilde{s}$, is not necessarily equal to s. Define

$$w\ =\ \frac{\mathrm{d}\widetilde{s}}{\mathrm{d}s} = \sqrt{g_{\mu\nu}(y)\,\frac{\mathrm{d}y^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}y^\nu}{\mathrm{d}s}}\ . \tag{15.40}$$

Then the second geodesic equation,

$$\frac{\mathrm{d}^2y^\lambda}{\mathrm{d}\widetilde{s}^2} + \Gamma^\lambda_{\mu\nu}(y)\,\frac{\mathrm{d}y^\mu}{\mathrm{d}\widetilde{s}}\,\frac{\mathrm{d}y^\nu}{\mathrm{d}\widetilde{s}} = 0\ , \tag{15.41}$$

may be rewritten as

$$w\,\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{1}{w}\,\frac{\mathrm{d}y^\lambda}{\mathrm{d}s}\right) + \Gamma^\lambda_{\mu\nu}(y)\,\frac{\mathrm{d}y^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}y^\nu}{\mathrm{d}s} = 0\ . \tag{15.42}$$

Using Equation (15.31), and assuming that both the deviation Δx^μ and its derivative with respect to s are small, we get to first order in Δx^μ that

$$\frac{\mathrm{d}^2(\Delta x^\lambda)}{\mathrm{d}s^2} - \frac{\mathrm{d}(\ln w)}{\mathrm{d}s}\,\frac{\mathrm{d}x^\lambda}{\mathrm{d}s} + \Gamma^\lambda_{\mu\nu,\rho}(x)\,\Delta x^\rho\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s} + 2\Gamma^\lambda_{\mu\nu}(x)\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}(\Delta x^\nu)}{\mathrm{d}s} = 0\ . \tag{15.43}$$

If we again regard the first particle as an observer of the motion of the second particle, the local time axis at one point on the observer's geodesic trajectory is the unit tangent vector

$$e^{\mu}_{(0)} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s} \ . \tag{15.44}$$

We now expand

$$\Delta x^{\mu} = \sum_{i=1}^{3} \xi^{(i)} e^{\mu}_{(i)} \ , \tag{15.45}$$

where $e^{\mu}_{(i)}$ for $i = 1, 2, 3$ are unit vectors orthogonal to the time axis $e^{\mu}_{(0)}$ and to each other. That is, for $\alpha, \beta = 0, 1, 2, 3$, and for any point $x^{\mu} = x^{\mu}(s)$ along the geodesic, we have that

$$g_{\mu\nu}(x)\, e^{\mu}_{(\alpha)}\, e^{\nu}_{(\beta)} = \eta_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{15.46}$$

Equation (15.31), the geodesic equation, means simply that the tangent vector $e^{\mu}_{(0)}$ is parallel transported along the curve. We now require that also the three normal vectors $e^{\mu}_{(i)}$ for $i = 1, 2, 3$ should be parallel transported, so that for $\alpha = 0, 1, 2, 3$ we have

$$\frac{\mathrm{d}(e^{\mu}_{(\alpha)})}{\mathrm{d}s} + \Gamma^{\mu}_{\sigma\rho}\, e^{\sigma}_{(\alpha)}\, \frac{\mathrm{d}x^{\rho}}{\mathrm{d}s} = \frac{\mathrm{d}(e^{\mu}_{(\alpha)})}{\mathrm{d}s} + \Gamma^{\mu}_{\sigma\rho}\, e^{\sigma}_{(\alpha)}\, e^{\rho}_{(0)} = 0 \ . \tag{15.47}$$

The physical meaning is that when the four unit vectors are parallel transported along the geodesic, they come as close as possible to defining a local inertial frame. We may call it the "local Lorentz frame", or with a longer expression, the "instantaneous rest frame of the first particle". Thus, the three spatial coordinates $\xi^{(1)}, \xi^{(2)}, \xi^{(3)}$, which we assume to be small, are what we may call "local Euclidean coordinates". The velocity and acceleration relative to the local Euclidean reference frame is respectively the first and second derivative of these coordinates with respect to the proper time $\tau = s/c$,

$$\frac{\mathrm{d}\xi^{(i)}}{\mathrm{d}\tau} = c\, \frac{\mathrm{d}\xi^{(i)}}{\mathrm{d}s} \ , \qquad \frac{\mathrm{d}^2\xi^{(i)}}{\mathrm{d}\tau^2} = c^2\, \frac{\mathrm{d}^2\xi^{(i)}}{\mathrm{d}s^2} \ . \tag{15.48}$$

In order to simplify a little bit, we ask for the acceleration $\mathrm{d}^2\xi^{(i)}/\mathrm{d}\tau^2$ at a moment when the velocity $\mathrm{d}\xi^{(i)}/\mathrm{d}\tau$ is zero. With this simplification we have that

$$\frac{\mathrm{d}(\Delta x^{\mu})}{\mathrm{d}s} = \frac{\mathrm{d}\xi^{(i)}}{\mathrm{d}s}\, e^{\mu}_{(i)} + \xi^{(i)}\, \frac{\mathrm{d}(e^{\mu}_{(i)})}{\mathrm{d}s} = -\xi^{(i)}\, \Gamma^{\mu}_{\sigma\rho}\, e^{\rho}_{(0)}\, e^{\sigma}_{(i)} \ , \tag{15.49}$$

and further that

$$\begin{aligned} \frac{\mathrm{d}^2(\Delta x^{\mu})}{\mathrm{d}s^2} &= \frac{\mathrm{d}^2\xi^{(i)}}{\mathrm{d}s^2}\, e^{\mu}_{(i)} + 2\, \frac{\mathrm{d}\xi^{(i)}}{\mathrm{d}s}\, \frac{\mathrm{d}(e^{\mu}_{(i)})}{\mathrm{d}s} + \xi^{(i)}\, \frac{\mathrm{d}^2(e^{\mu}_{(i)})}{\mathrm{d}s^2} \\ &= \frac{\mathrm{d}^2\xi^{(i)}}{\mathrm{d}s^2}\, e^{\mu}_{(i)} + \xi^{(i)}\, \frac{\mathrm{d}^2(e^{\mu}_{(i)})}{\mathrm{d}s^2} \ . \end{aligned} \tag{15.50}$$

Here we have that

$$\begin{aligned}\frac{\mathrm{d}^2(e^{\mu}_{(i)})}{\mathrm{d}s^2} &= -\frac{\mathrm{d}}{\mathrm{d}s}\left(\Gamma^{\mu}_{\sigma\rho}\, e^{\sigma}_{(i)}\, e^{\rho}_{(0)}\right) \\ &= -(\Gamma^{\mu}_{\sigma\rho,\kappa}\, e^{\kappa}_{(0)})\, e^{\sigma}_{(i)}\, e^{\rho}_{(0)} + \Gamma^{\mu}_{\sigma\rho}\,(\Gamma^{\sigma}_{\kappa\lambda}\, e^{\kappa}_{(i)}\, e^{\lambda}_{(0)})\, e^{\rho}_{(0)} + \Gamma^{\mu}_{\sigma\rho}\, e^{\sigma}_{(i)}\,(\Gamma^{\rho}_{\kappa\lambda}\, e^{\kappa}_{(0)}\, e^{\lambda}_{(0)}) \qquad (15.51)\\ &= \left(-\Gamma^{\mu}_{\sigma\rho,\kappa} + \Gamma^{\mu}_{\tau\rho}\,\Gamma^{\tau}_{\sigma\kappa} + \Gamma^{\mu}_{\sigma\tau}\,\Gamma^{\tau}_{\kappa\rho}\right) e^{\kappa}_{(0)}\, e^{\sigma}_{(i)}\, e^{\rho}_{(0)}\ .\end{aligned}$$

Altogether, Equation (15.43), which describes geodesic deviation, takes the following form at a moment when $\mathrm{d}\xi^{(i)}/\mathrm{d}s = 0$,

$$\begin{aligned}\frac{\mathrm{d}^2\xi^{(i)}}{\mathrm{d}s^2}\, e^{\lambda}_{(i)} - \frac{\mathrm{d}(\ln w)}{\mathrm{d}s}\, e^{\lambda}_{(0)} + \xi^{(i)}\Big(\Big(-\Gamma^{\lambda}_{\sigma\rho,\kappa} + \Gamma^{\mu}_{\tau\rho}\,\Gamma^{\tau}_{\sigma\kappa} + \Gamma^{\mu}_{\sigma\tau}\,\Gamma^{\tau}_{\kappa\rho}\Big)\, e^{\kappa}_{(0)}\, e^{\sigma}_{(i)}\, e^{\rho}_{(0)} \\ + \Gamma^{\lambda}_{\mu\nu,\rho}\, e^{\rho}_{(i)}\, e^{\mu}_{(0)}\, e^{\nu}_{(0)} - 2\Gamma^{\lambda}_{\mu\nu}(x)\, e^{\mu}_{(0)}\,\Gamma^{\mu}_{\sigma\rho}\, e^{\rho}_{(0)}\, e^{\sigma}_{(i)}\Big) = 0\ . \qquad (15.52)\end{aligned}$$

Introducing the Riemann curvature tensor, we write the equation as follows,

$$\frac{\mathrm{d}^2\xi^{(i)}}{\mathrm{d}s^2}\, e^{\lambda}_{(i)} - \frac{\mathrm{d}(\ln w)}{\mathrm{d}s}\, e^{\lambda}_{(0)} + \xi^{(i)}\, R^{\lambda}{}_{\mu\rho\nu}\, e^{\mu}_{(0)}\, e^{\rho}_{(i)}\, e^{\nu}_{(0)} = 0\ . \qquad (15.53)$$

Let us introduce the inverse of the 4×4 matrix $e^{\mu}_{(\alpha)}$, written as $e^{(\alpha)}_{\mu}$ and defined by either one of the two relations

$$e^{(\beta)}_{\mu}\, e^{\mu}_{(\alpha)} = \delta^{\beta}_{\alpha}\ , \qquad e^{\mu}_{(\alpha)}\, e^{(\alpha)}_{\nu} = \delta^{\mu}_{\nu}\ . \qquad (15.54)$$

We may then introduce the components of the Riemann curvature tensor relative to the local Lorentz frame,

$$R^{(\alpha)}{}_{(\beta)(\gamma)(\delta)} = R^{\lambda}{}_{\mu\rho\nu}\, e^{(\alpha)}_{\lambda}\, e^{\mu}_{(\beta)}\, e^{\rho}_{(\gamma)}\, e^{\nu}_{(\delta)}\ , \qquad (15.55)$$

and write the equation of geodesic deviation in the following simple way,

$$\frac{\mathrm{d}^2\xi^{(i)}}{\mathrm{d}\tau^2} + c^2\xi^{(j)}\, R^{(i)}{}_{(0)(j)(0)} = 0\ . \qquad (15.56)$$

This equation gives a direct physical meaning to the Riemann curvature tensor. In fact, the tidal forces experienced by an extended body moving in a gravitational field are described *locally* by a residual gravitational field of Newtonian type,

$$g^{(i)}(\xi^{(1)},\xi^{(2)},\xi^{(3)}) = -c^2\xi^{(j)}\, R^{(i)}{}_{(0)(j)(0)}\ . \qquad (15.57)$$

Remember that the two indices (0) refer to the local time axis, which is the tangent to the geodesic $x^{\mu}(s)$ followed by our reference particle.

Problems

1. Find explicitly the equations for geodesic lines in the following examples.

   a) The plane, with coordinates (x, y) and metric $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2$.

   b) Same as a), but in polar coordinates (r, φ).

   c) The surface of a sphere, in polar coordinates (θ, φ).

   Find the general solution in each example. (See the problems 3, 4 and 5 of Chapter 2.)

2. The acceleration of gravity on the surface of the Earth is $g = 9.81\ \mathrm{m/s^2}$, in this problem we take $g = 10\ \mathrm{m/s^2}$ for simplicity. A tiny ball thrown up into the air with a velocity of exactly 5 m/s will then come back down exactly one second later, ignoring the friction against the air.
   What is the proper time of the ball for the round trip up and down?
   Two effects influence the proper time: the special relativistic time dilation, and the gravitational field. What is the sign of each of the two effects, and which effect is the largest? Are your answers consistent with the principle of cosmic laziness? Why must the two effects necessarily be of roughly the same magnitude?
   Make similar calculations for an artificial sattelite in a circular orbit 500 km above the surface of the Earth, and for the Moon.

Chapter 16

Einstein's gravitational equation

The gravitational field influences point particles and other fields, such as the Klein–Gordon field and the Maxwell field, because the metric tensor $g_{\mu\nu}$ enters in the Lagrange functions for point particles and in the Lagrange densities for fields. In principle, the first derivatives of the metric tensor might enter through the connection coefficients $\Gamma^{\lambda}_{\mu\nu}$, but this does not happen in those examples we have seen so far.

Einstein's equation for the gravitational field is a second order differential equation for the metric tensor. The second order derivatives enter through the curvature tensor, which contains the connection coefficients and their first derivatives. The gravitational field equation gives a direct relation between the curvature of the four dimensional spacetime and the total energy momentum tensor of all particles and all fields, except for the gravitational field.

16.1 Einstein's field equation

According to Newton's gravitational theory, the divergence of the gravitational field vanishes when the mass density vanishes,

$$\nabla \cdot \boldsymbol{g} = 0 \, . \tag{16.1}$$

This is the field equation in vacuum. Applying this equation to the local gravitational field in Equation (15.57), computed from the equation of geodesic deviation, gives the field equation

$$R^{(i)}{}_{(0)(i)(0)} = 0 \, . \tag{16.2}$$

There is a sum over $i = 1, 2, 3$, but we may include $i = 0$ in the sum, because the antisymmetry property $R^{(\alpha)}{}_{(\beta)(\gamma)(\delta)} = -R^{(\alpha)}{}_{(\beta)(\delta)(\gamma)}$ implies that $R^{(0)}{}_{(0)(0)(0)} = 0$. Then we obtain the Ricci curvature tensor, and the field equation is

$$R^{(\alpha)}{}_{(0)(\alpha)(0)} = R_{(0)(0)} = R_{\mu\nu} \, e^{\mu}_{(0)} \, e^{\nu}_{(0)} = 0 \, . \tag{16.3}$$

This is the gravitational field equation in one particular local Lorentz frame, with a given local time axis $e^{\mu}_{(0)}$. But the requirement of invariance under general coordinate transformations implies that this equation must hold for an arbitrary timelike unit vector $e^{\mu}_{(0)}$, and the only way that can be true, is that the Ricci tensor vanishes identically,

$$R_{\mu\nu} = 0 \, . \tag{16.4}$$

This is Einstein gravitational field equation in vacuum. The Ricci tensor enters the equation because it is the trace of the Riemann tensor, which describes the residual gravitational field seen in a local inertial frame.

The field equation with a source term can be obtained by a similar reasoning, but there is some subtlety involved. Newton's equation, with the mass density ρ as the source,

$$\nabla \cdot \boldsymbol{g} = -4\pi G\rho \ , \tag{16.5}$$

would now take the form

$$c^2 R_{(0)(0)} = 4\pi G\rho \ . \tag{16.6}$$

Einstein wanted to replace the mass density ρ by the energy density ρc^2, which is the component $T_{(0)(0)}$ of the energy momentum tensor $T_{\mu\nu}$. This gives the equation

$$R_{(0)(0)} = \frac{4\pi G}{c^4}\, T_{(0)(0)} \ , \tag{16.7}$$

which holds in the case of a non-relativistic mass distribution, with $T_{(0)(0)} = \rho c^2$, with $T_{(i)(0)} = T_{(0)(i)} = 0$ and $T_{(i)(j)} = 0$ for $i, j = 1, 2, 3$. The obvious generalization would be the equation

$$R_{\mu\nu} = \frac{4\pi G}{c^4}\, T_{\mu\nu} \ , \tag{16.8}$$

and this was indeed Einstein's first guess. However, he soon discovered that it needs some modification, because it is inconsistent with the conservation of energy and momentum. We now jump directly to the correct equation.

Given a metric and symmetric connection, the *Einstein curvature tensor* is defined as

$$\boxed{G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R \ .} \tag{16.9}$$

It is symmetric, and due to the contracted Bianchi identity, Equation (6.54), it has zero divergence,

$$G^{\mu\nu}{}_{;\nu} = R^{\mu\nu}{}_{;\nu} - \frac{1}{2}\, g^{\mu\nu} R_{,\nu} = R^{\mu\nu}{}_{;\nu} - g^{\mu\nu} g^{\rho\sigma} R_{\rho\nu;\sigma} = R^{\mu\nu}{}_{;\nu} - R^{\sigma\mu}{}_{;\sigma} = 0 \ . \tag{16.10}$$

For these two reasons, and because it is a tensor containing second order derivatives of the metric tensor $g_{\mu\nu}$, Einstein proposed to place it on the left hand side of a field equation having the energy momentum tensor as a source, on the right hand side. This is Einstein's gravitational equation,

$$\boxed{G_{\mu\nu} = \kappa T_{\mu\nu} \ .} \tag{16.11}$$

Since the divergence of $G_{\mu\nu}$ vanishes identically, the equation guarantees the conservation of energy and momentum, in a similar way as Maxwell's equations guarantee the conservation of electric charge.

Einstein's gravitational constant κ is proportional to Newton's gravitational constant G. The relation between them can be found by comparison with Equation (16.7), as we will do immediately below, or equivalently by comparison with the approximate equations (15.5) and (15.14), as we will do in the next section.

The trace of the Einstein tensor is

$$g^{\mu\nu}G_{\mu\nu} = g^{\mu\nu}R_{\mu\nu} - \frac{1}{2}\,g^{\mu\nu}g_{\mu\nu}R = R - 2R = -R\,. \tag{16.12}$$

Hence, by taking the trace of Einstein's equation, we find that the scalar curvature R is proportional to the trace $T = g^{\mu\nu}T_{\mu\nu}$ of the energy momentum tensor,

$$R = -\kappa T\,. \tag{16.13}$$

This means that we may write the gravitational equation in another way which is equivalent,

$$\boxed{R_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}T\right)\,.} \tag{16.14}$$

A special case is the gravitational equation in vacuum, $R_{\mu\nu} = 0$.

In the non-relativistic case, Equation (16.14) gives the equation

$$R_{00} = \kappa\left(T_{00} - \frac{1}{2}\,g_{00}T\right) = \frac{\kappa}{2}\,T_{00} = \frac{\kappa}{2}\,\rho c^2\,. \tag{16.15}$$

Comparing with Equation (16.7), we see that

$$\boxed{\kappa = \frac{8\pi G}{c^4}\,.} \tag{16.16}$$

16.2 Static mass distribution and weak field

The first test for Einstein's theory is that it must reproduce the results from Newton's theory in the non-relativistic limit. Thus, we have to look for an approximate solution of Einstein's field equation corresponding to Newton's gravitational potential ϕ from masses lying at rest, and we start by assuming that

$$T^{00} = \rho c^2, \qquad \text{all other } T^{\mu\nu} = 0\,, \tag{16.17}$$

where ρ is the density of rest mass. Strictly speaking, the stress tensor T^{kl} with $k, l = 1, 2, 3$ can not vanish if the mass distribution is to be static, but we may anyway assume that it is negligible compared to the energy density T^{00}. In addition, we assume that the field is weak, in the sense that ϕ/c^2 is small. Although we make assumptions about the mass distribution that is the source of the field, we no longer make any assumptions about the velocities of the particles moving in the field.

That the field is weak, must mean that the metric is the special relativistic Minkowski metric $\eta_{\mu\nu}$ plus a small perturbation,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ . \tag{16.18}$$

We may imagine a Taylor series development to first order in a small parameter ϵ, such that $h_{\mu\nu} = \epsilon \widehat{h}_{\mu\nu}$, with $\widehat{h}_{\mu\nu}$ of the same order as $\eta_{\mu\nu}$. To zeroth order in ϵ we have $g^{\mu\nu} = \eta^{\mu\nu}$. We may easily verify that

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} \tag{16.19}$$

to first order in ϵ, when we define $h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}$. Furthermore, to first order in ϵ we have that

$$\Gamma^{\kappa}_{\lambda\mu} = \frac{1}{2}\,\eta^{\kappa\rho}\left(h_{\rho\lambda,\mu} + h_{\rho\mu,\lambda} - h_{\lambda\mu,\rho}\right) \ , \tag{16.20}$$

and that

$$R_{\lambda\nu} = \Gamma^{\kappa}_{\lambda\nu,\kappa} - \Gamma^{\kappa}_{\lambda\kappa,\nu} = \frac{1}{2}\,\eta^{\kappa\rho}\left(h_{\rho\nu,\lambda\kappa} - h_{\lambda\nu,\rho\kappa} - h_{\rho\kappa,\lambda\nu} + h_{\lambda\kappa,\rho\nu}\right) \ . \tag{16.21}$$

The scalar curvature is, to the same order,

$$R = \eta^{\lambda\nu}R_{\lambda\nu} = \eta^{\lambda\nu}\eta^{\kappa\rho}\left(h_{\rho\nu,\lambda\kappa} - h_{\lambda\nu,\rho\kappa}\right) \ , \tag{16.22}$$

whereas Einstein's tensor is $G_{\mu\nu} = R_{\mu\nu} - (1/2)\eta_{\mu\nu}R$.

Since the left hand side of the gravitational equation is of first order in the parameter ϵ, the energy momentum tensor, on the right hand side, must also be of first order in ϵ, i.e., $T^{\mu\nu} = \epsilon\widehat{T}^{\mu\nu}$ with $\widehat{T}^{\mu\nu}$ finite. We have to first order that $T_{\mu\nu} = \eta_{\mu\rho}\eta_{\nu\sigma}T^{\rho\sigma}$. The two equivalent forms of the gravitational equation, Equations (16.11) and (16.14), then take the following first order forms. Either

$$\eta^{\rho\sigma}\left(h_{\rho\mu,\nu\sigma} + h_{\rho\nu,\mu\sigma} - h_{\mu\nu,\rho\sigma} - h_{\rho\sigma,\mu\nu}\right) - \eta_{\mu\nu}\eta^{\alpha\beta}\eta^{\rho\sigma}\left(h_{\rho\alpha,\beta\sigma} - h_{\alpha\beta,\rho\sigma}\right) = 2\kappa T_{\mu\nu} \ , \tag{16.23}$$

or

$$\eta^{\rho\sigma}\left(h_{\rho\mu,\nu\sigma} + h_{\rho\nu,\mu\sigma} - h_{\mu\nu,\rho\sigma} - h_{\rho\sigma,\mu\nu}\right) = \kappa\left(2T_{\mu\nu} - \eta_{\mu\nu}T\right) \ . \tag{16.24}$$

Both are linear second order differential equations for $h_{\mu\nu}$.

The first order version of the identity $G^{\mu\nu}{}_{;\nu} = 0$ says that $G^{\mu\nu}{}_{,\nu} = 0$, where $G^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}G_{\rho\sigma}$. The conservation law for energy and momentum, $T^{\mu\nu}{}_{;\nu} = 0$, which is a consistency condition for the gravitational equation, also has a first order version, which is that $T^{\mu\nu}{}_{,\nu} = 0$, since the difference

$$T^{\mu\nu}{}_{;\nu} - T^{\mu\nu}{}_{,\nu} = \Gamma^{\mu}_{\rho\nu}T^{\rho\nu} + \Gamma^{\nu}_{\rho\nu}T^{\mu\rho} \tag{16.25}$$

is of second order. We may verify that the equation $T^{\mu\nu}{}_{,\nu} = 0$ is precisely the consistency condition for Equation (16.23), and hence also for Equation (16.24). But $T^{\mu\nu}{}_{,\nu} = 0$ is the special relativistic conservation law, in which the acceleration due to the gravitational field is not taken into account. Thus, to first order in ϵ there is no back reaction from the gravitational field on matter. With the energy momentum tensor of Equation (16.17), the four equations

$T^{\mu\nu}{}_{,\nu} = 0$ reduce to the one equation $T^{00}{}_{,0} = 0$, saying that the mass density ρ is time independent.

With a static mass distribution it is natural to look for a static solution of the field equation for the metric. Thus, let us assume that $h_{\mu\nu,0} = 0$, and use Equation (16.24), which is the simplest version to use here. We also insert $T = \eta^{\mu\nu}T_{\mu\nu} = T_{00} = \rho c^2$. Then there are altogether ten equations to be solved, we must e.g. let the indices μ and ν take all values $0 \le \mu \le \nu \le 3$.

Taking first $\mu = \nu = 0$, we get the equation

$$\nabla^2 h_{00} = \kappa\rho c^2 \; . \tag{16.26}$$

From the Equations (15.14) and (15.5) we know that

$$h_{00} = \frac{2\phi}{c^2} \; , \qquad \nabla^2\phi = 4\pi G\rho \; . \tag{16.27}$$

Thus far, we get the non-relativistic limit of Einstein's theory to agree with Newton's theory when we choose Einstein's gravitational constant to be given by Newton's constant as $\kappa = 8\pi G/c^4$, the same relation we found before.

In order to solve the remaining nine equations it is sufficient to consider the three typical cases $\mu\nu = 11$, 01 and 12, giving the equations

$$\begin{aligned} -2(h_{11,11} + h_{21,12} + h_{31,13}) + \nabla^2 h_{11} - h_{,11} &= \kappa\rho c^2 \; , \\ -h_{10,11} - h_{20,12} - h_{30,13} + \nabla^2 h_{01} &= 0 \; , \\ -h_{11,21} - h_{21,22} - h_{31,23} - h_{12,11} - h_{22,12} - h_{32,13} + \nabla^2 h_{12} - h_{,12} &= 0 \; . \end{aligned} \tag{16.28}$$

Here we have introduced

$$h = \eta^{\mu\nu}h_{\mu\nu} = h_{00} - h_{11} - h_{22} - h_{33} \; . \tag{16.29}$$

All these three equations, and in the same way all the six equations we did not write out explicitly, are solved by the first order metric

$$h_{00} = h_{11} = h_{22} = h_{33} = \frac{2\phi}{c^2} \; , \qquad \text{all other } h_{\mu\nu} = 0 \; . \tag{16.30}$$

16.3 Deflection of light

Now that we know all the components of the metric $g_{\mu\nu}$ to order ϕ/c^2, we are able to compute how a particle moves in a gravitational field when its velocity approaches the speed of light. In particular, we are able to compute the deflection of light.

The metric is, to first order in ϕ/c^2,

$$g_{00} = 1 + \frac{2\phi}{c^2}, \qquad g_{11} = g_{22} = g_{33} = -1 + \frac{2\phi}{c^2}, \qquad \text{all other } g_{\mu\nu} = 0 \; . \tag{16.31}$$

For simplicity we assume that the gravitational potential ϕ is static.

The connection coefficients are given by Equation (15.32), and to order ϕ/c^2 they are

$$\begin{aligned} \Gamma^0_{\mu\nu} &= \frac{1}{2}\left(g_{0\mu,\nu} + g_{0\nu,\mu}\right) \; , \\ \Gamma^j_{\mu\nu} &= -\frac{1}{2}\left(g_{j\mu,\nu} + g_{j\nu,\mu} - g_{\mu\nu,j}\right) \; . \end{aligned} \tag{16.32}$$

As usual, we let greek indices run from 0 to 3 and latin indices from 1 to 3. The nonvanishing coefficients to order ϕ/c^2 are

$$\begin{aligned}\Gamma^0_{0j} &= \Gamma^0_{j0} = \Gamma^j_{00} = \frac{1}{2}\, g_{00,j} = \frac{1}{c^2}\,\phi_{,j}\ ,\\ \Gamma^j_{kl} &= -\frac{1}{c^2}\left(\delta_{jk}\,\phi_{,l} + \delta_{jl}\,\phi_{,k} - \delta_{kl}\,\phi_{,j}\right)\ .\end{aligned} \tag{16.33}$$

For the time coordinate x^0 the geodesic equation gives that

$$\frac{\mathrm{d}^2x^0}{\mathrm{d}s^2} = -\Gamma^0_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = -\frac{2}{c^2}\,\phi_{,j}\,\frac{\mathrm{d}x^0}{\mathrm{d}s}\,\frac{\mathrm{d}x^j}{\mathrm{d}s} = -\frac{2}{c^2}\,\frac{\mathrm{d}x^0}{\mathrm{d}s}\,\frac{\mathrm{d}\phi}{\mathrm{d}s}\ . \tag{16.34}$$

This equation may be integrated once exactly, and gives then that

$$\frac{\mathrm{d}x^0}{\mathrm{d}s} = K\mathrm{e}^{-2\phi/c^2} = K\left(1 - \frac{2\phi}{c^2}\right), \tag{16.35}$$

where K is an integration constant. We may for example choose $K = c$, then the parameter s corresponds approximately to the time t. We compute all the way through to first order in ϕ/c^2.

For the space coordinate x^j the geodesic equation gives that

$$\begin{aligned}\frac{\mathrm{d}^2x^j}{\mathrm{d}s^2} &= -\Gamma^j_{00}\left(\frac{\mathrm{d}x^0}{\mathrm{d}s}\right)^2 - \Gamma^j_{kl}\,\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^l}{\mathrm{d}s}\\ &= -\frac{1}{c^2}\,\phi_{,j}\left(\frac{\mathrm{d}x^0}{\mathrm{d}s}\right)^2 + \frac{2}{c^2}\,\phi_{,k}\,\frac{\mathrm{d}x^j}{\mathrm{d}s}\,\frac{\mathrm{d}x^k}{\mathrm{d}s} - \frac{1}{c^2}\,\phi_{,j}\,\delta_{kl}\,\frac{\mathrm{d}x^k}{\mathrm{d}s}\,\frac{\mathrm{d}x^l}{\mathrm{d}s}\ .\end{aligned} \tag{16.36}$$

When the curve $x^\mu = x^\mu(s)$ describes a light ray, its tangent $\mathrm{d}x^\mu/\mathrm{d}s$ is a null vector,

$$0 = g_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \left(1 + \frac{2\phi}{c^2}\right)\left(\frac{\mathrm{d}x^0}{\mathrm{d}s}\right)^2 - \left(1 - \frac{2\phi}{c^2}\right)\delta_{jk}\,\frac{\mathrm{d}x^j}{\mathrm{d}s}\,\frac{\mathrm{d}x^k}{\mathrm{d}s}\ . \tag{16.37}$$

With $\mathrm{d}x^0/\mathrm{d}s$ as above, and with $K = c$, this means that

$$\delta_{jk}\,\frac{\mathrm{d}x^j}{\mathrm{d}s}\,\frac{\mathrm{d}x^k}{\mathrm{d}s} = c^2\ . \tag{16.38}$$

Thus we arrive at the following equation, to first order,

$$\frac{\mathrm{d}^2x^j}{\mathrm{d}s^2} = -2\phi_{,j} + \frac{2}{c^2}\,\phi_{,k}\,\frac{\mathrm{d}x^j}{\mathrm{d}s}\,\frac{\mathrm{d}x^k}{\mathrm{d}s}\ . \tag{16.39}$$

That the field is weak, means that the light ray is approximately a straight line. We choose our coordinate system and our zero point for the parameter s such that the straight line is given by the equations $x = cs$, $y = y_0 =$ constant and $z = z_0 =$ constant. The equation of motion for x,

$$\frac{\mathrm{d}^2x}{\mathrm{d}s^2} = -2\,\frac{\partial\phi}{\partial x} + \frac{2}{c^2}\,\frac{\mathrm{d}x}{\mathrm{d}s}\left(\frac{\partial\phi}{\partial x}\,\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial\phi}{\partial y}\,\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial\phi}{\partial z}\,\frac{\mathrm{d}z}{\mathrm{d}s}\right), \tag{16.40}$$

holds when we insert $x = cs$ and $\mathrm{d}y/\mathrm{d}s = \mathrm{d}z/\mathrm{d}s = 0$. The deflection is given by the equations for y and z,

$$\begin{aligned} \frac{\mathrm{d}^2 y}{\mathrm{d}s^2} &= -2\,\frac{\partial\phi}{\partial y} + \frac{2}{c^2}\,\frac{\partial\phi}{\partial x}\,\frac{\mathrm{d}y}{\mathrm{d}s}\,\frac{\mathrm{d}x}{\mathrm{d}s} = -2\,\frac{\partial\phi}{\partial y}\,, \\ \frac{\mathrm{d}^2 z}{\mathrm{d}s^2} &= -2\,\frac{\partial\phi}{\partial z} + \frac{2}{c^2}\,\frac{\partial\phi}{\partial x}\,\frac{\mathrm{d}z}{\mathrm{d}s}\,\frac{\mathrm{d}x}{\mathrm{d}s} = -2\,\frac{\partial\phi}{\partial z}\,. \end{aligned} \tag{16.41}$$

It is interesting to compare the deflection computed here with the one that Newton's equation of motion would give. According to Newton, the acceleration of a particle in a gravitational field is

$$\frac{\mathrm{d}^2 x^j}{\mathrm{d}t^2} = -\phi_{,j}\,, \tag{16.42}$$

independent of the mass and velocity of the particle. If the velocity equals the speed of light c, and if we choose coordinates such that the light ray follows approximately the straight line $x = ct$, $y = y_0$, $z = z_0$, then the deflection is given, according to Newton, by the equations

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -\frac{\partial\phi}{\partial y}\,, \qquad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{\partial\phi}{\partial z}\,. \tag{16.43}$$

We see that Einstein's theory predicts twice as much deflection as Newton's theory for a particle moving with the speed of light. Remember that the most important requirement that Einstein imposed on his new theory, was that it had to agree with Newton's theory in the case of a particle moving with a small velocity. The reason for the factor of two between the predictions of the two theories when the velocities are large, is not too difficult to see. Newton's theory is the non-relativistic limit of Einstein's theory, and in this limit the gravitational potential ϕ is retained only in the time component g_{00} of the metric. As we have seen, the gravitational potential gives equally large first order corrections to all the four diagonal elements of the metric, but the corrections to the spatial components g_{jk} of the metric give noticeable contributions in the equation of motion only for particles having velocities comparable to the speed of light.

As a concrete example, let ϕ be the gravitational potential from a point mass M at the origin,

$$\phi = -G\,\frac{M}{r}\,, \tag{16.44}$$

with $r = \sqrt{x^2 + y^2 + z^2}$. We choose our coordinate system such that $z = z_0 = 0$, and get the following equation to solve,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}s^2} = -2\,\frac{\partial\phi}{\partial y} = -2G\,\frac{My_0}{(x^2 + {y_0}^2)^{3/2}}\,. \tag{16.45}$$

The total deflection is

$$\Delta\left(\frac{\mathrm{d}y}{\mathrm{d}s}\right) = -2GMy_0 \int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(c^2 s^2 + {y_0}^2)^{3/2}} = -\frac{4GM}{cy_0}\,, \tag{16.46}$$

since

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(c^2 s^2 + y^2)^{3/2}} = \left.\frac{s}{y^2\sqrt{c^2 s^2 + y^2}}\right|_{-\infty}^{\infty} = \frac{2}{cy^2}\,. \tag{16.47}$$

The angle of deflection, found by division with $\mathrm{d}x/\mathrm{d}s = c$, is

$$\alpha = -\frac{4GM}{c^2 y_0} \, . \tag{16.48}$$

For a light ray passing just above the surface of the Sun, the angle is

$$|\alpha| = 8,5 \times 10^{-6} = 1,75'' \, . \tag{16.49}$$

We have inserted here the mass of the Sun, $M = 2,0 \times 10^{30}$ kg, and the radius of the Sun, $y_0 = 7,0 \times 10^8$ m. According to Newton's theory the deflection should be half as much. Compare the angle of $1,75''$ to the angular diameters of the discs of the Sun and the Moon, both about $30' = 1800''$, that is, about one thousand times larger.

Arthur Eddington measured the deflection of the light from a star at the edge of the Sun during a total solar eclipse in 1919. He concluded that Einstein's theory is correct. However, his measurement was not an easy one, and did perhaps not convince everybody. Today, radio astronomers are able to measure angles with such impressive precision that they have to correct for the deflection of radio waves in the gravitational field of the Sun, even when observing in directions at 90 degrees relative to the Sun.

16.4 Linearized gravitational theory

The linearized gravitational Equation (16.23), or the equivalent Equation (16.24), is an interesting field equation in itself, irrespective of its derivation as an approximation to Einstein's equation. We have seen one way we may use it to compare the theories of Einstein and Newton. If the souce $T^{\mu\nu}$ is not static, but oscillates, then the same equation predicts a new phenomenon, gravitational radiation. We will not study gravitational radiation here, instead we will take a look at how the invariance of Einstein's equation under general coordinate transformations is expressed in the linearized equation.

Then we consider a coordinate transformation of the form

$$x^\mu \mapsto \widetilde{x}^\mu = x^\mu + \xi^\mu \, , \tag{16.50}$$

where $\xi^\mu = \xi^\mu(x) = \epsilon\widehat{\xi}^\mu(x)$ is a small quantity, proportional to the small parameter ϵ. As usual, we compute only to first order in ϵ.

The energy momentum tensor $T^{\mu\nu}$ is already of first order in ϵ, and it is invariant to first order under the small coordinate transformation, since every possible correction has to be second order. The metric tensor is transformed as follows,

$$\widetilde{g}_{\mu\nu}(\widetilde{x}) = \frac{\partial x^\rho}{\partial \widetilde{x}^\mu} \, \frac{\partial x^\sigma}{\partial \widetilde{x}^\nu} \, g_{\rho\sigma}(x) \, . \tag{16.51}$$

The matrix $\partial x^\rho / \partial \widetilde{x}^\mu$ is the inverse of

$$\frac{\partial \widetilde{x}^\mu}{\partial x^\rho} = \delta^\mu_\rho + \xi^\mu{}_{,\rho} \, , \tag{16.52}$$

and it is, to first order,

$$\frac{\partial x^\rho}{\partial \widetilde{x}^\mu} = \delta^\rho_\mu - \xi^\rho{}_{,\mu} \, . \tag{16.53}$$

Thus we have, to first order,

$$\widetilde{g}_{\mu\nu}(x) = (\delta^\rho_\mu - \xi^\rho{}_{,\mu})\,(\delta^\sigma_\nu - \xi^\sigma{}_{,\nu})\,g_{\rho\sigma}(x-\xi)\ . \tag{16.54}$$

Since $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $h_{\mu\nu}$ proportional to ϵ, we have to first order in ϵ that

$$g_{\mu\nu}(x-\xi) = g_{\mu\nu}(x) - \xi^\rho g_{\mu\nu,\rho}(x) = g_{\mu\nu}(x) - \xi^\rho h_{\mu\nu,\rho}(x) = g_{\mu\nu}(x)\ . \tag{16.55}$$

Thus, the field $h_{\mu\nu}$ transforms as follows, with $\xi_\mu = \eta_{\mu\rho}\xi^\rho$,

$$\widetilde{h}_{\mu\nu} = h_{\mu\nu} - \xi^\sigma{}_{,\nu}\,\eta_{\rho\mu} - \xi^\rho{}_{,\mu}\,\eta_{\sigma\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}\ . \tag{16.56}$$

We see that a small coordinate transformation acts as a kind of gauge transformation of the linearized gravitational field, in analogy to the gauge transformationen $\widetilde{A}_\mu = A_\mu - \chi_{,\mu}$ of the electromagnetic four vector potential. We see also directly that the left hand sides of both Equations (16.23) and (16.24) are invariant under this kind of gauge transformation.

16.5 The geodesic equation follows from the field equation

We postulated from the beginning that a "test particle", i.e. a point particle having such a small mass that it does not contribute essentially to the gravitational field, moves along a geodesic curve in spacetime. This postulate is actually superfluous, because it follows from the field equation $G_{\mu\nu} = \kappa T_{\mu\nu}$.

As we have seen, the field equation implies that the energy momentum tensor is conserved,

$$T^{\mu\nu}{}_{;\nu} = 0\ , \tag{16.57}$$

since $G^{\mu\nu}{}_{;\nu} = 0$ identically. That the field equation had to guarantee the conservation of the source, was an important clue to Einstein in his search for the right equation. His model in this respect was Maxwell's equations, which guarantee the conservation of electric charge. An important difference between the two cases is that the electric charge is a scalar, and hence its conservation is expressed in one single continuity equation. Energy and momentum, on the other hand, are components of a four vector, and as a consequence, the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ gives four equations, which are no less than complete equations of motion for the particles that are the sources of the field.

To see this, we consider as an example a collection of "dust", or to put it differently, an ideal gas without pressure, i.e. at zero temperature. If there are sufficiently many particles we may describe the gas approximately as a continuum, and if they are sufficiently small, they will have no appreciable influence on the metric. One single particle of mass m will follow an orbit in spacetime which may be parametrized by the proper time τ of this particle, as $x^\mu = x^\mu(\tau)$. The energy and momentum of the particle are the four componentene of the four vector $p^\mu = mu^\mu$, where u^μ is the *four velocity*, defined as

$$u^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\ . \tag{16.58}$$

The definition of the proper time τ is that

$$u_\mu u^\mu = c^2 = \text{constant}\ , \tag{16.59}$$

and the equation of motion for the particle that we want to derive, is the geodesic equation

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\rho\sigma} \frac{\mathrm{d}x^\rho}{\mathrm{d}\tau} \frac{\mathrm{d}x^\sigma}{\mathrm{d}\tau} = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma = 0 \, . \tag{16.60}$$

We asume that all the particles together may be described by a velocity field $u^\mu = u^\mu(x)$, and in addition a function $\rho = \rho(x)$ which is the mass density in the rest system of the particles passing by the spacetime point x. We want to show that Equation (16.60) follows from Equation (16.57) when the energy momentum tensor is

$$T^{\mu\nu} = \rho u^\mu u^\nu \, . \tag{16.61}$$

Inserting in Equation (16.57) we get that

$$(\rho u^\nu)_{;\nu} \, u^\mu + \rho u^\nu u^\mu{}_{;\nu} = 0 \, . \tag{16.62}$$

Multiplying this equation by u_μ gives that

$$(\rho u^\nu)_{;\nu} = 0 \, , \tag{16.63}$$

since $u_\mu u^\mu = c^2$, and hence

$$u_\mu u^\mu{}_{;\nu} = \frac{1}{2} \, (u_\mu u^\mu)_{;\nu} = 0 \, . \tag{16.64}$$

It follows that

$$u^\nu u^\mu{}_{;\nu} = 0 \, . \tag{16.65}$$

This equation is precisely the geodesic equation, since

$$u^\nu u^\mu{}_{;\nu} = \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau} \left(u^\mu{}_{,\nu} + \Gamma^\mu_{\rho\nu} u^\rho \right) = \frac{\mathrm{d}u^\mu}{\mathrm{d}\tau} + \Gamma^\mu_{\rho\nu} u^\rho u^\nu \, . \tag{16.66}$$

16.6 The cosmological constant

Einstein arrived at his field equation in 1915, but modified it in 1917, because he wanted the equation to have a static solution which could describe the Universe as a whole. His modification was to introduce an extra term, containing a new constant Λ, called the *cosmological constant*, such that the equation takes the form

$$\boxed{G^{\mu\nu} - \Lambda g^{\mu\nu} = \kappa T^{\mu\nu} \, .} \tag{16.67}$$

He regretted later the introduction of the new term, and called it "the blunder of my life". In fact, the cosmological constant did not solve the problem it was intended to solve. Formally, it results in the existence of solutions describing a static Universe, but these solutions can not be realized physically, because they are unstable.

If Einstein had not changed the equation, he would have predicted the cosmic redshift of light from distant galaxies, detected by Edwin Hubble during the next decade, and showing

that the Universe is expanding. However, the question of whether or not the cosmological constant is exactly zero, is in the end an experimental question. Since the modified gravitational equation may be written as

$$G^{\mu\nu} = \kappa \left(T^{\mu\nu} + \frac{\Lambda}{\kappa} g^{\mu\nu} \right), \tag{16.68}$$

we may interpret Λ/κ as an extra energy density, which is then thought of as a *vacuum energy density*. A crude upper limit for this energy density is the average mass density in a typical cluster of galaxies, which is around $\rho_0 = 10^{-26}$ kg/m^3. The fact that Newton's equations seem to be at least approximately valid in a galactic cluster, indicates that the energy density in vacuum is at most of the order of $\rho_0 c^2$. This gives the upper limit that

$$|\Lambda| \leq \kappa \rho_0 c^2 = \frac{8\pi G \rho_0}{c^2} = 1.9 \times 10^{-52} \text{ m}^{-2} . \tag{16.69}$$

Since $G^{\mu\nu}{}_{;\nu} = 0$ identcally, and also $g^{\mu\nu}{}_{;\rho} = 0$ identcally, we get by taking the divergence of Equation (16.67) that

$$-\Lambda_{,\nu}\, g^{\mu\nu} = \kappa T^{\mu\nu}{}_{;\nu} . \tag{16.70}$$

Thus, Λ has to be constant, $\Lambda_{,\nu} = 0$, if the energy and momentum conservation law $T^{\mu\nu}{}_{;\nu} = 0$ holds. Note that the introduction of a variable "cosmological constant" Λ would be a possible way of breaking the conservation law for energy and momentum.

Problems

1. What is the number of components of Einstein's curature tensor $G_{\mu\nu}$?
   What is the number of *independent* components of $G_{\mu\nu}$?
   How many independent field equations represent Einstein's gravitational equation?

2. Show that the Lagrange function

$$L = C g_{\mu\nu}(x) \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} \ ,$$

   where C is an arbitrary constant, gives the geodesic equation.
   Use this to compute the deflection of light in a weak gravitational field, without computing the connection coefficients explicitly.

Chapter 17

Lagrange formalism for the gravitational field

In this chapter we will see how to derive Einstein's gravitational equation, as a field equation for the metric tensor, from a variational principle. This derivation gives the gravitational equation rather directly and uniquely from a few simple assumptions, among which the most important is the invariance under general coordinate transformations.

Hilbert formulated a variational principle for gravitation in 1915, with the components $g_{\mu\nu} = g_{\nu\mu}$ of the metric tensor as the field variables. Palatini extended Hilbert's variational principle in 1919, using the same Lagrange density, but treating also the connection coefficients $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ as free variables together with $g_{\mu\nu}$. The field equations derived by variation of the connection coefficients are simply the standard relations between the connection coefficients and the metric.

In Hilbert's variational principle, with the components of the metric tensor as the only field variables, the Lagrange density contains both first and second order derivatives of the field. Normally, second order derivatives in the Lagrange density should give fourth order field equations, but the Hilbert action nevertheless gives second order equations. There is of course a natural explanation for this minor miracle, and one way to understand it is to observe that it is possible to transform the variational integral by partial integration in such a way that only first order derivatives remain.

In Palatini's variational principle, with the connection coefficients upgraded from auxiliary quantities to fully fledged field variables, there are only first order derivatives of the connection coefficients, and actually no derivatives at all of the metric. Normally, first order derivatives in the Lagrange density should give second order field equations. However, in this particular case it gives only first order field equations for the metric together with the connection coefficients. In these equations we may eliminate the connection coefficients, so as to recover the second order equations for the metric.

17.1 Hilbert's variational principle

Einstein's field equation for the gravitational field, including the cosmological constant Λ, may be derived from a variational principle in which the action integral is ekstremalized by variation of the metric $g_{\mu\nu}$. The action integral over a four dimensional spacetime region Ω

is of the form

$$\boxed{S = \frac{1}{c}\int_\Omega \mathrm{d}^4x\,\mathcal{L}\ .} \tag{17.1}$$

The integral is invariant under general coordinate transformations if the Lagrange density $\mathcal{L}$ transforms as a scalar density. The standard construction of a scalar density is to take $\mathcal{L} = \sqrt{|g|}\,\mathcal{L}_s$ with $\mathcal{L}_s$ a scalar, the "Lagrange scalar".

The invariance of the variational integral under general coordinate transformations guarantees the invariance of the Euler–Lagrange equations. However, it is worth noting that the Euler–Lagrange equations may be invariant under coordinate transformations even when the "Lagrange density" is not a scalar density. A sufficient condition is that the change in the "Lagrange density" under an arbitrary infinitesimal transformation is a divergence, because this means that the change in the variational integral is a surface integral, and hence does not modify the Euler–Lagrange equations.

The Lagrange density $\mathcal{L}$ of gravitation coupled to matter and to other fields, is a sum,

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_E + \mathcal{L}_M + \dots\ , \tag{17.2}$$

where $\mathcal{L}_G$ represents the gravitational field, $\mathcal{L}_E$ the electromagnetic field, $\mathcal{L}_M$ matter, and so on, if necessary. Note that all the three terms mentioned here, $\mathcal{L}_G$, $\mathcal{L}_E$, and $\mathcal{L}_M$, contain the metric. Hence, by varying the metric we get contributions from all of them, and the contributions from all the terms apart from $\mathcal{L}_G$ add up to the total energy momentum tensor. In general, $\mathcal{L}_M$ will also contain the electromagnetic vector potential A_μ, and by varying A_μ in $\mathcal{L}_M$ we will obtain the current density which is the source of the electromagnetic field. And so on.

Hilbert's Lagrange density for the gravitational field is

$$\boxed{\mathcal{L}_G = -\frac{1}{2\kappa}\sqrt{|g|}\,(R + 2\Lambda)\ .} \tag{17.3}$$

The variation of the corresponding action integral is

$$\delta S_G = \frac{1}{c}\int_\Omega \mathrm{d}^4x\ \delta\mathcal{L}_G = -\frac{1}{2c\kappa}\int_\Omega \mathrm{d}^4x\ \delta\Big(\sqrt{|g|}\,(R + 2\Lambda)\Big)\ . \tag{17.4}$$

We use Equation (2.81) to get that

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}\,g_{\mu\nu}\,\delta g^{\mu\nu}\ . \tag{17.5}$$

Furthermore,

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = \delta g^{\mu\nu}R_{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}\ . \tag{17.6}$$

The last term here is the covariant divergence of a vector, and a simple proof uses the same trick as we used in the proof of the Bianchi identity. We concentrate on one single, arbitrary

point, and choose a coordinate system in which $\Gamma^\lambda_{\mu\nu} = 0$ at this single point. Then we have, at this point,

$$g^{\mu\nu}{}_{,\kappa} = g^{\mu\nu}{}_{;\kappa} = 0 \; . \tag{17.7}$$

Hence also $g_{,\kappa} = 0$, and

$$g^{\mu\nu}\delta R_{\mu\nu} = g^{\mu\nu}(\delta\Gamma^\kappa_{\mu\nu,\kappa} - \delta\Gamma^\lambda_{\mu\lambda,\nu}) = \frac{\partial}{\partial x^\kappa}(\delta w^\kappa) = \frac{1}{\sqrt{|g|}}\,\frac{\partial}{\partial x^\kappa}\left(\sqrt{|g|}\;\delta w^\kappa\right) \; , \tag{17.8}$$

with

$$\delta w^\kappa = g^{\mu\nu}\,\delta\Gamma^\kappa_{\mu\nu} - g^{\mu\kappa}\,\delta\Gamma^\lambda_{\mu\lambda} \; . \tag{17.9}$$

Note that δw^κ is a vector. Even though the connection $\Gamma^\kappa_{\mu\nu}$ is not a tensor, its *variation* $\delta\Gamma^\kappa_{\mu\nu}$ actually is. We have now proved that

$$g^{\mu\nu}\delta R_{\mu\nu} = \frac{1}{\sqrt{|g|}}\,\frac{\partial}{\partial x^\kappa}\left(\sqrt{|g|}\;\delta w^\kappa\right) \tag{17.10}$$

at one point in one special coordinate system. But since both sides of the equation are scalars, the equation holds in all coordinate systems, and since the point was arbitrary, it holds everywhere.

Altogether we have that

$$\delta S_G = -\frac{1}{2c\kappa}\int_\Omega \mathrm{d}^4x\left(\sqrt{|g|}\;\delta g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}(R+2\Lambda)\right) + \frac{\partial}{\partial x^\kappa}\left(\sqrt{|g|}\;\delta w^\kappa\right)\right) . \tag{17.11}$$

The integral of the last term may be rewritten as a three dimensional surface integral over the boundary $\partial\Omega$ of Ω,

$$\int_\Omega \mathrm{d}^4x\;\frac{\partial}{\partial x^\kappa}\left(\sqrt{|g|}\;\delta w^\kappa\right) = \frac{1}{6}\int_{\partial\Omega}\sqrt{|g|}\;\delta w^\kappa\epsilon_{\kappa\lambda\mu\nu}\,\mathrm{d}x^\lambda\wedge\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu \; . \tag{17.12}$$

The boundary integral vanishes, unless we vary $g_{\mu\nu}$ and its derivatives on the boundary $\partial\Omega$, which we have no reason to do.

In order that $\delta S_G = 0$ for arbitrary variations $\delta g^{\mu\nu}$ in the interiour of Ω, with the symmetry restriction that $\delta g^{\mu\nu} = \delta g^{\nu\mu}$, a necessary and sufficient condition is that

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}(R+2\Lambda) = 0 \; . \tag{17.13}$$

This is the gravitational equation in vacuum. All other fields than the gravitational field, such as the electromagnetic field, contribute to the energy momentum tensor, standing on the right hand side in the gravitational equation as the source of the gravitational field.

17.2 Palatini's variational principle

The above Lagrange density $\mathcal{L}_G$ is a function of the metric alone, as long as we require that the connection is given by the metric as in Equation (15.32). However, Palatini observed that it is unnecessary to postulate the relation between metric and connection, because it follows from the same Lagrange density when we vary the metric $g_{\mu\nu}$ and the connection coefficients

$\Gamma^\lambda_{\mu\nu}$ as independent fields. One condition for the validity of Palatini's variational principle is that the connection coefficients do not occur in the Lagrange density of any other field than the gravitational field. This is not too serious a restriction, since it holds for example in the cases of the Klein–Gordon field and the Maxwell field, as we have seen earlier.

We assume, as always, that the metric tensor is symmetric, $g_{\mu\nu} = g_{\nu\mu}$. In addition, we assume that the connection is symmetric, $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$. The Lagrange density is, as a function of the 10 independent components of the metric tensor and the 40 independent connection coefficients,

$$\mathcal{L}_G = -\frac{1}{2\kappa}\sqrt{|g|}\left(g^{\lambda\nu}\left(\Gamma^\kappa_{\lambda\nu,\kappa} - \Gamma^\kappa_{\lambda\kappa,\nu} + \Gamma^\kappa_{\rho\kappa}\Gamma^\rho_{\lambda\nu} - \Gamma^\kappa_{\rho\nu}\Gamma^\rho_{\lambda\kappa}\right) + 2\Lambda\right) . \tag{17.14}$$

It contains no derivatives of the metric, which may therefore be regarded as a kind of Lagrange multiplier. Note that we have no right to assume that the Ricci tensor $R_{\lambda\nu}$ is symmetric, since even though we postulate that the connection is symmetric, we do not know a priori that it is metric. What we do know is that the scalar curvature R in the Lagrange density depends only on the symmetric part of the Ricci tensor,

$$R = g^{\lambda\nu}\,R_{\lambda\nu} = \frac{1}{2}\,g^{\lambda\nu}(R_{\lambda\nu} + R_{\nu\lambda}) . \tag{17.15}$$

Variation of the metric gives Einstein's gravitational equation for the symmetric part of the Ricci tensor. The computation is essentially the same as above,

$$\delta\mathcal{L}_G = -\frac{1}{4\kappa}\sqrt{|g|}\;\delta g^{\mu\nu}\left(R_{\mu\nu} + R_{\nu\mu} - \,g_{\mu\nu}(R + 2\Lambda)\right) . \tag{17.16}$$

Variation of the connection, independent of the metric, gives that

$$\delta\mathcal{L}_G = -\frac{1}{4\kappa}\sqrt{|g|}\;g^{\mu\nu}\left(\delta R_{\mu\nu} + \delta R_{\nu\mu}\right) . \tag{17.17}$$

Above, when we varied the metric under the assumption that it defined the connection coefficients uniquely, we found that the variation of the Ricci tensor, $\delta R_{\mu\nu}$, did not contribute to the field equations. Here we find otherwise, and one may ask what is new. The new element is that the last two equalities in Equation (17.8) need not hold any more, in the Palatini scheme, where it is no longer assumed that the metric is covariantly constant, i.e. that $g_{\mu\nu;\rho} = 0$. Hence, we have no guarantee that $g_{\mu\nu,\rho} = 0$ at a given point, even though the connection coefficients may vanish there, and this destroys our earlier proof that $\delta R_{\mu\nu}$ does not contribute to the field equations.

Palatini's variation of the connection gives the following Euler–Lagrange equations,

$$\frac{\partial\mathcal{L}_G}{\partial\Gamma^\alpha_{\beta\gamma}} - \frac{\mathrm{d}}{\mathrm{d}x^\mu}\left(\frac{\partial\mathcal{L}_G}{\partial\Gamma^\alpha_{\beta\gamma,\mu}}\right) = 0 . \tag{17.18}$$

If all the 64 fields $\Gamma^\alpha_{\beta\gamma}$ had been independent, we would have had 64 equations, but since we postulated that the connection should be symmetric, the number is reduced to 40. One way to treat the symmetry conditions is to differentiate with respect to $\Gamma^\alpha_{\beta\gamma}$ and $\Gamma^\alpha_{\beta\gamma,\mu}$ without paying attention to the symmetry $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$, but symmetrize afterwards the resulting expression in the indices β and γ. That gives the following symmetrized Euler–Lagrange equation,

$$\begin{aligned}\sqrt{|g|}\left(g^{\lambda\nu}\left(\delta^\gamma_\alpha\,\Gamma^\beta_{\lambda\nu} + \delta^\beta_\alpha\,\Gamma^\gamma_{\lambda\nu}\right) + 2\left(g^{\beta\gamma}\,\Gamma^\kappa_{\alpha\kappa} - g^{\lambda\gamma}\,\Gamma^\beta_{\lambda\alpha} - g^{\beta\nu}\,\Gamma^\gamma_{\alpha\nu}\right)\right)\\ = \frac{\partial}{\partial x^\mu}\left(\sqrt{|g|}\left(2\,g^{\beta\gamma}\,\delta^\mu_\alpha - g^{\beta\mu}\,\delta^\gamma_\alpha - g^{\gamma\mu}\,\delta^\beta_\alpha\right)\right) .\end{aligned} \tag{17.19}$$

These are 40 equations for 40 unknown connection coefficients. In order to solve them we first contract the indices α and γ, finding that

$$\sqrt{|g|}\, g^{\lambda\nu}\, \Gamma^{\beta}_{\lambda\nu} = -\frac{\partial}{\partial x^{\mu}} \left(\sqrt{|g|}\, g^{\beta\mu} \right) , \tag{17.20}$$

which we insert back into the Euler–Lagrange equation. We get that

$$\sqrt{|g|} \left(g^{\beta\gamma}\, \Gamma^{\kappa}_{\alpha\kappa} - g^{\lambda\gamma}\, \Gamma^{\beta}_{\lambda\alpha} - g^{\beta\nu}\, \Gamma^{\gamma}_{\alpha\nu} \right) = \frac{\partial}{\partial x^{\alpha}} \left(\sqrt{|g|}\, g^{\beta\gamma} \right) . \tag{17.21}$$

Here we contract the indices β and γ, to get that

$$\sqrt{|g|}\, \Gamma^{\kappa}_{\alpha\kappa} = \frac{1}{2}\, g_{\beta\gamma}\, \frac{\partial}{\partial x^{\alpha}} \left(\sqrt{|g|}\, g^{\beta\gamma} \right) = \frac{\partial}{\partial x^{\alpha}} \sqrt{|g|} \, , \tag{17.22}$$

which we insert back into Equation (17.21). The final result is an equation which is just the condition that the connection is metric,

$$g^{\beta\gamma}{}_{,\alpha} + g^{\lambda\gamma}\, \Gamma^{\beta}_{\lambda\alpha} + g^{\beta\nu}\, \Gamma^{\gamma}_{\alpha\nu} = 0 \, . \tag{17.23}$$

17.3 Uniqueness of Einstein's gravitational equation

A reformulation of Hilbert's variational principle

Hilbert used the curvature scalar

$$R = g^{\lambda\nu} \left(\Gamma^{\kappa}_{\lambda\nu,\kappa} - \Gamma^{\kappa}_{\lambda\kappa,\nu} + \Gamma^{\kappa}_{\rho\kappa} \Gamma^{\rho}_{\lambda\nu} - \Gamma^{\kappa}_{\rho\nu} \Gamma^{\rho}_{\lambda\kappa} \right) \tag{17.24}$$

in his invariant variational principle. This scalar quantity contains second order derivatives of the metric tensor, when the connection coefficients are expressed in terms of the metric and its first order derivatives, as in Equation (15.32). The second order derivatives in the Lagrange density $\mathcal{L}_G$ come from the expression

$$\mathcal{L}_2 = \sqrt{|g|}\, g^{\lambda\nu} \left(\Gamma^{\kappa}_{\lambda\nu,\kappa} - \Gamma^{\kappa}_{\lambda\kappa,\nu} \right) . \tag{17.25}$$

However, this expression may be rewritten as $\mathcal{L}_2 = \mathcal{L}'_2 + \mathcal{L}''_2$, with

$$\begin{aligned} \mathcal{L}'_2 &= -\left(\frac{\partial}{\partial x^{\rho}} \left(\sqrt{|g|}\, g^{\lambda\nu} \right) \right) \Gamma^{\rho}_{\lambda\nu} + \left(\frac{\partial}{\partial x^{\rho}} \left(\sqrt{|g|}\, g^{\lambda\rho} \right) \right) \Gamma^{\kappa}_{\lambda\kappa} \, , \\ \mathcal{L}''_2 &= \frac{\partial}{\partial x^{\rho}} \left(\sqrt{|g|} \left(g^{\lambda\nu}\, \Gamma^{\rho}_{\lambda\nu} - g^{\lambda\rho}\, \Gamma^{\kappa}_{\lambda\kappa} \right) \right) \, . \end{aligned} \tag{17.26}$$

$\mathcal{L}'_2$ is a function of only the metric and its first order derivatives, whereas $\mathcal{L}''_2$ is a divergence and does not contribute to the field equations. We have that

$$\frac{\partial}{\partial x^{\rho}} \sqrt{|g|} = \frac{1}{2} \sqrt{|g|}\, g^{\lambda\kappa} g_{\kappa\lambda,\rho} \, , \qquad \frac{\partial}{\partial x^{\rho}}\, g^{\mu\nu} = -g^{\mu\kappa} g^{\nu\lambda} g_{\kappa\lambda,\rho} \, . \tag{17.27}$$

Hence,

$$\frac{\partial}{\partial x^{\rho}} \left(\sqrt{|g|}\, g^{\mu\nu} \right) = \sqrt{|g|} \left(\frac{1}{2}\, g^{\mu\nu} g^{\lambda\kappa} - g^{\mu\kappa} g^{\nu\lambda} \right) g_{\kappa\lambda,\rho} \, . \tag{17.28}$$

Since the covariant derivative of the metric vanishes, $g_{\kappa\lambda;\rho} = 0$, we have that

$$g_{\kappa\lambda,\rho} = \Gamma^{\sigma}_{\kappa\rho}\, g_{\sigma\lambda} + \Gamma^{\sigma}_{\lambda\rho}\, g_{\kappa\sigma} \ , \tag{17.29}$$

and therefore

$$\frac{\partial}{\partial x^{\rho}} \left(\sqrt{|g|}\, g^{\mu\nu} \right) = \sqrt{|g|} \left(g^{\mu\nu}\, \Gamma^{\kappa}_{\kappa\rho} - g^{\mu\kappa}\, \Gamma^{\nu}_{\kappa\rho} - g^{\nu\kappa}\, \Gamma^{\mu}_{\kappa\rho} \right) \ . \tag{17.30}$$

Altogether, we find that

$$\mathcal{L}'_2 = -2 \sqrt{|g|}\, g^{\lambda\nu} \left(\Gamma^{\kappa}_{\rho\kappa} \Gamma^{\rho}_{\lambda\nu} - \Gamma^{\kappa}_{\rho\nu} \Gamma^{\rho}_{\lambda\kappa} \right) \ . \tag{17.31}$$

This means that Hilbert's Lagrange density $\mathcal{L}_G$ is equivalent to the Lagrange density

$$\mathcal{L}'_G = \frac{1}{2\kappa} \sqrt{|g|} \left(g^{\lambda\nu} \left(\Gamma^{\kappa}_{\rho\kappa} \Gamma^{\rho}_{\lambda\nu} - \Gamma^{\kappa}_{\rho\nu} \Gamma^{\rho}_{\lambda\kappa} \right) - 2\Lambda \right) \ . \tag{17.32}$$

Here the connection coefficients are not independent variables, but are given by the metric as in Equation (15.32). Explicitly, we have that

$$\begin{aligned} g^{\lambda\nu} \Gamma^{\kappa}_{\rho\kappa} \Gamma^{\rho}_{\lambda\nu} &= \frac{1}{4}\, g^{\lambda\nu} g^{\kappa\alpha} g^{\rho\beta} \left(g_{\alpha\rho,\kappa} + g_{\alpha\kappa,\rho} - g_{\rho\kappa,\alpha} \right) \left(g_{\beta\lambda,\nu} + g_{\beta\nu,\lambda} - g_{\lambda\nu,\beta} \right) \\ &= \frac{1}{4}\, g^{\lambda\nu} g^{\kappa\alpha} g^{\rho\beta} g_{\alpha\kappa,\rho} \left(2 g_{\beta\lambda,\nu} - g_{\lambda\nu,\beta} \right) \ , \\ g^{\lambda\nu} \Gamma^{\kappa}_{\rho\nu} \Gamma^{\rho}_{\lambda\kappa} &= \frac{1}{4}\, g^{\lambda\nu} g^{\kappa\alpha} g^{\rho\beta} \left(g_{\alpha\rho,\nu} + g_{\alpha\nu,\rho} - g_{\rho\nu,\alpha} \right) \left(g_{\beta\lambda,\kappa} + g_{\beta\kappa,\lambda} - g_{\lambda\kappa,\beta} \right) \\ &= \frac{1}{4}\, g^{\lambda\nu} g^{\kappa\alpha} g^{\rho\beta} \left(-g_{\alpha\rho,\nu} g_{\beta\kappa,\lambda} + 2 g_{\alpha\nu,\rho} g_{\beta\lambda,\kappa} \right) \ . \end{aligned} \tag{17.33}$$

And finally,

$$\begin{aligned} \mathcal{L}'_G = \frac{1}{8\kappa} \sqrt{|g|} \Big(g^{\kappa\lambda} g^{\mu\nu} g^{\rho\sigma} \big(& g_{\kappa\mu,\rho}\, g_{\lambda\nu,\sigma} - 2 g_{\kappa\mu,\rho}\, g_{\lambda\sigma,\nu} \\ & - g_{\kappa\lambda,\rho}\, g_{\mu\nu,\sigma} + 2 g_{\kappa\lambda,\rho}\, g_{\mu\sigma,\nu} \big) - 8\Lambda \Big) \ . \end{aligned} \tag{17.34}$$

This alternative "density" contains no higher than first order derivatives of the metric tensor. The price we pay is that it no longer transforms as a scalar density under general coordinate transformations, because we dropped the divergence term $\mathcal{L}''_2$, given in Equation (17.26). Now it does not matter if a coordinate transformation gives an extra divergence term in the Lagrange density, since such a term anyway does not contribute to the field equations. Thus, even though $\mathcal{L}'_G$ is not strictly speaking a scalar density, it is good enough for our purpose of formulating an invariant variational principle.

The general case

It is of course no accident that Hilbert's Lagrange density contains second order derivatives. It is difficult, not to say impossible, to construct a true scalar density, if it is required to depend on the first derivatives of the metric, but not on any higher derivatives.

For example, there exists no scalar density depending on the metric and its first derivatives, and quadratic in the derivatives, for the fundamental reason that we may always transform the value of such an expression into zero at an arbitrary point in spacetime, by choosing a

coordinate system in which all the first order derivatives of the metric vanish at this one point.

On the other hand, we found above precisely such an expression which is an acceptable Lagrange "density", because it transforms *almost* as a scalar density, only with an extra term which is a divergence. Having got the idea, we may just as well ask for the most general expression quadratic in the first derivatives of the metric, and transforming in a similar way under general coordinate transformations. This leads us to try with

$$\begin{aligned}\mathcal{L} \,=\, \sqrt{|g|}\, g^{\kappa\lambda} g^{\mu\nu} g^{\rho\sigma}\, (C_1\, g_{\kappa\mu,\rho}\, g_{\lambda\nu,\sigma} + C_2\, g_{\kappa\mu,\rho}\, g_{\lambda\sigma,\nu} \\ + C_3\, g_{\kappa\lambda,\rho}\, g_{\mu\nu,\sigma} + C_4\, g_{\kappa\lambda,\rho}\, g_{\mu\sigma,\nu} + C_5\, g_{\kappa\rho,\lambda}\, g_{\mu\sigma,\nu})\,, \end{aligned} \tag{17.35}$$

where C_1 to C_5 are constants. It is obvious that $\mathcal{L}$ transforms as a scalar density under *linear* coordinate transformations, but we demand certain transformation properties under *general* coordinate transformations, such that the field equations are invariant. The question is whether we are able to choose the coefficients C_1 to C_5 so as to obtain the right transformation properties.

It is sufficient to study an infinitesimal coordinate transformation, of the form

$$x^\mu \mapsto x^\mu + \xi^\mu\ , \tag{17.36}$$

with ξ^μ infinitesimal. The metric tensor transforms as follows, to first order in ξ^μ,

$$g_{\mu\nu} \mapsto \widetilde{g}_{\mu\nu} = g_{\mu\nu} + \Delta g_{\mu\nu} = g_{\mu\nu} - \xi^\kappa_{,\mu}\, g_{\kappa\nu} - \xi^\lambda_{,\nu}\, g_{\mu\lambda} - \xi^\sigma g_{\mu\nu,\sigma}\ . \tag{17.37}$$

Differentiation gives that

$$\Delta g_{\mu\nu,\rho} = -\xi^\kappa_{,\mu}\, g_{\kappa\nu,\rho} - \xi^\lambda_{,\nu}\, g_{\mu\lambda,\rho} - \xi^\sigma_{,\rho}\, g_{\mu\nu,\sigma} - \xi^\sigma g_{\mu\nu,\sigma\rho} - \xi^\kappa_{,\mu\rho}\, g_{\kappa\nu} - \xi^\lambda_{,\nu\rho}\, g_{\mu\lambda}\ . \tag{17.38}$$

Clearly, $g_{\mu\nu,\rho}$ would have been a tensor if it had not been for the last two terms in $\Delta g_{\mu\nu,\rho}$, containing the second derivatives of the coordinate transformation. In that case, $\mathcal{L}$ would have been a scalar density, transforming as follows,

$$\Delta\mathcal{L} = -\frac{\partial}{\partial x^\mu}\,(\xi^\mu \mathcal{L})\ . \tag{17.39}$$

Instead, we have now that

$$\Delta\mathcal{L} = -\frac{\partial}{\partial x^\mu}\,(\xi^\mu \mathcal{L}) + \Delta\mathcal{L}'\,, \tag{17.40}$$

where $\Delta\mathcal{L}'$ contains the second derivatives of the coordinate transformation,

$$\begin{aligned}\Delta\mathcal{L}' \,=\, -\sqrt{|g|}\, g^{\kappa\lambda} g^{\mu\nu} \Big((4C_1 + 2C_2)\, \xi^\rho_{,\kappa\mu}\, g_{\rho\lambda,\nu} + 2C_2\, \xi^\rho_{,\kappa\mu}\, g_{\lambda\nu,\rho} + (4C_3 + C_4)\, \xi^\rho_{,\rho\kappa}\, g_{\mu\nu,\lambda} \\ + 2(C_4 + C_5)\, \xi^\rho_{,\rho\kappa}\, g_{\lambda\mu,\nu} + C_4\, \xi^\rho_{,\kappa\lambda}\, g_{\mu\nu,\rho} + 2C_5\, \xi^\rho_{,\kappa\lambda}\, g_{\rho\mu,\nu} \Big)\,. \end{aligned} \tag{17.41}$$

A sufficient condition for the infinitesimal coordinate transformation to be a symmetry, is that $\Delta\mathcal{L}'$ is a divergence, in other words, that there exists a quantity $\mathcal{K}^\alpha$ such that $\Delta\mathcal{L}' = \mathcal{K}^\alpha_{,\alpha}$.

There are a limited number of possibilities if we try to guess what $\mathcal{K}^\alpha$ might be. We could have most generally that

$$\begin{aligned}\mathcal{K}^\alpha = \sqrt{|g|}\Big(&A_1\, g^{\kappa\lambda}\xi^\alpha_{,\kappa\lambda} + A_2\, g^{\alpha\lambda}\xi^\kappa_{,\kappa\lambda} + g^{\kappa\lambda}g^{\mu\nu}(A_3\,\xi^\alpha_{,\kappa}\,g_{\lambda\mu,\nu} + A_4\,\xi^\alpha_{,\kappa}\,g_{\mu\nu,\lambda})\\ &+g^{\alpha\lambda}g^{\mu\nu}(A_5\,\xi^\kappa_{,\kappa}\,g_{\lambda\mu,\nu} + A_6\,\xi^\kappa_{,\kappa}\,g_{\mu\nu,\lambda} + A_7\,\xi^\kappa_{,\lambda}\,g_{\kappa\mu,\nu} + A_8\,\xi^\kappa_{,\lambda}\,g_{\mu\nu,\kappa}\\ &+A_9\,\xi^\kappa_{,\mu}\,g_{\kappa\lambda,\nu} + A_{10}\,\xi^\kappa_{,\mu}\,g_{\kappa\nu,\lambda} + A_{11}\,\xi^\kappa_{,\mu}\,g_{\lambda\nu,\kappa})\Big)\,,\end{aligned} \tag{17.42}$$

with arbitrary constant coefficients A_1 to A_{11}. The conditions for $\mathcal{K}^\alpha_{,\alpha}$ to contain no third derivatives of ξ^μ and no second derivatives of $g_{\mu\nu}$, are that

$$A_1 + A_2 = A_3 + A_{11} = A_4 + A_8 = A_5 = A_6 = A_7 + A_9 = A_{10} = 0\,. \tag{17.43}$$

But $\mathcal{K}^\alpha_{,\alpha}$ should also contain no terms quadratic in the derivatives of the metric tensor. We ensure that by setting $A_3 = A_4 = A_7 = 0$, such that

$$\mathcal{K}^\alpha = A_1\,\sqrt{|g|}\,(g^{\kappa\lambda}\xi^\alpha_{,\kappa\lambda} - g^{\alpha\lambda}\xi^\kappa_{,\kappa\lambda}). \tag{17.44}$$

Then the equation $\Delta\mathcal{L}' = \mathcal{K}^\alpha_{,\alpha}$ holds if and only if

$$4C_1 = -2C_2 = -4C_3 = 2C_4 = A_1\,, \qquad C_5 = 0\,. \tag{17.45}$$

Comparison with Equation (17.34) shows that $A_1 = 1/(2\kappa)$.

We conclude that the requirement that the field equations should be invariant under general coordinate transformations, leads to a unique Lagrange density for the gravitational field which is quadratic in the first order derivatives of the metric tensor, and does not contain higher derivatives. It is given by Equation (17.34), excluding the cosmological constant Λ.

The general Euler–Lagrange equations

Deriving the Euler–Lagrange equations from the general Lagrange density in Equation (17.35) is a tedious exercise, complicated by the symmetry of the metric tensor. We have to compute

$$\mathcal{M}^{\alpha\beta} = \frac{\partial\mathcal{L}}{\partial g_{\alpha\beta}} - \frac{\mathrm{d}}{\mathrm{d}x^\gamma}\left(\frac{\partial\mathcal{L}}{\partial g_{\alpha\beta,\gamma}}\right). \tag{17.46}$$

The symmetry $g_{\alpha\beta} = g_{\beta\alpha}$ means that 10 of the 16 components of the metric are independent, and that the number of independent Euler–Lagrange equations is also 10. The symmetry must imply that $\mathcal{M}^{\alpha\beta} = \mathcal{M}^{\beta\alpha}$, if we differentiate correctly. In practice, however, it may be simplest to differentiate in Equation (17.46) as if $g_{\alpha\beta}$ and $g_{\beta\alpha}$ were independent variables. If the resultat is not symmetric, we have to symmetrize explicitly afterwards by writing the Euler–Lagrange equations as

$$\mathcal{M}^{\alpha\beta} + \mathcal{M}^{\beta\alpha} = 0\,. \tag{17.47}$$

Here is a brief reasoning to justify the procedure. When we vary $g_{\alpha\beta}$, we may admit only variations having the correct symmetry $\delta g_{\alpha\beta} = \delta g_{\beta\alpha}$, and hence the quantity occuring in the variation δS of the variational integral $S = (1/c)\int \mathrm{d}^4x\,\mathcal{L}$ is

$$\mathcal{M}^{\alpha\beta}\,\delta g_{\alpha\beta} = \frac{1}{2}\left(\mathcal{M}^{\alpha\beta} + \mathcal{M}^{\beta\alpha}\right)\delta g_{\alpha\beta}\,. \tag{17.48}$$

In order to compute $\mathcal{M}^{\alpha\beta}$, we use that

$$\frac{\partial}{\partial g_{\alpha\beta}}\sqrt{|g|} = \frac{1}{2}\sqrt{|g|}\,g^{\beta\alpha}\;, \qquad \frac{\partial}{\partial g_{\alpha\beta}}\,g^{\gamma\delta} = -g^{\gamma\alpha}g^{\delta\beta}\;, \tag{17.49}$$

and we use Equation (17.27). We find that $\mathcal{M}^{\alpha\beta}+\mathcal{M}^{\beta\alpha}$ is a sum of the following four kinds of expressions,

$$\begin{aligned}
&\text{(I)} && \sqrt{|g|}\,g^{\alpha\beta}g^{\cdot\cdot}\,g^{\cdot\cdot}\,g^{\cdot\cdot}\,g_{\cdot\cdot,\cdot}\,g_{\cdot\cdot,\cdot}\;,\\
&\text{(II)} && \sqrt{|g|}\,g^{\alpha\cdot}\,g^{\beta\cdot}\,g^{\cdot\cdot}\,g^{\cdot\cdot}\,g_{\cdot\cdot,\cdot}\,g_{\cdot\cdot,\cdot}\;,\\
&\text{(III)} && \sqrt{|g|}\,g^{\alpha\beta}\,g^{\cdot\cdot}\,g^{\cdot\cdot}\,g_{\cdot\cdot,\cdot\cdot}\;,\\
&\text{(IV)} && \sqrt{|g|}\,g^{\alpha\cdot}\,g^{\beta\cdot}\,g^{\cdot\cdot}\,g_{\cdot\cdot,\cdot\cdot}\;.
\end{aligned} \tag{17.50}$$

Taking into account the symmetry in the indices α and β, we get 5 possible terms of type (I), 11 of type (II), 2 of type (III), and 3 of type (IV). The Euler–Lagrange equations get the following form, after symmetrization and division by $\sqrt{|g|}$,

$$\begin{aligned}
&2g^{\alpha\beta}g^{\kappa\lambda}g^{\mu\nu}g^{\rho\sigma}\,(D_1\,g_{\kappa\mu,\rho}\,g_{\lambda\nu,\sigma}+D_2\,g_{\kappa\mu,\rho}\,g_{\lambda\sigma,\nu}\\
&\qquad +D_3\,g_{\kappa\lambda,\rho}\,g_{\mu\nu,\sigma}+D_4\,g_{\kappa\lambda,\rho}\,g_{\mu\sigma,\nu}+D_5\,g_{\kappa\rho,\lambda}\,g_{\mu\sigma,\nu})\\
&+\left(g^{\alpha\kappa}g^{\beta\lambda}+g^{\alpha\lambda}g^{\beta\kappa}\right)g^{\mu\nu}g^{\rho\sigma}\,(D_6\,g_{\kappa\mu,\rho}\,g_{\lambda\nu,\sigma}+D_7\,g_{\kappa\mu,\rho}\,g_{\lambda\sigma,\nu}+D_8\,g_{\kappa\mu,\nu}\,g_{\lambda\rho,\sigma}\\
&\qquad +D_9\,g_{\kappa\mu,\rho}\,g_{\nu\sigma,\lambda}+D_{10}\,g_{\kappa\mu,\nu}\,g_{\rho\sigma,\lambda}+D_{11}\,g_{\mu\rho,\kappa}\,g_{\nu\sigma,\lambda}+D_{12}\,g_{\mu\nu,\kappa}\,g_{\rho\sigma,\lambda}\\
&\qquad +D_{13}\,g_{\kappa\lambda,\mu}\,g_{\rho\sigma,\nu}+D_{14}\,g_{\kappa\lambda,\mu}\,g_{\nu\rho,\sigma}+D_{15}\,g_{\kappa\mu,\lambda}\,g_{\nu\rho,\sigma}+D_{16}\,g_{\kappa\mu,\lambda}\,g_{\rho\sigma,\nu})\\
&+2g^{\alpha\beta}g^{\kappa\lambda}g^{\mu\nu}\,(D_{17}\,g_{\kappa\lambda,\mu\nu}+D_{18}\,g_{\kappa\mu,\lambda\nu})\\
&+\left(g^{\alpha\kappa}g^{\beta\lambda}+g^{\alpha\lambda}g^{\beta\kappa}\right)g^{\mu\nu}\,(D_{19}\,g_{\kappa\lambda,\mu\nu}+D_{20}\,g_{\kappa\mu,\lambda\nu}+D_{21}\,g_{\mu\nu,\kappa\lambda})=0\;.
\end{aligned} \tag{17.51}$$

The coefficients D_1 to D_{21} become linear combinations of the coefficients C_1 to C_5 in the Lagrange density. A straightforward, but tedious computation gives that

$$\begin{aligned}
D_1 &= \frac{C_1}{2}+2C_3\;, \quad D_2=\frac{C_2}{2}+C_4\;, \quad D_3=-\frac{C_3}{2}\;, \quad D_4=2C_3\;, \quad D_5=C_4+\frac{C_5}{2}\;,\\
D_6 &= 2C_1\;, \quad D_7=C_2\;, \quad D_8=C_5\;, \quad D_9=2C_5\;, \quad D_{10}=-C_5\;,\\
D_{11} &= -C_1+C_4\;, \quad D_{12}=-C_3-\frac{C_4}{2}\;, \quad D_{13}=-C_1\;, \quad D_{14}=2C_1\;,\\
D_{15} &= 2C_2\;, \quad D_{16}=-C_2\;, \quad D_{17}=-2C_3\;, \quad D_{18}=-C_4\;,\\
D_{19} &= -2C_1\;, \quad D_{20}=-2(C_2+C_5)\;, \quad D_{21}=-C_4\;.
\end{aligned} \tag{17.52}$$

Problems

1. Use Noether's theorem to derive a conservation law following from the general covariance of the theory defined by the Lagrange "density" $\mathcal{L}'_G$ in Equation (17.34).

2. What conditions must be imposed on the coefficients D_1 til D_{21} in Equation (17.51) to make the equation generally covariant (i.e. invariant under general coordinate transformations)?
   Do these conditions ensure that the equation can be derived from a Lagrange "density" (such as $\mathcal{L}'_G$)?

Chapter 18

Rotationally symmetric gravitational field

The first exact solution of Einstein's gravitational equation was, quite naturally, the one with full rotational symmetry, found by Schwarzschild immediately after Einstein proposed the equation. The few exact solutions that exist, and in particular the Schwarzschild solution, are very useful for the purpose of understanding the contents of the theory. The Schwarzschild solution is even a realistic representation of the gravitational field of the Sun, and is used for computing the general relativistic corrections to planetary orbits.

In order to derive the Schwarzschild solution, we will now proceed to compute the Einstein tensor $G_{\mu\nu}$ with no other simplification than the assumption that the metric $g_{\mu\nu}$ is spherically symmetric. In particular, we will not assume that it is static.

18.1 Spherically symmetric metric

We introduce polar coordinates $x^0 = ct$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \varphi$, in terms of which the Cartesian coordinates are

$$x = r\sin\theta\cos\varphi\ , \qquad y = r\sin\theta\sin\varphi\ , \qquad z = r\cos\theta\ .$$

The most general spherically symmetric metric may be written as

$$\mathrm{d}s^2 = A(t,r)\,c^2\mathrm{d}t^2 + 2B(t,r)\,c\,\mathrm{d}t\,\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} + C(t,r)\,(\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r})^2 + D(t,r)\,\mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r}\ , \tag{18.1}$$

where A, B, C and D are arbitrary functions of t and r. The assumption of spherical symmetry reduces a four dimensional problem to a two dimensional one. We have that

$$\begin{aligned} \boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} &= \frac{1}{2}\,\mathrm{d}(\boldsymbol{r}\cdot\boldsymbol{r}) = \frac{1}{2}\,\mathrm{d}(r^2) = r\,\mathrm{d}r\ , \\ \mathrm{d}\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{r} &= \mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2\ . \end{aligned} \tag{18.2}$$

Hence,

$$\mathrm{d}s^2 = Ac^2\mathrm{d}t^2 + 2rBc\,\mathrm{d}t\,\mathrm{d}r + (r^2C + D)\,\mathrm{d}r^2 + r^2D\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\ , \tag{18.3}$$

or in matrix form,

$$g_{\mu\nu} = \begin{pmatrix} A & rB & 0 & 0 \\ rB & r^2C + D & 0 & 0 \\ 0 & 0 & r^2D & 0 \\ 0 & 0 & 0 & r^2D\sin^2\theta \end{pmatrix} . \tag{18.4}$$

The metric is diagonal, except that possibly $g_{01} \neq 0$, and in addition we have $g_{33} = g_{22}\sin^2\theta$. This is quite generally the form of a covariant tensor field of rank two, expressed in polar coordinates, when it is symmetric under interchange of the two indices, and also rotationally symmetric. Thus, spherical symmetry means that the Ricci tensor $R_{\mu\nu}$, the Einstein tensor $G_{\mu\nu}$, and the energy momentum tensor $T_{\mu\nu}$ must all be of a similar form.

We still have a certain freedom in choosing coordinates, and we may use it to simplify the metric a little further. We assume that $D < 0$, and then we may introduce a new radial coordinate

$$\widetilde{r} = r\sqrt{-D} \, . \tag{18.5}$$

Note that the relation between r and $\widetilde{r}$ is time dependent if D is time dependent, and this change of coordinates removes any time dependence there might be in the angular components of the metric, g_{22} and g_{33}. We get that

$$\mathrm{d}s^2 = \widetilde{A}c^2\mathrm{d}t^2 + 2\widetilde{B}c\,\mathrm{d}t\,\mathrm{d}\widetilde{r} + \widetilde{C}\,\mathrm{d}\widetilde{r}^2 - \widetilde{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) , \tag{18.6}$$

when we introduce new functions $\widetilde{A} = \widetilde{A}(t,\widetilde{r})$, $\widetilde{B} = \widetilde{B}(t,\widetilde{r})$ and $\widetilde{C} = \widetilde{C}(t,\widetilde{r})$, given by A, B, C and D. Note that the angular part of the metric determines the area of the two dimentional surface t = constant, $\widetilde{r}$ = constant to be $4\pi\widetilde{r}^{\,2}$. This expression for the area defines the particular radial coordinate $\widetilde{r}$.

We may make the metric completely diagonal by introducing a new time coordinate $\widetilde{t}$, such that

$$c\,\mathrm{d}\widetilde{t} = E\left(\widetilde{A}c\,\mathrm{d}t + \widetilde{B}\,\mathrm{d}\widetilde{r}\right) . \tag{18.7}$$

Here $E = E(t,\widetilde{r})$ is an arbitrary integrating factor, such a factor exists because there are only two independent variables t and $\widetilde{r}$. In fact, all we have to do in order to solve Equation (18.7), with a function $E(t,\widetilde{r})$ not specified beforehand, is to define a function $\widetilde{t} = \widetilde{t}(t,\widetilde{r})$ such that $\widetilde{t}$ is constant on every integral curve in the $(t,\widetilde{r})$ plane of the ordinary differential equation

$$\widetilde{A}c\,\mathrm{d}t + \widetilde{B}\,\mathrm{d}\widetilde{r} = 0 \, . \tag{18.8}$$

Then we get that

$$\mathrm{d}s^2 = \frac{c^2}{\widetilde{A}E^2}\,\mathrm{d}\widetilde{t}^{\,2} + \left(\widetilde{C} - \frac{\widetilde{B}^2}{\widetilde{A}}\right)\mathrm{d}\widetilde{r}^{\,2} - \widetilde{r}^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) . \tag{18.9}$$

Even now we have the freedom to choose another time coordinate which is an arbitrary function of $\widetilde{t}$, without thereby destroying the diagonal form of the metric. We will use this freedom later on.

Let us now change our notation and write (t,r) for $(\widetilde{t},\widetilde{r})$, and furthermore

$$\mathrm{d}s^2 = \mathrm{e}^{2a}c^2\mathrm{d}t^2 - \mathrm{e}^{2b}\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2\right) , \tag{18.10}$$

where $a = a(x^0, x^1) = a(ct, r)$ and $b = b(x^0, x^1) = b(ct, r)$ are unknown functions to be determined from the field equations.

As we have now seen, it is possible in general, by means of a suitable choice of coordinates, to write a spherically symmetric metric in the form

$$g_{\mu\nu} = \begin{pmatrix} \mathrm{e}^{2a} & 0 & 0 & 0 \\ 0 & -\mathrm{e}^{2b} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} . \tag{18.11}$$

The only angular dependece here is in g_{33}, and in particular the metric is independent of the asimuthal angle $x^3 = \varphi$. The determinant of the metric tensor is

$$g = \det(g_{\mu\nu}) = -\mathrm{e}^{2(a+b)}\, r^4 \sin^2\theta \ , \tag{18.12}$$

and we define

$$\gamma = \frac{1}{2} \ln|g| = a + b + 2\ln r + \ln\sin\theta \ , \tag{18.13}$$

with the derivatives

$$\gamma_{,0} = a_{,0} + b_{,0} \ , \quad \gamma_{,1} = a_{,1} + b_{,1} + \frac{2}{r} \ , \quad \gamma_{,2} = \cot\theta \ , \quad \gamma_{,3} = 0 \ . \tag{18.14}$$

18.2 The Ricci tensor

We will use here Equation (6.55) to compute the Ricci tensor $R_{\mu\nu}$ for the spherically symmetric metric. For a vector field with constant componenta A^μ we have that

$$R_{\lambda\nu} A^\lambda A^\nu = R^{(\mathrm{I})} + R^{(\mathrm{IIa})} + R^{(\mathrm{IIb})} + R^{(\mathrm{III})} \ , \tag{18.15}$$

with

$$\begin{aligned} R^{(\mathrm{I})} &= B^\kappa{}_{,\kappa} + \gamma_{,\kappa}\, B^\kappa \ , \\ R^{(\mathrm{IIa})} &= \frac{1}{4}\, g^{\mu\sigma} g^{\kappa\rho} (A_{\kappa,\sigma} - A_{\sigma,\kappa})(A_{\rho,\mu} - A_{\mu,\rho}) \ , \\ R^{(\mathrm{IIb})} &= -\frac{1}{4}\, g^{\mu\sigma} g^{\kappa\rho} (A^\nu g_{\kappa\sigma,\nu})(A^\lambda g_{\rho\mu,\lambda}) \ , \\ R^{(\mathrm{III})} &= -\gamma_{,\lambda\nu}\, A^\lambda A^\nu \ , \end{aligned} \tag{18.16}$$

and with

$$B^\kappa = A^\nu A^\kappa{}_{;\nu} = g^{\kappa\rho} A^\nu \left(A_{\rho,\nu} - \frac{1}{2}\, A_{\nu,\rho} \right) . \tag{18.17}$$

We have immedately that

$$\begin{aligned} R^{(\mathrm{III})} &= -(a_{,00} + b_{,00})(A^0)^2 - 2(a_{,01} + b_{,01}) A^0 A^1 \\ &\quad - \left(a_{,11} + b_{,11} - \frac{2}{r^2} \right) (A^1)^2 + \frac{1}{\sin^2\theta}\, (A^2)^2 \ . \end{aligned} \tag{18.18}$$

Next,

$$A_\mu = (A_0, A_1, A_2, A_3) = (\mathrm{e}^{2a} A^0\,,\, -\mathrm{e}^{2b} A^1\,,\, -r^2 A^2\,,\, -r^2 \sin^2\theta\, A^3)\,. \tag{18.19}$$

The partial derivatives of A_μ not vanishing identically are

$$\begin{aligned} A_{0,0} &= 2a_{,0}\,\mathrm{e}^{2a} A^0\,, \quad & A_{0,1} &= 2a_{,1}\,\mathrm{e}^{2a} A^0\,, \\ A_{1,0} &= -2b_{,0}\,\mathrm{e}^{2b} A^1\,, \quad & A_{1,1} &= -2b_{,1}\,\mathrm{e}^{2b} A^1\,, \\ A_{2,1} &= -2r A^2\,, & & \\ A_{3,1} &= -2r\sin^2\theta\, A^3\,, \quad & A_{3,2} &= -2r^2 \sin\theta\cos\theta\, A^3\,. \end{aligned} \tag{18.20}$$

This gives that

$$\begin{aligned} B^0 &= \mathrm{e}^{-2a}\left(\frac{1}{2}\,A^0 A_{0,0} + A^1\left(A_{0,1} - \frac{1}{2}\,A_{1,0}\right)\right) \\ &= a_{,0}\,(A^0)^2 + 2a_{,1}\,A^0 A^1 + b_{,0}\,\mathrm{e}^{2(b-a)}\,(A^1)^2\,, \\ B^1 &= -\mathrm{e}^{-2b}\left(A^0\left(A_{1,0} - \frac{1}{2}\,A_{0,1}\right) + \frac{1}{2}\,A^1 A_{1,1} - \frac{1}{2}\,A^2 A_{2,1} - \frac{1}{2}\,A^3 A_{3,1}\right) \\ &= a_{,1}\,\mathrm{e}^{2(a-b)}\,(A^0)^2 + 2b_{,0}\,A^0 A^1 + b_{,1}\,(A^1)^2 - r\,\mathrm{e}^{-2b}\left((A^2)^2 + \sin^2\theta\,(A^3)^2\right)\,, \\ B^2 &= -\frac{1}{r^2}\left(A^1 A_{2,1} - \frac{1}{2}\,A^3 A_{3,2}\right) = \frac{2}{r}\,A^1 A^2 - \sin\theta\cos\theta\,(A^3)^2\,. \end{aligned} \tag{18.21}$$

We do not bother to compute B^3. In fact, it does not contribute to $R^{(\mathrm{I})}$, because the coordinate $x^3 = \varphi$ is cyclic, that is, the metric does not depend on φ. We have that

$$\begin{aligned} R^{(\mathrm{I})} &= \left(a_{,00} + (a_{,0})^2 + a_{,0}\,b_{,0} + \left(a_{,11} + 3(a_{,1})^2 - a_{,1}\,b_{,1} + \frac{2a_{,1}}{r}\right)\mathrm{e}^{2(a-b)}\right)(A^0)^2 \\ &\quad + 2\left(a_{,01} + b_{,01} + a_{,0}\,a_{,1} + 2a_{,1}\,b_{,0} + b_{,0}\,b_{,1} + \frac{2b_{,0}}{r}\right)A^0 A^1 \\ &\quad + \left(\left(b_{,00} - a_{,0}\,b_{,0} + 3(b_{,0})^2\right)\mathrm{e}^{2(b-a)} + b_{,11} + a_{,1}\,b_{,1} + (b_{,1})^2 + \frac{2b_{,1}}{r}\right)(A^1)^2 \\ &\quad - (3 + ra_{,1} - rb_{,1})\,\mathrm{e}^{-2b}\left((A^2)^2 + \sin^2\theta\,(A^3)^2\right) \\ &\quad + \frac{2\cot\theta}{r}\,A^1 A^2 + (3\sin^2\theta - 2)\,(A^3)^2\,. \end{aligned} \tag{18.22}$$

We have further that

$$\begin{aligned} R^{(\mathrm{IIa})} &= -\frac{\mathrm{e}^{-2(a+b)}}{2}\,(A_{0,1} - A_{1,0})^2 + \frac{\mathrm{e}^{-2b}}{2r^2}\,(A_{2,1})^2 \\ &\quad + \frac{\mathrm{e}^{-2b}}{2r^2\sin^2\theta}\,(A_{3,1})^2 + \frac{1}{2r^4\sin^2\theta}\,(A_{3,2})^2 \\ &= -2(a_{,1})^2\,\mathrm{e}^{2(a-b)}\,(A^0)^2 - 4a_{,1}\,b_{,0}\,A^0 A^1 - 2(b_{,0})^2\,\mathrm{e}^{2(b-a)}\,(A^1)^2 \\ &\quad + 2\mathrm{e}^{-2b}\left((A^2)^2 + \sin^2\theta\,(A^3)^2\right) + 2\cos^2\theta\,(A^3)^2\,. \end{aligned} \tag{18.23}$$

The metric tensor is diagonal, and for the diagonal elements we have that

$$\begin{aligned} A^\nu g_{00,\nu} &= 2\mathrm{e}^{2a}\,(a_{,0}\,A^0 + a_{,1}\,A^1)\,, \\ A^\nu g_{11,\nu} &= -2\mathrm{e}^{2b}\,(b_{,0}\,A^0 + b_{,1}\,A^1)\,, \\ A^\nu g_{22,\nu} &= -2rA^1\,, \\ A^\nu g_{33,\nu} &= -2r\sin^2\theta\,A^1 - 2r^2\sin\theta\cos\theta\,A^2\,. \end{aligned} \tag{18.24}$$

Furthermore,

$$\begin{aligned} R^{(\mathrm{IIb})} &= -\frac{\mathrm{e}^{-4a}}{4}\,(A^\nu g_{00,\nu})^2 - \frac{\mathrm{e}^{-4b}}{4}\,(A^\nu g_{11,\nu})^2 \\ &\quad -\frac{1}{4r^4}\,(A^\nu g_{22,\nu})^2 - \frac{1}{4r^4\sin^4\theta}(A^\nu g_{33,\nu})^2\ . \\ &= -((a_{,0})^2 + (b_{,0})^2)\,(A^0)^2 - 2(a_{,0}\,a_{,1} + b_{,0}\,b_{,1})\,A^0A^1 \\ &\quad -\left((a_{,1})^2 + (b_{,1})^2 + \frac{2}{r^2}\right)\,(A^1)^2 - \frac{2\cot\theta}{r}A^1A^2 - \cot^2\theta\,(A^2)^2\ . \end{aligned} \tag{18.25}$$

The sum of the four contributions is

$$\begin{aligned} R_{\lambda\nu}A^\lambda A^\nu &= \left(-b_{,00} - (b_{,0})^2 + a_{,0}\,b_{,0} + \left(a_{,11} + (a_{,1})^2 - a_{,1}\,b_{,1} + \frac{2a_{,1}}{r}\right)\mathrm{e}^{2(a-b)}\right)\,(A^0)^2 \\ &\quad + \left(\left(b_{,00} + (b_{,0})^2 - a_{,0}\,b_{,0}\right)\mathrm{e}^{2(b-a)} - a_{,11} - (a_{,1})^2 + a_{,1}\,b_{,1} + \frac{2b_{,1}}{r}\right)(A^1)^2 \\ &\quad + \frac{4b_{,0}}{r}\,A^0A^1 + \left(1 - (1 + ra_{,1} - rb_{,1})\,\mathrm{e}^{-2b}\right)\left((A^2)^2 + \sin^2\theta\,(A^3)^2\right). \end{aligned} \tag{18.26}$$

We may read the components $R_{\lambda\nu}$ of the Ricci tensor out of this formula. The scalar curvature is

$$\begin{aligned} R &= g^{\lambda\nu}R_{\lambda\nu} = \mathrm{e}^{-2a}R_{00} - \mathrm{e}^{-2b}R_{11} - \frac{R_{22}}{r^2} - \frac{R_{33}}{r^2\sin^2\theta} \\ &= -2(b_{,00} + (b_{,0})^2 - a_{,0}\,b_{,0})\,\mathrm{e}^{-2a} + 2(a_{,11} + (a_{,1})^2 - a_{,1}\,b_{,1})\,\mathrm{e}^{-2b} \\ &\quad -\frac{2}{r^2}\left(1 - (1 + 2ra_{,1} - 2rb_{,1})\,\mathrm{e}^{-2b}\right)\ . \end{aligned} \tag{18.27}$$

Since the metric is diagonal, we have that $G_{\lambda\nu} = R_{\lambda\nu}$ for $\lambda \neq \nu$. The diagonal elements of $G_{\lambda\nu}$ are

$$\begin{aligned} G_{00} &= R_{00} - \frac{\mathrm{e}^{2a}}{2}\,R = \frac{\mathrm{e}^{2a}}{r^2}\left(1 + (2rb_{,1} - 1)\,\mathrm{e}^{-2b}\right) = \frac{\mathrm{e}^{2a}}{r^2}\,\frac{\partial}{\partial r}\left(r\left(1 - \mathrm{e}^{-2b}\right)\right)\ , \\ G_{11} &= R_{11} + \frac{\mathrm{e}^{2b}}{2}\,R = \frac{1}{r^2}\left(2ra_{,1} + 1 - \mathrm{e}^{2b}\right)\ , \\ G_{22} &= R_{22} + \frac{r^2}{2}\,R \\ &= -r^2(b_{,00} + (b_{,0})^2 - a_{,0}\,b_{,0})\,\mathrm{e}^{-2a} + \left(r^2(a_{,11} + (a_{,1})^2 - a_{,1}\,b_{,1}) + r(a_{,1} - b_{,1})\right)\mathrm{e}^{-2b}\ , \\ G_{33} &= G_{22}\sin^2\theta\ . \end{aligned} \tag{18.28}$$

18.3 The Schwarzschild metric

The Schwarzschild metric is the spherically symmetric solution of the equation

$$G_{\lambda\nu} = 0\ , \tag{18.29}$$

which is Einstein's gravitational equation in vacuum. It is worth pointing out that we may choose to restrict our solution of the equation to a limited interval in the radius r. Hence, the

Schwarzschild metric may be interpreted quite generally as, for example, the gravitational field outside an arbitrary spherically symmetric mass distribution.

The equation $G_{01} = 2b_{,0}/r = 0$ means that b is constant in time, $b = b(r)$. Then the equation $G_{00} = 0$ has the solution

$$\mathrm{e}^{-2b} = 1 - \frac{R_M}{r} \,, \tag{18.30}$$

where R_M is a time independent integration constant with the dimension of length. The index "M" is written here to suggest that R_M is somehow related to a gravitational mass M which is the source of the field. Here and now this is a suspicion, to be confirmed shortly.

The two equations $G_{00} = 0$ and $G_{11} = 0$ together give that

$$a_{,1} + b_{,1} = \frac{r}{2}\left(\mathrm{e}^{2(b-a)}G_{00} + G_{11}\right) = 0 \,, \tag{18.31}$$

which means that $a + b = f(t)$ for an arbitrary function $f(t)$, so that

$$\mathrm{e}^{2a}\,\mathrm{d}t^2 = \mathrm{e}^{-2b}\mathrm{e}^{2f(t)}\,\mathrm{d}t^2 \,. \tag{18.32}$$

Without any loss of generality we may assume that $f(t) = 0$, since otherwise we may simply replace t by a new time coordinate $\widetilde{t}$ such that

$$\mathrm{d}\widetilde{t} = \mathrm{e}^{f(t)}\,\mathrm{d}t \,. \tag{18.33}$$

Thus we get that

$$\mathrm{e}^{2a} = \mathrm{e}^{-2b} = 1 - \frac{R_M}{r} \,. \tag{18.34}$$

This solves the three equations $G_{00} = G_{11} = G_{01} = 0$. The two remaining equations, $G_{22} = G_{33} = 0$, reduce to one, $G_{22} = 0$, reducing further to the equation

$$ra_{,11} + (ra_{,1} + 1)(a_{,1} - b_{,1}) = 0 \,. \tag{18.35}$$

It holds identically with the above expressions for a and b.

Physical interpretation

In order to interpret the integration constant R_M we let r become large, because we know that $g_{00} = \mathrm{e}^{2a}$ is then given by Newton's gravitational potensial ϕ as

$$\mathrm{e}^{2a} = 1 + \frac{2\phi}{c^2} = 1 - \frac{2GM}{c^2 r} \,, \tag{18.36}$$

where G is Newton's gravitational constant, and M is a constant corresponding to the gravitational mass in Newton's theory. We see that

$$\boxed{R_M = \frac{2GM}{c^2} \,.} \tag{18.37}$$

This is a characteristic length defined by the mass M, called the *gravitational radius*, or the *Schwarzschild radius*.

The Earth, for example, has a mass of 6.0×10^{24} kg and a gravitational radius of 8.9 mm. The Sun has a mass of 2.0×10^{30} kg and a gravitational radius of 3.0 km.

Thus, the Schwarzschild metric having the mass M as its source is

$$\boxed{\mathrm{d}s^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{2GM}{c^2 r}} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\ .} \tag{18.38}$$

It describes the gravitational field in vacuum outside an arbitrary spherically symmetric mass distribution with a total gravitational mass M. Inside the mass distribution the metric is modified and differs from the Schwarzschild metric, but on the outside the metric is uniquely given, apart from the freedom we always have to choose space and time coordinates. This result, which is proved by our present computation, that the metric is uniquely given when the source is spherically symmetric and the total mass is given, is known as *Birkhoff's theorem*. Note that the proof is based on the implicit assumption that the metric must be spherically symmetric when the mass distribution is. Birkhoff's theorem corresponds to the result demonstrated by Newton, that the gravitational field outside a spherically symmetric mass distribution is the same as if all the mass were collected at the centre.

The Schwarzschild metric is *static*, as seen from the coordinate system we have chosen, and this is true regardless of whether the source of the field is static, as long as all the motions maintain the spherical symmetry. This means, first of all, that the total gravitational mass M is concerved.

Furthermore, since a static field can not describe radiation, and since there exist no other solutions, it means that a spherically symmetric mass distribution is incapable of emitting gravitational radiation. Like electromagnetic radiation, gravitational radiation can not be monopole radiation. Note that if energy is radiated by other means than by gravitational radiation, e.g. electromagnetically, then there is no longer vacuum outside the mass distribution, and we have $G_{\mu\nu} = \kappa T_{\mu\nu} \neq 0$. Then the gravitational field is modified, and the metric is no longer the Schwarzschild metric.

18.4 Planetary orbits

The Schwarzschild metric ought to be a good approximation to the gravitational field of the Sun, so that we may use it in order to compute the orbits of the planets. The simplest approach is to derive the equations of motion directly from the variational principle, without computing explicitly the connection coefficients. Our task is then to extremalize the action integral $S = -mc\int \mathrm{d}u\, w$, where m is the mass of the planet, u is an arbitrary parameter along the orbit, and

$$\begin{aligned} w = \frac{\mathrm{d}s}{\mathrm{d}u} &= \sqrt{\left(1 - \frac{R_M}{r}\right) c^2\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{R_M}{r}} - r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)} \\ &= \sqrt{\left(1 - \frac{R_M}{r}\right) c^2\dot{t}^2 - |\dot{\boldsymbol{r}}|^2 - \frac{R_M(\boldsymbol{r}\cdot\dot{\boldsymbol{r}})^2}{r^2(r - R_M)}}\ . \end{aligned} \tag{18.39}$$

We write $\dot{t} = \mathrm{d}t/\mathrm{d}u$, $\dot{r} = \mathrm{d}r/\mathrm{d}u$, etc..

This problem obviously has enough symmetry to be exactly solvable, although it may be impossible to express the general solution in terms of elementary functions. First, w is not explicitly dependent on the "coordinate" t, and this fact implies a conservation law for the canonically conjugate momentum, which has a certain right to the name of "energy". Second, w is rotationally invariant, which means that the angular momentum,

$$\boldsymbol{L} = -mc\,\boldsymbol{r} \times \frac{\partial w}{\partial \dot{\boldsymbol{r}}} = \frac{mc}{w}\,\boldsymbol{r} \times \dot{\boldsymbol{r}}\ , \tag{18.40}$$

is conserved. Since $\boldsymbol{r} \cdot \boldsymbol{L} = 0$, it follows that the orbit lies in a plane through the origin, perpendicular to the constant vector $\boldsymbol{L}$. Therefore we may choose our coordinate system in such a way that the orbital plane has the equation $\theta = \pi/2$.

For the three variables t, θ and φ we get the explicit Euler–Lagrange equations

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial w}{\partial \dot{t}}\right) = \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{1}{w}\left(1 - \frac{R_M}{r}\right)c^2\dot{t}\right), \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial w}{\partial \dot{\theta}}\right) - \frac{\partial w}{\partial \theta} = -\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{r^2\dot{\theta}}{w}\right) + \frac{r^2 \sin\theta\cos\theta\,\dot{\varphi}^2}{w}\ , \\ 0 &= \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{\partial w}{\partial \dot{\varphi}}\right) = -\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{r^2\sin^2\theta\,\dot{\varphi}}{w}\right). \end{aligned} \tag{18.41}$$

The first equation has the solution

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\dot{t}}{w} = \frac{A}{c\left(1 - \frac{R_M}{r}\right)}\ , \tag{18.42}$$

where A is a dimensionless integration constant. The quantity

$$-mc\,\frac{\partial w}{\partial \dot{t}} = -mc^2 A \tag{18.43}$$

is the momentum canonically conjugate to t, so that mc^2A is to be interpreted as the energy.

The third equation has the solution

$$\frac{\mathrm{d}\varphi}{\mathrm{d}s} = \frac{\dot{\varphi}}{w} = \frac{BR_M}{r^2\sin^2\theta}\ , \tag{18.44}$$

where B is another dimensionless integration constant. B is proportional to the angular momentum about the z axis, L_z, which is the canonically conjugate momentum of the asimuthal angle φ,

$$L_z = -mc\,\frac{\partial w}{\partial \dot{\varphi}} = mcR_M B\ . \tag{18.45}$$

We see that we may solve the second equation by setting $\theta = \pi/2$. As already mentioned, every solution of the equation of motion in the Schwarzschild metric may be described in this way, by a suitable choice of coordinates.

Strangely enough, there is no need to write out the Euler–Lagrange equation for r, because we find a first integral of it simply by squaring w, and substituting the above expressions for $\dot{t}$ and $\dot{\varphi}$, as well as $\theta = \pi/2$. We get that

$$w^2 = \frac{w^2 A^2}{1 - \frac{R_M}{r}} - \frac{\dot{r}^2}{1 - \frac{R_M}{r}} - \frac{w^2 B^2 {R_M}^2}{r^2} \,. \tag{18.46}$$

Here we may introduce the dimensionless variable $\psi = R_M/r$, and write

$$\dot{r} = \dot{\varphi}\,\frac{\mathrm{d}r}{\mathrm{d}\varphi} = \frac{wBR_M}{r^2}\,\frac{\mathrm{d}r}{\mathrm{d}\varphi} = -wB\,\frac{\mathrm{d}\psi}{\mathrm{d}\varphi} \,. \tag{18.47}$$

The result is the following equation for ψ,

$$\boxed{\left(\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right)^2 = \frac{A^2 - 1 + \psi}{B^2} - \psi^2 + \psi^3 \,.} \tag{18.48}$$

By differentiating this equation we obtain a second order differential equation for ψ,

$$\boxed{\frac{\mathrm{d}^2\psi}{\mathrm{d}\varphi^2} = \frac{1}{2B^2} - \psi + \frac{3}{2}\,\psi^2 \,.} \tag{18.49}$$

Comparison with Newton's theory

Newton's equations of motion for a planet of mass m in the equatorial plane $\theta = \pi/2$ follow from the conservation of angular momentum

$$L_z = mr^2\,\frac{\mathrm{d}\varphi}{\mathrm{d}t} \tag{18.50}$$

and energy

$$E = \frac{m}{2}\left(\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\right) - \frac{GmM}{r} \,. \tag{18.51}$$

We may introduce the same dimensionless variable $\psi = R_M/r$, and write the energy equation in the dimensionless form

$$\begin{aligned} \frac{2E}{mc^2} &= \frac{1}{c^2}\left(\frac{\mathrm{d}\varphi}{\mathrm{d}t}\right)^2\left(\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + r^2\right) - \frac{2GM}{c^2 r} \\ &= \left(\frac{L_z}{mcr^2}\right)^2\left(\left(\frac{\mathrm{d}r}{\mathrm{d}\varphi}\right)^2 + r^2\right) - \frac{R_M}{r} \\ &= \left(\frac{L_z}{mcR_M}\right)^2\left(\left(\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right)^2 + \psi^2\right) - \psi \,. \end{aligned} \tag{18.52}$$

Or equivalently,

$$\left(\frac{\mathrm{d}\psi}{\mathrm{d}\varphi}\right)^2 = \frac{2mE{R_M}^2}{{L_z}^2} + \left(\frac{mcR_M}{L_z}\right)^2 \psi - \psi^2 \ . \tag{18.53}$$

This non-relativistic equation has the same form as equation (18.48), except for the absence of the term ψ^3. By comparing the two equations we see what the two integration constants A and B correspond to in the non-relativistic limit,

$$B = \frac{L_z}{mcR_M} \ , \qquad A = \sqrt{1 + \frac{2E}{mc^2}} \ . \tag{18.54}$$

The dimensionless variable $\psi = R_M/r$ is very small for all the planets in the solar system, in fact $\psi < 10^{-7}$, since $R_M = 3\,\mathrm{km}$ for the Sun, whereas $r \approx 5 \times 10^7\,\mathrm{km}$ for the innermost planet Mercury. Therefore the extra term ψ^3 in Equation (18.48) is always a small perturbation.

Approximate solution

By simply neglecting the last, quadratic, term in Equation (18.49) we find the non-relativistic solution,

$$\psi = \frac{1 + e\cos(\varphi - \varphi_0)}{2B^2} \ , \tag{18.55}$$

where e and φ_0 are integration constants. We may always choose $e \geq 0$. Equation (18.48) without the term ψ^3 gives the following relation between the integration constants A, B, and e,

$$e^2 = 4B^2(A^2 - 1) + 1 \ . \tag{18.56}$$

Equation (18.55) is the equation for a conic section with one focus at the origin, and with eccentricity e: a circle if $e = 0$, an ellipse if $0 < e < 1$, a parabola if $e = 1$, or a hyperbola if $e > 1$.

To see this a little more explicitly, we may define

$$x = -r\cos(\varphi - \varphi_0) \ , \qquad y = r\sin(\varphi - \varphi_0) \ , \tag{18.57}$$

such that the equation takes the form $2B^2R_M = r - ex$. The equation for an ellipse with eccentricity e, one focus at the origin, and the second focus at $(x, y) = (2ea, 0)$, where $a > 0$, is

$$r + \sqrt{(x - 2ea)^2 + y^2} = 2a \ . \tag{18.58}$$

Here a is the major half axis of the ellipse. The equation may be rewritten as follows,

$$(x - 2ea)^2 + y^2 = (2a - r)^2 \ , \tag{18.59}$$

or further as

$$(1 - e^2)a = r - ex \ . \tag{18.60}$$

Thus, the major half axis is, when we use Equation (18.56),

$$a = \frac{2B^2 R_M}{1-e^2} = \frac{R_M}{2(1-A^2)} = \frac{GM}{c^2(1-A^2)} \; . \tag{18.61}$$

That point of the orbit where r is minimal, and hence ψ maximal, is called *perihelion*. Equation (18.55) describes an orbit with its perihelion at $\varphi = \varphi_0$. If the orbit is an ellipse (or a circle), all values are allowed for the angle φ, and the orbit is a closed curve, since the perihelion angle after n periods is

$$\varphi_n = \varphi_0 + 2n\pi \; . \tag{18.62}$$

The most accurate way to linearize Equation (18.49), however, is not to neglect the term ψ^2 altogether, but rather to write $\psi = \psi_0 + \Delta\psi$, with $\psi_0 = R_M/r_0$ constant, and then neglect $(\Delta\psi)^2$. This gives a slightly different approximate equation linear in $\Delta\psi$,

$$\frac{\mathrm{d}^2}{\mathrm{d}\varphi^2}\Delta\psi = \frac{1}{2B^2} - \psi_0 + \frac{3}{2}\,{\psi_0}^2 - (1-3\psi_0)\,\Delta\psi \; . \tag{18.63}$$

We choose ψ_0 as a solution of the equation

$$\frac{1}{2B^2} - \psi_0 + \frac{3}{2}\,{\psi_0}^2 = 0 \; . \tag{18.64}$$

Or as a more practical approach, we choose first ψ_0 and let this equation determine the integration constant B. That gives the solution

$$\Delta\psi = e\psi_0 \cos\Bigl(\sqrt{1-3\psi_0}\,(\varphi-\varphi_0)\Bigr) \; . \tag{18.65}$$

According to this more accurate solution, the perihelion angle after n periods is

$$\varphi_n = \varphi_0 + \frac{2n\pi}{\sqrt{1-3\psi_0}} = \varphi_0 + 2n\pi + 3n\pi\psi_0 \; . \tag{18.66}$$

Thus, every time the planet completes a full circuit around the Sun, the perihelion point moves a small angle

$$\boxed{\Delta\varphi = 3\pi\psi_0 = \frac{3\pi R_M}{r_0} = \frac{6\pi GM}{c^2 r_0} \; .} \tag{18.67}$$

In the solar system, the advance in the perihelion point is largest and most easily observable for Mercury, because it is the planet closest to the Sun, and because its orbit is rather eccentric. The solar mass is $M = 1.989 \times 10^{30}\,\mathrm{kg}$. Mercury has an orbital period of 87.969 days, and a mean distance from the Sun of $r_m = 5.791 \times 10^{10}\,\mathrm{m}$. However, the quantity needed here is the mean *inverse* distance, $\psi_0 = R_M/r_0$. The eccentricity is $e = 0.205\,63$, which means that the distance from the Sun varies between $r_1 = r_0/(1+e) = 0.8294\,r_0$ and $r_2 = r_0/(1-e) = 1.2589\,r_0$. Thus, the mean distance is $r_m = 1.0442\,r_0$, and hence $r_0 = 5.546 \times 10^{10}\,\mathrm{m}$. During 100 Earth years, or 36 525 days, Mercury completes $36\,525/87.969 = 415.20$ turns around the

Sun, and its accumulated perihelion advance due to the general relativistic correction to its equation of motion is

$$415.20\,\Delta\varphi = 42.98'' \; . \tag{18.68}$$

The observed perihelion advance of Mercury during one hundred years amounts to

$$5599.74'' \pm 0.41'' \tag{18.69}$$

according to data from 1947. This figure has to be corrected for the precession of the rotation axis of the Earth, with a period of 26 000 years. The precession implies that the *tropical* year, which is the year followed by our calendar, is 20 minutes and 24 seconds shorter than the *siderial* year, which is measured relative to the fixed stars, and which is the relevant time reckoning for measuring the perihelion advance of Mercury. This contribution to the directly observed perihelion advance was in 1947 computed to be

$$5025.65'' \pm 0.50'' \; \left(\approx 2\pi \, \frac{100\,\text{years}}{26\,000\,\text{years}} \right) \tag{18.70}$$

per one hundred years. Another contribution, due to the gravitational influence of the other planets, mainly Jupiter, was computed to be $531.54'' \pm 0,68''$. The residual angle of $42.56'' \pm 0,94''$ is compatible with the prediction made by the general theory of relativity. One quantity which used to be uncertain, and which influences the perihelion advance, is the oblateness of the interior of the Sun due to rotation.

18.5 Generalizations of the Schwarzschild metric

The Reissner–Nordström metric

Not long after Schwarzschild found his solution it was generalized by Reissner and Nordström to the case when there is also a spherically symmetric electric field, but no electric charge in the region where we solve the equation. The metric is still almost as simple,

$$\boxed{\mathrm{d}s^2 = \left(1 - \frac{R_M}{r} + \frac{{R_Q}^2}{r^2}\right) c^2 \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{1 - \frac{R_M}{r} + \frac{{R_Q}^2}{r^2}} - r^2(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) \; .} \tag{18.71}$$

The electric field is radial, and the radial component is

$$\boxed{E_r = cF_{01} = -cF^{01} = \frac{Q}{4\pi\epsilon_0 \, r^2} \; .} \tag{18.72}$$

Using Gauss's law on a spherical surface of constant r, and remembering that the angular part of the metric is the same as in Euclidean space, we see that the constant Q is simply the

total electric charge inside the surface. The new constant R_Q in the metric is a characteristic length given by the charge Q,

$$\boxed{{R_Q}^2 = \frac{GQ^2}{4\pi\epsilon_0\, c^4}\ .} \tag{18.73}$$

Using (without any good justification) the usual relation between the metric $g_{\mu\nu}$ and Newton's gravitational potential ϕ, that $g_{00} \approx 1 + (2\phi/c^2)$, we have here that

$$\phi \approx -\frac{R_M\, c^2}{2r} + \frac{{R_Q}^2 c^2}{2r^2}\ . \tag{18.74}$$

Then the gravitational force on a mass m at a distance r is

$$-m\,\frac{\mathrm{d}\phi}{\mathrm{d}r} \approx -\frac{mR_M\, c^2}{2r^2} + \frac{m{R_Q}^2 c^2}{r^3}\ . \tag{18.75}$$

By this formula, the gravitational force is *repelling* for

$$r < \frac{2{R_Q}^2}{R_M} = \frac{Q^2}{4\pi\epsilon_0\, Mc^2}\ . \tag{18.76}$$

This surprising resultat may be understood as follows. The electric field has an energy density spread out all over space, and diverging in the limit $r \to 0$ so strongly that the total energy of the electric field, from $r = 0$ to $r = \infty$, becomes infinite. In order to make the total gravitational mass M finite, we must compensate the infinite field energy by another infinite and *negative* point mass at the origin, and it is this negative mass which is felt when we get close enough.

It may be interesting to compare the two characteristic lengths R_M and R_Q e.g. for an electron, with $Q = -|e|$. The gravitational length of the electron is

$$R_M = \frac{2Gm_e}{c^2} = 1.4 \times 10^{-57}\,\mathrm{m}\ , \tag{18.77}$$

whereas the "charge length" is

$$R_Q = \sqrt{\frac{R_M\, r_e}{2}} = 1.4 \times 10^{-36}\,\mathrm{m} = 10^{21}\, R_M\ . \tag{18.78}$$

Here $r_e = Q^2/(4\pi\epsilon_0\, m_e c^2) = 2.8 \times 10^{-15}\,\mathrm{m}$ is the classical electron radius.

The Kerr–Newman metric

If the mass which is the source of the gravitational field rotates, then the field no longer has the full rotational symmetry, but has only an axial symmetry. The generalization of the Schwarzschild solution to this case is rather more complicated, it was found in 1963, by Kerr. Newman found the corresponding solution with an electromagnetic field.

18.6 Black and white holes

In the extreme case where all the gravitational mass M is concentrated inside a radius smaller than the Schwarzschild radius R_M, the Schwarzschild metric describes a so called *black hole*. The idea that such objects might exist is actually much older than Einstein and Schwarzschild. An Englishman, John Michell, and a Frenchman, Laplace, speculated already two hundred years ago that there might exist planets or stars with such strong gravitational fields that the escape velocity exceeds the speed of light c. In Newton's gravitational theory the escape velocity v_0 at a distance r from the mass M is given by the relation

$$\frac{{v_0}^2}{2} - \frac{GM}{r} = 0 \; . \tag{18.79}$$

This is the condition that the kinetic plus potential energy of a particle is exactly zero, so that the particle has just enough kinetic energy to escape to $r = \infty$. By accident, the Schwarzschild radius R_M is precisely the radius where the non-relativistic escape velocity v_0 equals the speed of light.

The Schwarzschild metric is singular at the Schwarzschild radius, and t and r exchange their roles inside $r = R_M$, in such a way that r becomes a time coordinate and t a space coordinate! This so called "Schwarzschild singularity" at $r = R_M$ caused much confusion, until it became clear that it is not really a singularity of the spacetime geometry, it is only a singularity of the metric appearing because we have chosen singular coordinates. The metric singularity at $r = R_M$ is no more serious than the singularities at the North Pole $\theta = 0$ and the South Pole $\theta = \pi$ in polar coordinates. Spacetime itself with the Schwarzschild metric is no more singular at $r = R_M$ than the surface of a sphere is singular at $\theta = 0$ or $\theta = \pi$. One indication that the geometry is nonsingular is perhaps the fact that the determinant of the metric is nonsingular and nonzero in the limit $r \to R_M$.

The significance of the gravitational radius $r = R_M$ is that it is a *horizon*. It is impossible to observe from the outside anything beyond the horizon. Everything crossing this radius from the outside, sails out of sight of any observer staying outside $r = R_M$. In particular, a radial light signal starting at $r = R_M$ in the outward direction will stay forever at the same distance $r = R_M$ from the origin. And a light signal starting inside $r = R_M$, in any direction whatsoever, will fall towards $r = 0$.

Nonsingular coordinates

The time coordinate t may be identified as the source of the singularity in the Schwarzschild metric in its original form. The simplest way to transform away the singularity is to introduce a new time coordinate

$$v = ct + r + R_M \ln \left| \frac{r - R_M}{R_M} \right| \; . \tag{18.80}$$

This gives that

$$c \, \mathrm{d}t = \mathrm{d}v - \frac{r \, \mathrm{d}r}{r - R_M} \; , \tag{18.81}$$

and hence,

$$\mathrm{d}s^2 = \left(1 - \frac{R_M}{r} \right) \mathrm{d}v^2 - 2 \, \mathrm{d}v \, \mathrm{d}r - r^2 (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) \; . \tag{18.82}$$

This is known as the Eddington–Finkelstein form of the Schwarzschild metric, and it is clearly as simple as it can be made. The price we pay for obtaining a non-singular form of the metric is to introduce an off-diagonal term $\mathrm{d}v\,\mathrm{d}r$, which means that the time reversal symmetry of the metric is no longer explicit.

The physical meaning of the new time coordinate is that $v =$ constant is the equation of an incoming radial light signal. If we introduce

$$v' = ct - r - R_M \ln\left|\frac{r - R_M}{R_M}\right| \tag{18.83}$$

instead of v, then $v' =$ constant is the equation of an outgoing radial light signal, and the time reversal transformation $t \mapsto -t$ corresponds to $v \mapsto -v'$.

Another set of coordinates that are nonsingular at $r = R_M$ are the Kruskal–Szekeres coordinates u and v, replacing r and t and defined by the relations

$$\begin{aligned} u^2 - v^2 \;&=\; f(r) = \left(\frac{r}{R_M} - 1\right)\mathrm{e}^{r/R_M}\;, \\ \frac{v}{u} \;&=\; g(t) \;=\; \begin{cases} \tanh(ct/(2R_M)) & \text{for } r > R_M\;, \\ \coth(ct/(2R_M)) & \text{for } r < R_M\;. \end{cases} \end{aligned} \tag{18.84}$$

This definition is consistent, since it gives that

$$u^2 = \frac{f}{1 - g^2} \geq 0\;, \qquad v^2 = \frac{fg^2}{1 - g^2} \geq 0\;. \tag{18.85}$$

Writing $f' = \mathrm{d}f/\mathrm{d}r$ and $\dot{g} = \mathrm{d}g/\mathrm{d}t$, we have that

$$2u\,\mathrm{d}u - 2v\,\mathrm{d}v = \mathrm{d}f = f'\,\mathrm{d}r\;, \qquad \frac{\mathrm{d}v}{u} - \frac{v\,\mathrm{d}u}{u^2} = \mathrm{d}g = \dot{g}\,\mathrm{d}t\;. \tag{18.86}$$

Substituting for $\mathrm{d}r$ and $\mathrm{d}t$ in the line element $\mathrm{d}s^2$, we find after some manipulations that

$$\mathrm{d}s^2 = \frac{4{R_M}^3}{r}\,\mathrm{e}^{-r/R_M}\,(\mathrm{d}v^2 - \mathrm{d}u^2) - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\;, \tag{18.87}$$

both for $r > R_M$ and for $r < R_M$. Again the apparent singularity at $r = R_M$ is eliminated.

On the other hand, there is no doubt that the metric is singular at the origin, $r = 0$. This singularity can not be transformed away. That such a singularity inside a closed horizon is inevitable, under certain "reasonable" assumptions on the energy momentum tensor, was proved by Roger Penrose in 1965.

The Kruskal–Szekeres coordinates give an illuminating picture of the geometry of spacetime with the Schwarzschild metric, when we imagine that the source is a point mass lying at rest at the origin. See Figure 18.1. Perhaps the most interesting phenomenon is that the whole spacetime is doubled. In fact, to one point (r, t) in space and time there corresponds two values (u, v) and $(-u, -v)$ of the new coordinates. One way out of this dilemma might be to say that (u, v) and $(-u, -v)$ describe the same point in spacetime, but if we do so, we create confusion about the concept of causality, as we shall see. The most direct interpretation is to say that (u, v) and $(-u, -v)$ represent two different points in space and time, even though it means that we make spacetime twice as big.

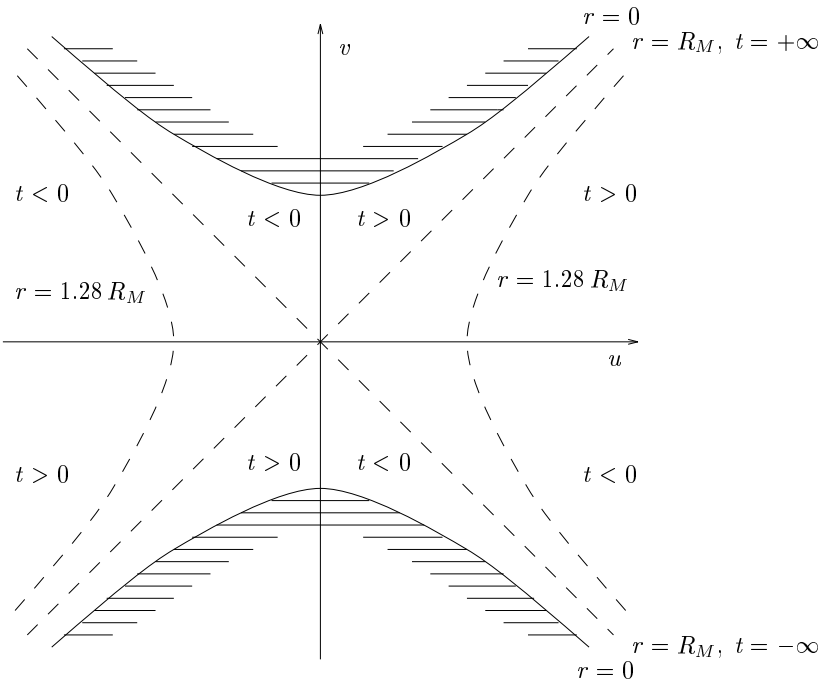

Figure 18.1: The Kruskal–Szekeres coordinates u and v.

We may e.g. impose the arbitrary convention that "our own" outside of the black hole is the region $u > |v|$. The region $u < -|v|$ is however another outside of the same black hole, simply another universe!

The Schwarzschild radius $r = R_M$ corresponds to the two straight lines $v = \pm u$ in the (u, v) plane. It is interesting to observe that the two conditions $r = R_M$ and t variable but finite, which apparently defines a line in an (r, t)-diagram, i.e. a three dimensional surface in the four dimensional spacetime, actually define one single point $u = v = 0$ in the (u, v) plane. The surface in spacetime is actually two dimensional and not three dimensional. So what about the two lines $u = |v|$ in the (u, v) plane, limiting "our" universe and also having $r = R_M$? Well, one line, $u = -v > 0$, has $r = R_M$ and $t = -\infty$, whereas the other line, $u = v > 0$, has $r = R_M$ and $t = +\infty$. Both these lines represent three dimensional surfaces in the four dimensional spacetime. This picture of what the horizon $r = R_M$ "really looks like" is rather unexpected.

The inside of the black hole, that is, the part of spacetime lying inside the horizon, has $u^2 - v^2 < 0$. This inequality again defines two separate regions in the (u, v) plane, one where $v > |u|$ and one where $v < -|u|$. In particular, the singularity at the origin $r = 0$ is split into two separate sigularities, with $v = \sqrt{1 + u^2}$ and with $v = -\sqrt{1 + u^2}$.

The characteristic property of a light signal is that $\mathrm{d}s^2 = 0$. In particular, for a radial light signal we have that $\mathrm{d}\theta = \mathrm{d}\varphi = 0$, and hence $\mathrm{d}v = \pm \mathrm{d}u$, which means that a radial light signal moves along a straight line in the (u, v) plane at an angle of $\pm 45°$.

In "our" universe with $r > R_M$, that is, in the region $u > |v|$, the positive time direction, with $\mathrm{d}t > 0$, is the same as the positive v direction, with $\mathrm{d}v > 0$. But when this is true in the region $u > |v|$, it must be true everywhere in the (u, v) plane, since otherwise it might happen that two observers meeting somewhere and sometime disagree as to what is past and future.

It seems a reasonable requirement that the concepts of past and future should have a unique meaning. But it is violated if we try to identify the opposite points (u, v) and $(-u, -v)$, since that means identifying $\mathrm{d}v$ at the point (u, v) with $-\mathrm{d}v$ at the point $(-u, -v)$.

That positive v direction represents future, means in particular that the singularity at $v = \sqrt{1+u^2}$ is a *future singularity*, a *black* hole, whereas the singularity at $v = -\sqrt{1+u^2}$ is a *past singularity*, a *white* hole. Nothing can emerge from the future singularity, and nothing can fall into the past singularity. Nothing falling in through the Schwarzschild radius, which means crossing the line $v = |u|$, can avoid falling into the future singularity. And similarly, nothing can come out through the Schwarzschild radius, meaning that it passes the line $v = -|u|$, if it has not first come out of the past singularity.

Gravitational collapse

The Sun and other stars may burn for millions or billions of years thanks to the energy liberated when protons and other light atomic nuclei fusion into heavier nuclei. Matter inside an ordinary star is almost an ideal gas, even though the high pressures may compress it into densities many times the density of a heavy metal like gold under laboratory conditions on Earth. The luminosity is very nearly constant during most of the lifetime of a star, because there exists a stable equilibrium between the gravitational force compressing the gas, and the gas pressure opposing the compression. In the most massive stars also the radiation pressure due to the gas of photons plays a role.

The equilibrium is stable because a small local compression of the gas would increase both the temperature and the pressure, as described by the equation of state. In addition, inside the core where energy production by nuclear processes takes place, compression would increase the rate of energy production, thereby heating the gas and raising the pressure even more. Yet another effect would be an increase in radiation pressure, because the matter would become more opaque. A small expansion would have the opposite effects, thus the effects of either compression or expansion tend to restore the same equilibrium conditions as before.

When the star has spent its nuclear fuel, it starts cooling because the energy keeps radiating away from its surface, and as a result, the gravitational forces compress it. The compression leads to heating of the gas, and if the star is sufficiently massive, it may reach the threshold temperature of another nuclear fusion process. However, in the end all possible fusion processes will be completed, and the only effect which may stop the compression, in the end, is the circumstance that the particles bulding up the matter inside the star, are fermions. N electrons, for example, can not be compressed into a phase space volume smaller than $Nh^3/2$, where h is Planck's constant, and the factor 2 appears because every electron may have 2 spin states. Thus, the N electrons will fill a certain minimal volume in three dimensional space, inversely proportional to the volume available in momentum space. When the electron gas is compressed as much, it is said to be *degenerate*, and the star can not contract any more. It has then become a *white dwarf star*.

When the Sun reaches the white dwarf stage, it will have a radius of about one per cent of its present radius, or roughly the radius of the Earth. Several hundred stars have been identified as white dwarfs, and there exist several thousand "candidates". The closest white dwarf is Sirius B, orbiting Sirius A, with an orbital period of 50 years. Sirius A and B are among the very closest stars, at a distance of only 9 light years, and Sirius A is the brightest of all the stars in the sky, with the exception of the Sun.

When a star contracts, after having spent its nuclear fuel, the gas is heated more the

larger mass the star has, and as a result the available volume in momentum space increases. This means that the density of a white dwarf is larger the larger its mass, and actually so much larger that its radius is *smaller* the larger its mass. There exists a mass, about 1.4 times the solar mass, at which the radius of a white dwarf would be zero, or more precisely, the central density of the star would become infinite. This mass is called the *Chandrasekhar mass*, and it is an upper limit for the mass of a white dwarf star. If a star has a larger mass, the pressure from the degenerate electron gas is insufficient to stop the contraction.

When a star of mass exceding the Chandrasekhar mass contracts, the electrons will get a high enough energy that they start reacting with the protons to produce neutrons and neutrinoes. The neutrinoes simply escape from the star, bringing with them a not negligible amount of energy, whereas the neutrons stay. If the mass is not too much larger than the Chandrasekhar mass, the result may be a *neutron star*, consisting mostly of degenerate neutrons. A neutron star of the same mass as the Sun would have a radius of about 10 km, which is only about three times the gravitational radius, and a density around 10^{15} times the density of water, about the same deneity as in an atomic nucleus. A neutron star is essentially a giant atomic nucleus.

Neutron stars are observed as *pulsars*, and more than a thousand are now known. They are very small, by stellar measures, and hence radiate little light from their surfaces, even though they are hot. However, they radiate in all parts of the electromagnetic spectrum, from radio waves to X rays and even γ rays, mostly in brief pulses, many pulses per second. The pulse period of one pulsar is very nearly constant, except for "glitches" from time to time, at which the period suddenly increases in small jumps.

The pulsing is believed to result from the rotation of the neutron star, with a rotation period of a small fraction of a second. The enormously strong magnetic field is not parallel to the rotation axis, and the rotating magnetic field will accelerate charged particles, especially electron, which emit electromagnetic radiation.

There exists an upper limit to the mass of a neutron star, corresponding to the Chandrasekhar mass. This upper limit is not very accurately known, because the exact equation of state is not very well known at the high densities found inside a neutron star, but it is certainly no more than a few solar masses, and probably no more than about two solar masses. A more massive star has only two alternatives open, as far as is known: it must either shed enough mass maybe in a supernova explosion, that it gets below the critical limit, or it must collapse into a black hole.

A black hole formed by gravitational collapse of a mass distribution without any singularities does not correpond directly to a black hole as described by the Schwarzschild metric. Apart from the fact that in practise it could never be exactly spherically symmetric, the most important difference is that it has only a future singularity, and no past singularity. Fortunately, this means that the extra universe contained in the Schwarzschild metric is not present.

The only observable objects left as remainders of a star falling inside its own Schwarzschild horizon, will be the gravitational field, and possibly an electromagnetic field. The gravitational field will symmetrize itself by gravitational radiation, most of which will disappear into the hole, in such a way that the field left will be described to a good approximation by the Kerr–Newman solution of the combined Einstein's and Maxwell's equations. Such a black hole is completely described by only three parameters, mass, spin and electric charge. This result was formulated by Wheeler in the following way,

"*Black holes have no hair*".

An important clue when looking for black holes, and attempting to distinguish them from ordinary stars, neutron stars and other sources of electromagnetic radiation, is that they possess a mechanism for converting mass into radiation which is much more efficient than other known mechanisms. The fusion of atomic nuclei inside stars is capable of converting less than one per mille of the mass into radiation energy, an important reason for the limitation being that baryon number is conserved: protons and neutrons may be converted into each other, but never destroyed, at least not in ordinary nuclear reactions. On the other hand, it is estimated that when matter falls into a black hole, as much as 20% of the rest energy of the mass may be reradiated. In a sense, baryon number is not conserved in such a process, because the baryons disappear into the black hole, and the radiated energy comes from the gravitational potential energy converted into kinetic energy when the matter is falling in.

There are two cases where astronomers believe they see the signatures of black holes. One kind of black holes are those that are formed when the most massive stars burn out and collapse in supernova explosions. A small number of likely black hole candidates of this kind have been identified. They are members of double star systems, and are observed, by Kepler's third law relating the sum of the masses of the two stars to the orbital period and the distance between them, to have masses several times the solar mass. Furthermore, they are all sources of X-ray radiation, implying that they have high surface temperatures, but at ther same time they are not directly visible, implying that they must be very compact. Presumably the X-ray radiation is produced when mass transferred from the other star accumulates into a flat accretion disc and eventually falls into the black hole.

Much more massive black holes seem to exist at the centres of many galaxies, perhaps even in all galaxies, including our own Milky Way galaxy. Especially quasars ("quasi stellar", i.e. starlike, objects) seem to produce enormous amounts of radiation energy from comparatively very small volumes, which it is hard to explain by any other known mechanism than that of matter falling into a black hole. There are indications that the quasars are the galactic nuclei of young galaxies.

18.7 Hawking radiation, or grey, red and blue holes

A spherically symmetric black hole has a surface area of

$$\boxed{A = 4\pi {R_M}^2 = \frac{16\pi G^2 M^2}{c^4}} \ , \tag{18.88}$$

since the angular part of the Schwarzschild metric is the same as in flat space, in polar coordinates, $r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)$. More generally, the *horizon* around a (stationary) black hole has always a well defined area, even if it is not the surface of a sphere.

Stephen Hawking proved in 1971 a result which resembles the second law of thermodynamics, and which is therefore called *the second law of black hole dynamics:*

The total area of a black hole can never decrease.

For example, if two black holes of masses M_1 and M_2 collide and merge into one, the following inequality must hold for the mass M of the resulting single black hole,

$$M^2 \geq {M_1}^2 + {M_2}^2 \,. \tag{18.89}$$

This sets an upper limit to the amount of energy that may be radiated during the collision, in the form of gravitational radiation or other types of radiation.

This law, like the second law of thermodynamics, breaks time reversal invariance, and one may ask how that is possible, since it is proved from laws that are invariant under time reversal. The point is that it is valid for *black* holes, which are future singularities, formed from non-singular mass distributions by gravitational collapse. For *white* holes, formed in the big bang when the Universe was born, and disappearing after a finite time, there must be a corresponding law saying that the sum of areas of white hole horizons can never increase. White holes are problematic for theoretical reasons, and the simplest assumption is that they do not exist. They have not yet been observed, and it is also somewhat unclear what properties they should have if they do exist.

It would seem tempting to postulate that the second law of black hole dynamics is nothing but another example of the second law of thermodynamics. This would be the case if a black hole has an entropy proportional to its area. Jacob Bekenstein proposed to take this idea seriously, but met opposition from most other experts, not least from Hawking. One problem is that a black hole having entropy must also have a finite temperature, and hence emit temperature radiation. Thus, it is not black, it has one or another of the colours of the rainbow, depending on its temperature.

However, Hawking soon turned around and accepted the idea of black hole entropy and finite temperature, after he followed up an idea of Yakov Zel'dovich and Alexander Starobinsky, calculating a *quantum mechanical* process in which spontaneous particle radiation is emitted from a black hole. He convinced himself that the effect is real, and the spectral distribution which he calculated was precisely the one coming from black body radiation of the temperature a black hole must have if its entropy is proportional to its area. Thus, two *classical* theories in combination, the general theory of relativity and termodynamics, led to the idea of temperature radiation from a black hole, which is a *quantum mechanical* effect.

The Hawking radiation from a black hole correspond to its entropy being

$$\boxed{\mathcal{S} = \frac{k_B c^3}{4\hbar G}\, A = \frac{4\pi k_B G M^2}{\hbar c} = \frac{4\pi k_B M^2}{{m_P}^2} \,.} \tag{18.90}$$

Here Newton's gravitational constant G, Planck's constant $h = 2\pi\hbar$, Boltzmann's constant k_B and the speed of light c are combined in one formula. The mass $m_P = 2.18 \times 10^{-8}\,\mathrm{kg}$ is the Planck mass. The inverse temperature $1/T$ is the derivative of the entropy with respect to the energy $E = Mc^2$,

$$\boxed{\frac{1}{T} = \frac{\mathrm{d}\mathcal{S}}{\mathrm{d}(Mc^2)} = \frac{8\pi k_B G M}{\hbar c^3} = \frac{8\pi k_B M}{{m_P}^2 c^2} \,.} \tag{18.91}$$

The temperature of a black hole of solar mass is around $10^{-7}\,\mathrm{K}$, which is many orders of magnitude less than the temperature $2.73\,\mathrm{K}$ of the background radiation left over from the

big bang. Thus, such a massive black hole is not going to emit Hawking radiation, on the contrary, it will absorb more radiation than it will emit.

The power radiated is proportional to T^4, that is, to $1/M^2$, according to the Stefan–Boltzmann law. Hence, all the energy Mc^2 will be radiated away during a finite time. Clearly the time is more than astronomical for a black hole of solar mass, but it is less than a second for a mass of 10^5 kg. When 10^5 kg is converted into radiation energy in less than a second, the explosion corresponds to quite a few hydrogen bombs of the megaton class. Such an explosion should be observable if it happens within our solar system, or not too far outside, even though the most intense radiation would come in a very brief burst.

Problems

1. Compute the deviation of light in the Schwarzschild metric.
   (See Problem 2 in Chapter 16.)

2. Generalize the Schwarzschild metric to the case when the cosmological constant $\Lambda \neq 0$.
   That is, find the general solution of the equation $G_{\mu\nu} - \Lambda g_{\mu\nu} = 0$ when the metric is completely rotationally symmetric.
   Is this gravitational field attractive or repulsive?
   (Study this e.g. by studying radial motion of a test particle.)

3. Derive the Reissner–Nordström metric.
   That is, solve the gravitational equation together with Maxwell's equations when the source of the gravitational field is a rotationally symmetric electric field.
   Assume that both the electric field and the metric are static, and that the electromagnetic field tensor is given as a 2-form

   $$\mathbf{F} = \frac{1}{2} F_{\mu\nu} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = E_r(r) \, \mathrm{d}t \wedge \mathrm{d}r \ ,$$

   as expressed in polar coordinates.
   Assume that the electric charge density and current density vanish outside the origin.

4. The critical energy density in the universe is $\rho_c \approx 2 \times 10^{-26}$ kg/m^3. This is the average energy density which is just large enough to stop the expansion of the universe.
   Compute the numerical value of an electric field having precisely this energy density.
   Compute also the numerical value of the magnetic flux density if the purely magnetic field has precisely this energy density.

Chapter 19

Cosmology

Cosmology is the science of *everything*, including cosmologists and cosmology, in brief, the past, present and future of the Universe. Obviously, even the tiny knowledge we have is too much for one brief chapter, so we have to limit our topic drastically. We will discuss a very much simplified model of the Universe, which is called the *standard model*, because it is consistent with observations, more so than any other model proposed so far. According to this model, the Universe has a finite age, about 14 billion years by the newest estimates, and it started in an explosion, known as "the Big Bang" since Fred Hoyle introduced this name in 1950. Hoyle favoured his own "steady state" model, in which the expansion of the Universe is compensated for by the spontaneous creation of matter.

Remarkably, it is possible to use the known laws of nature to compute what happened in the Big Bang, during the first few seconds, days, and thousands of years of the history of the Universe, and afterwards confront the results of the computations with observations to verify that the theoretical model has at least some resemblance to the real world. The computations are much simplified because matter and radiation in the early Universe was densely packed, the temperature was high, and most of the time all the matter and radiation was in a state of thermodynamic equilibrium. Even though the Universe expanded very fast, the processes needed for maintaining the equilibrium were even faster, as a rule, and therefore it is a valid approximation to assume that the Universe expanded *quasistatically*, from one equilibrium state to the next. In addition, most of the time the expansion was approximately "friction free", so that entropy was conserved.

Important deviations from equilibrium happened, for example during the first few minutes, when protons and neutrons fused into helium nuclei and a very small number of other light atomic nuclei. We have good reasons to celebrate the lucky circumstance that the Universe at times expanded fast enough to get out of thermodynamic equilibrium. Otherwise it would have been a very uniform and boring place, where of course nobody would have been around to get bored.

This chapter is organized as follows. We summarize very briefly the main observational evidence, before introducing the metric and the energy momentum tensor. Then we write down the gravitational equations and discuss their solution. We will also have a brief look at the synthesis of primordial elements heavier than hydrogen, mainly helium.

19.1 Observational basis

Olbers' paradox

One of the most important observations in cosmology is that it is dark at night. The paradox hidden in this trivial fact was recognized by Olbers some 200 years ago, and perhaps by other people, including Johannes Kepler, before him.

The paradox is that the directly observed brightness of a star at a distance r away from us is proportional to $1/r^2$, whereas the number of stars within the distance r is proportional to r^3, under the assumption that the density of stars is constant. Hence, if the Universe is homogeneous and infintely large, and in addition the stars are infinitely old, we would expect that the amount of starlight received on the Earth should be infinite, and the night sky should not be dark.

Many effects may contribute to a resolution of the paradox, such as the eclipsing by one star of all the stars accidentally lined up behind it, the grouping of stars into galaxies, and the cosmological redshift of starlight due to the expansion of the Universe. But the more fundamental problem raised by the paradox is simply that the idea of a an infinitely old Universe, whether finite or infinite in extension, is not easily reconciled with the first and second laws of thermodynamics. The first law states that energy is conserved, the second law that all physical processes are more or less irreversible, so that entropy is increasing all the time. An infinitely old Universe ought to be in thermodynamic equilibrium, with the same temperature everywhere. The natural conclusion to be drawn must be that the Universe we observe, which is clearly not in thermodynamic equilibrium, has a finite age.

The Universe is expanding

The one observation pointing most directly towards the cosmological standard model, and also the most important observation historically, is the fact that the spectral lines of the light from distant galaxies are displaced towards the red end of the visible spectrum. This *cosmological redshift* is seen as a direct proof that the Universe is expanding.

It is customary to measure redshift by a quantity called z. By definition, $1+z$ is the factor by which the wave length of the light is increased, or equivalently, by which the frequency is reduced. The same factor applies to every spectral line in the light from one source. If the spacetime of our Universe could be described as a flat Minkowski spacetime, then the redshift would have a straightforward interpretation as a Doppler shift, given by the formula

$$1 + z = \sqrt{\frac{c+v}{c-v}} \,, \tag{19.1}$$

where v is the velocity of the light source relative to the observer, and c is the speed of light.

The significance of the cosmological redshift became clear when it turned out that the galaxies move away from us with velocities, as calculated from the Doppler formula, that are roughly proportional to their distances. Edwin Hubble observed the linear relation between velocity v and distance r,

$$v = Hr \,, \tag{19.2}$$

at the end of the 1920s. The proportionality constant H is called the Hubble constant, or better, the Hubble parameter, since it is not constant, but varies with time.

The simplest and most direct way to explain the Hubble relation is that the whole Universe is expanding. Imagine the Universe as a dough of raisin bread with the galaxies as the inbaked raisins. Then as the dough is expanding the raisins will move apart, in such a way that at any given moment the relative velocity v of any two raisins will be proportional to the distance r between them. The expansion will satisfy a "Hubble-relation" $v = Hr$, with a time dependent proportionality factor $H = H(t)$ telling how fast the dough is expanding at a given time t.

As soon as we accept that the Universe is actually expanding, the logical next step is to extrapolate backwards in time, and to conclude that the expansion must have started with an explosion of a "cosmic fireball" in which the density of matter and radiation, and also the temperature, were extremely high.

It is possible to imagine that the expansion we observe follows a previous contraction phase, during which the galaxies got close together, but not necessarily close enough to collide. However, this hypothesis is inconsistent with the general theory of relativity, in particular the rather general *singularity theorems* proved by Penrose and Hawking in the 1960s. Assuming certain "natural" positivity properties of the energi momentum tensor, they proved that, according to general relativity, an extrapolation backwards in time must end in a singularity. Even though the theory certainly breaks down before the density becomes literally infinite, it seems hard to avoid the conclusion that the expansion must have started from a state of extremely high density.

The physical dimension of the Hubble parameter H is inverse time, and the *Hubble time* $t_H = 1/H$ is simply the age of the Universe in the simple case when H does not vary with time. Since the expansion presumably must be braked by the gravitational attraction, we would expect the age to be somewhat less than t_H. We will see that the age of the Universe is $(2/3)t_H$ in the cosmological standard model, if there is no cosmological constant, and if the energy density of matter and radiation equals the critical density which is just enough to make the expansion stop. This assumption does not hold exactly, but it is also not terribly wrong.

Traditionally, H is measured in the strange unit of km/s per megaparsec (Mpc). One parsec (pc) is defined as the distance from which the distande between the Earth and the Sun, the *astronomical unit* $A = 149.60$ million km, extends an angle of one second of arc, which is $\pi/(180 \times 3600)$ of a radian. Thus we have that

$$1\,\text{parsec} = \frac{648\,000}{\pi}\,149.60 \times 10^6\,\text{km} = 3.0857 \times 10^{16}\,\text{m} = 3.2616\,\text{light years}\,. \tag{19.3}$$

Until this time there has been much controversy about the precise value of the Hubble parameter, and it has been customary to give its value as

$$H_0 = 100h\,\frac{\text{km}}{\text{s Mpc}}\,, \tag{19.4}$$

where the index 0 means that this is the value at the present time, and where h is an "ignorance factor" which is now settling down to an accepted value of 0.7.

Even if disagreeing about its exact value, astronomers have always agreed about the intricate unit in which to measure the Hubble parameter. The unit, when straightened out, is simply inverse time, and the Hubble time measured in everyday units is

$$t_H = \frac{1}{H} = \frac{9.8 \times 10^9\,\text{years}}{h}\,. \tag{19.5}$$

Thus, the accepted value $h = 0.7$ gives $t_H = 14$ billion years.

The naive interpretation of cosmological redshifts as a Doppler effect, and nothing more, would have been valid if the flat Minkowski spacetime had been a valid model of our Universe on cosmological scales. However, according to our present standard cosmology, spacetime is not flat, but is better described by a simple geometrical model proposed by Friedmann in 1923, in which the three dimensional space itself is expanding. This model is such that spacetime would have had a nonzero curvature even if there had been a perfect vacuum. In an expanding Friedmann model the main contribution to the cosmological redshift can be understood very simply as the stretching of the light waves due to the expansion of three dimensional space.

The cosmic background radiation

The breakthrough for the theory of the "Big Bang" happened as late as 1965, with an unexpected discovery by Arno Penzias and Robert Wilson. When perfecting a receiver for satellite communication they observed an electromagnetic *cosmic background radiation* in the microwave range, that is, with wave lengths in the millimeter and centimeter range. The spectrum of this radiation has now been observed with extreme precision from satellites and balloons, and it is seen to be a perfect Planck type black body spectrum corresponding to a temperature of 2.735 K.

This radiation is interpreted as a direct observation of the cosmic fireball, at a time when the Universe was a few hundred thousand years old. At that time, the temperature had dropped to a few thousand kelvin, sufficiently to allow what was until then a plasma of free electrons, protons and ionized helium atoms, to be converted into a gas of neutral atoms. This so called recobination meant that the matter in the Universe rather suddenly became transparent, in other words, the fog that had filled space until then, suddenly cleared.

Of all the electromagnetic radiation present when this phase transition took place, most photons are still present, and have not been absorbed or scattered since. The only change that has taken place is that all wave lengths have been stretched by a factor of around one thousand, as the Universe has expanded by the same factor. As a result of the redshift, the temperature of the black body radiation has been reduced, again by the same factor, from a few thousand kelvin to 2.735 K. Of all the radiation from space being observed in the millimeter and centimeter range, only a small fraction is younger.

The presence of the background radiation was actually predicted theoretically by George Gamow and his collaborators in 1948, at roughly the correct temperature, about two or three times higher than what was later observed. The observation of a perfect Planck spectrum is a very precise *quantitative* test of the standard cosmological model. No alternative explanation has been proposed that is nearly as simple and natural.

Helium and other light elements

The third type of observations, also giving quantitative tests, is the abundance of the very lightest atomic nuclei in the Universe as a whole. The interesting isotopes are hydrogen (H), deuterium (D $= {}^2$H), helium 3 and 4 (^{3}He and ^{4}He), and lithium 6 and 7 (^{6}Li and ^{7}Li). Of the total mass observed, in the form of stars and clouds of gass and dust, three quarters is hydrogen, or more precisely 73 %, as measured in kg. One quarter, or 24.5 %, is helium 4, whereas no more than 2–3 % are heavier elements, mostly oxygen.

The main energy source of the shining stars are the nuclear reactions in their interiors, the most important reaction being the fusion of hydrogen nuclei into helium nuclei. But even though some of the helium produced inside stars is afterwards scattered in space when massive stars explode as supernovae, the helium from stars can make up at most a few per cent of the amount of helium which is observed.

The total amount of helium that has been produced in stars may be estimated from observations by a method which is simple in principle, though not quite as simple in practice. Since essentially all elements heavier than helium have been produced in stars, at least according to the Big Bang model itself, one may study the correlation between the abundances of helium and for example oxygen, in the spectra of stars and gas clouds. The abundance of oxygen shows local variations, as is to be expected if all the oxygen has been made in stars. One finds a clear correlation between the abundancies of oxygen and helium, and by extrapolating to zero oxygen content one may estimate the primordial helium abundance, that is, how much helium was produced in the Big Bang itself. This kind of analysis shows clearly that most of the helium which is observed, must have existed before any stars existed.

19.2 The metric

The cosmic background radiation may serve as a clock showing the age of the Universe, since it diminishes in intensity as the Universe expands. Since it fills the Universe, all observers have, at least in principle, local clocks that are syncronized in an absolute way, no matter how far apart they may be,

The observed radiation is *isotropic*, that is, equally intense in all directions, except that there are direction dependent fluctuations of order 10^{-5}. But the isotropy can hold only for an observer who moves with a uniquely defined velocity, one may say that an observer seeing an isotropic background radiation is at rest relative to the radiation. In our part of the Universe, such an observer moves at 600 km/s in one particular direction relative to us. Any other observer, including ourselves, will see the radiation coming from one direction blueshifted towards shorter wave lengths, and the radiation from the opposite direction redshifted towards longer wave lengths.

Assuming, as the simplest hypothesis, that the isotropy holds always and everywhere, it means that the cosmic background radiation defines not only a universal time, but also a unique rest system at every point in spacetime. In a certain sense it gives meaning to concepts like absolute motion and absolute simultaneity. Thus, it defines a natural coordinate system in spacetime, with a universal time coordinate $x^0 = ct$, and with three spatial coordinates $(x^1, x^2, x^3) = (x, y, z)$, which we may choose in such a way that they follow the expansion of the Universe. By the definition of the coordinate system, an observer seeing all the time an isotropic background radiation follows an orbit given by the equations $x^j(t) =$ constant for $j = 1, 2, 3$.

It is natural to assume further that the metric,

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 + 2cg_{0i}\,\mathrm{d}t\,\mathrm{d}x^i + g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\ , \tag{19.6}$$

where the indices i, j are summed over from 1 to 3, has the same isotropy as the background radiation. First, this means that $g_{01} = g_{02} = g_{03} = 0$, because otherwise these three components of the metric would define a privileged direction in space. Second, the three dimensional

metric,

$$\mathrm{d}\sigma^2 = -g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j\ , \tag{19.7}$$

must be isotropic. To se what this condition means, let us consider the two dimensional case, which is more easily visualized.

Example: two dimensional isotropic and homogeneous spaces

Flat space. Isotropy about the origin, for example, means invariance under arbitrary rotations about the origin. A rotation about the origin by an angle α transforms an arbitrary point (x, y) into $(\widetilde{x}, \widetilde{y})$, where

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha\ x - \sin\alpha\ y \\ \sin\alpha\ x + \cos\alpha\ y \end{pmatrix}. \tag{19.8}$$

The Euclidean metric in the plane, $\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2$, is invariant under so called *Euclidean* transformations that are combined rotations and translations. When the metric tensor is prenserved by a transformation, it means that not only infinitesimal distances are preserved, but also the distance between two arbitrary points is preserved. A transformation preserving the metric is called in general an *isometry*. The most general Euclidean transformation in the plane has the form

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} = \begin{pmatrix} d + \cos\alpha\ x - \sin\alpha\ y \\ e + \sin\alpha\ x + \cos\alpha\ y \end{pmatrix}, \tag{19.9}$$

and it contains three parameters: one rotation angle α, in addition to two displacemants d and e in the x and y diretion, respectively. If $\cos\alpha \neq 1$, then the point $(x, y) = (x_0, y_0)$, with

$$\begin{aligned} x_0 &= \frac{d}{2} - \frac{e\,\sin\alpha}{2(1-\cos\alpha)}\ , \\ y_0 &= \frac{e}{2} + \frac{d\,\sin\alpha}{2(1-\cos\alpha)}\ , \end{aligned} \tag{19.10}$$

is a fixed point of the transformation, that is, $(\widetilde{x}_0, \widetilde{y}_0) = (x_0, y_0)$. The combined rotation and translation can thus be interpreted as a pure rotation by the angle α about the point (x_0, y_0).

Note that the combination of two rotations, first a rotation by an angle $-\alpha$ about the origin, and then a rotation by an angle α about (x_0, y_0), is the same as a pure translation $(\widetilde{x}, \widetilde{y}) = (x + d, y + e)$. This is an important result, because it proves that a space which is isotropic as seen from any arbitrary point, is also *homogeneous*, that is, the metric is invariant under arbitrary translations.

Surface of a sphere. The Euclidean plane is not the only example of a two dimensional space which is isotropic and homogeneous. Another well known example is the two dimensional surface of a sphere in a three dimensional space, for example the surface

$$x^2 + y^2 + u^2 = a^2\ , \tag{19.11}$$

where the radius of the sphere is the constant a, whereas x and y are coordinates on the spherical surface, and u is the third coordinate. At a point on the surface u is a function of x and y,

$$u = \sqrt{a^2 - x^2 - y^2}\ . \tag{19.12}$$

The particular coordinates (x, y) introduced here become singular at the equator, where $u = 0$. The metric on the spherical surface is

$$\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}u^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \frac{(x\,\mathrm{d}x + y\,\mathrm{d}y)^2}{a^2 - x^2 - y^2} \,. \tag{19.13}$$

It is invariant under an arbitrary rotation about an arbitrary axis through the origin in the three dimensional space in which the two dimensional surface is embedded.

A rotation about the u axis has the same form as before, see Equation (19.8).

A rotation about the y axis in the three dimensional space has the form

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ u \end{pmatrix} . \tag{19.14}$$

It transforms only the x coordinate, since

$$\widetilde{x} = \cos\beta\, x + \sin\beta \sqrt{a^2 - x^2 - y^2} \,, \quad \widetilde{y} = y \,. \tag{19.15}$$

Thus, it is a transformation of the surface of the sphere corresponding to a translation in the x direction in the Euclidean plane. Let $a \to \infty$ and $\beta \to 0$ in such a way that $a\beta \to d$. Then the rotation about the y axis will become a pure translation in the x direction, $\widetilde{x} = x + d$, $\widetilde{y} = y$.

Similarly, a rotation about the x axis has the form

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ u \end{pmatrix} . \tag{19.16}$$

It transforms only the y coordinate,

$$\widetilde{x} = x \,, \quad \widetilde{y} = \cos\gamma\, y - \sin\gamma \sqrt{a^2 - x^2 - y^2} \,. \tag{19.17}$$

And when $a \to \infty$ and $\gamma \to 0$ such that $a\gamma \to -e$, it becomes a pure translation in the y direction, $\widetilde{x} = x$, $\widetilde{y} = y + e$.

In the Euclidean plane all translations commute. That is no longer true for the surface of the sphere, since two rotations in three dimensions commute essentially only when they have the same rotation axis. This is a manifestation of the fact that the surface is curved. It is characteristic of the surface of a sphere that it has a constant and positive scalar curvature.

Hyperboloid. Based on the previous example it is simple enough to find a metric in two dimensional space having a scalar curvature which is constant and negative. We need only change the sign of the metric in the third dimension, and constrain ourselves to the two dimensional hyperboloid

$$x^2 + y^2 - u^2 = -a^2 \,, \tag{19.18}$$

where a is a constant, and u is a third coordinate, like before. In a point on the surface we have that

$$u = \sqrt{a^2 + x^2 + y^2} \,. \tag{19.19}$$

The metric on the surface is

$$\mathrm{d}\sigma^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \mathrm{d}u^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - \frac{(x\,\mathrm{d}x + y\,\mathrm{d}y)^2}{a^2 + x^2 + y^2}\ . \tag{19.20}$$

It is again invariant under an arbitrary rotation about the u axis, as given by Equation (19.8). In addition, it is invariant under Lorentz transformations in three dimensional space, corresponding to translations in two dimensional space.

A Lorentz transformation in the x direction has the form

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \\ \widetilde{u} \end{pmatrix} = \begin{pmatrix} \cosh\beta & 0 & \sinh\beta \\ 0 & 1 & 0 \\ \sinh\beta & 0 & \cosh\beta \end{pmatrix} \begin{pmatrix} x \\ y \\ u \end{pmatrix}. \tag{19.21}$$

It transforms only the x coordinate, since

$$\widetilde{x} = \cosh\beta\ x + \sinh\beta\ \sqrt{a^2 + x^2 + y^2}\ , \quad \widetilde{y} = y\ . \tag{19.22}$$

When $a \to \infty$, $\beta \to 0$, and $a\beta \to d$, then this three dimensional Lorentz transformation becomes a pure translation in the x direction, $\widetilde{x} = x + d$, $\widetilde{y} = y$.

Similarly, a Lorentz transformation in the y direction transforms only the y coordinate,

$$\widetilde{x} = x\ , \quad \widetilde{y} = \cosh\gamma\ y + \sinh\gamma\ \sqrt{a^2 + x^2 + y^2}\ . \tag{19.23}$$

When $a \to \infty$, $\gamma \to 0$, and $a\gamma \to e$, we are left with a pure translation in the y direction, $\widetilde{x} = x$, $\widetilde{y} = y + e$.

Summary. Substituting $x \to ax$ and $y \to ay$ in the last two examples, we may summarize all three examples above in the single formula

$$\mathrm{d}\sigma^2 = a^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2 + k\,\frac{(x\,\mathrm{d}x + y\,\mathrm{d}y)^2}{1 - k(x^2 + y^2)} \right), \tag{19.24}$$

where $k = 0$ for the flat space, $k = 1$ for the spherical surface, and $k = -1$ for the hyperboloid. This formula generalizes immediately to three spatial dimensions.

The Friedmann–Robertson–Walker metric

In three spatial dimensions there exist the same three possibilities for making a metric which is isotropic and homogeneous. We may write

$$\mathrm{d}\sigma^2 = a^2 \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + k\,\frac{(x\,\mathrm{d}x + y\,\mathrm{d}y + z\,\mathrm{d}z)^2}{1 - k(x^2 + y^2 + z^2)} \right), \tag{19.25}$$

with $k = 0$ for a flat three dimensional space, $k = 1$ for a three dimensional spherical surface, and $k = -1$ for a hyperboloid. Introducing the usual polar coordinates r, θ, φ we get that

$$\mathrm{d}\sigma^2 = a^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\varphi^2 \right). \tag{19.26}$$

The curvature scalar for this three dimensional metric is

$$R^{(3)} = \frac{6k}{a^2} \, . \tag{19.27}$$

In this way we obtain the four dimensional metric known as the Friedmann–Robertson–Walker, or FRW, metric,

$$\mathrm{d}s^2 = c^2 \, \mathrm{d}t^2 - a(t)^2 \left(\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 \mathrm{d}\theta^2 + r^2 \sin^2\theta \, \mathrm{d}\varphi^2 \right) . \tag{19.28}$$

It is a basic ingredient in the cosmological standard model. Because it has maximal symmetry, it contains only two free variables, the constant k giving the sign of the three dimensional curvature scalar, and the time dependent scale factor $a(t)$, which is the length scale in the three dimensional space. That the Universe expands, means simply that $a(t)$ increases with time. The equations determining how $a(t)$ varies with time are the gravitational equation and the equations of state for matter and radiation. The same equations also determine whether $k = 0$, $k = 1$, or $k = -1$.

19.3 The energy momentum tensor

According to the cosmological standard model, the cosmic background radiation is isotropic at an arbitrary point in spacetime, or more precisely, it looks isotropic to an observer who is at rest relative to the preferred frame of reference defined by the radiation. Such an observer has time independent coordinates r, θ and φ in the coordinate system which is used in the FRW metric, equation (19.28).

The same observer may choose his local coordinate system as a so called Lorentz system, or Minkowski system, in which the metric *at one single point* has the standard form

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{19.29}$$

The time axis in this local coordinate system will be the same as in the global FRW system, and only the three spatial coordinates will need to be redefined. We definee n^μ as the unit vector along the common time axis of the two coordinate systems, in both systems it has the components

$$n^\mu = (1, 0, 0, 0) \, . \tag{19.30}$$

In the local "isotropic" coordinates the energy momentum tensor must be isotropic, hence it must have the diagonal form

$$T_\mu{}^\nu = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & -P & 0 & 0 \\ 0 & 0 & -P & 0 \\ 0 & 0 & 0 & -P \end{pmatrix} = (\rho c^2 + P) n_\mu n^\nu - P \delta_\mu^\nu \, . \tag{19.31}$$

Here ρc^2 is the energy density in the local Lorentz system, and we choose to include a faktor c^2 so that ρ will be the mass density in a Universe containing only non-relativistic matter.

The spatial diagonal elements must be equal, because of the assumed isotropy, and we call them $-P$, because P then represents *pressure*, e.g. in a gas in thermodynamic equilibrium.

Both ρ and P depend only on the universal time t and are independent of the spatial coordinates, either in the local Lorentz system or in the FRW system, thus $\rho = \rho(t)$ and $P = P(t)$. By definition, they transform as scalars under coordinate transformations, whereas n^μ transforms as a contravariant vector. In an arbitrary coordinate system the energy momentum tensor may therefore be written in the following way,

$$T_{\mu\nu} = (\rho c^2 + P) n_\mu n_\nu - P g_{\mu\nu} \; . \tag{19.32}$$

The name "pressure" may perhaps sometimes be misleading, since an isotropic energy momentum tensor will have the same form independent of whether it describes a physical system in thermodynamic equilibrium. Furthermore, the quantinty P describes two somewhat different physical realities if we compare e.g. air to a gas of photons, such as the cosmic background radiation. The air molecules interact so strongly that one volume of air exerts a pressure on a nighbouring volume by collisions between the molecules in the two volumes. In the photon gas, on the other hand, the interactions are negligible, and the momentum transfer between two neighbouring volumes takes place because photons from one volume cross the border into the other volume.

We have seen that the electromagnetic energy momentum tensor is traceless, that is, we have $T_\mu{}^\mu = 0$. Now one might object that a non-zero electric or magnetic field always points in some direction in space and hence can never be isotropic and homogeneous. However, the electromagnetic energy momentum tensor may very well be isotropic and homogeneous, to a good approximation, if we average it over a small volume and a small time interval. Therefore there is no contradiction in the assumption that the cosmic background radiation is isotropic and homogeneous. Thus, for the background radiation, and for any other electromagnetic field which is isotropic and homogeneous, whether in thermodynamic equilibrium or not, we will have that

$$\rho c^2 - 3P = 0 \; . \tag{19.33}$$

This relation is actually more general. It depends on two special properties of photons, that they move with the speed of light, and that they interact not very strongly. But other particles may have the same properties, if not exactly, then at least approximately. Examples are neutrinos, and even electrons, protons and any other particles when they are relativistic, that is, when every particle has an energy much larger than its own rest mass. In all such cases the relation $\rho c^2 = 3P$ will hold, independent of whether the distribution of particles in phase space corresponds to a state of thermodynamic equilibrium.

The opposite limiting case is the non-relativistic one, when the particles move with velocities much smaller than the speed of light. Then we have that $P \approx \rho v^2$, where v is a typical particle velocity, so that it will be a good approximation to take

$$P = 0 \; . \tag{19.34}$$

A third interesting case is an energy momentum tensor corresponding to a non-zero value of the cosmological constant Λ, of the form $T_{\mu\nu} = (\Lambda/\kappa)\, g_{\mu\nu}$ with $\kappa = 8\pi G/c^4$. One possible physical realization is by means of a so called Higgs field, which is a Klein–Gordon field ϕ with a modified Lagrange density. The Higgs field may have a constant value $\phi_0 \neq 0$ in the

vacuum state, that is, in the state in which the energy density is minimal. The characteristic property of such a vacuum energy is that it gives

$$P = -\rho c^2 \ . \tag{19.35}$$

We may introduce a dimensionless constant w, and combine all the three special cases mentioned above in one common equation of state,

$$P = w\rho c^2 \ , \tag{19.36}$$

which may describe states where there is not necessarily thermodynamic equilibrium.

For a gas of relativistic particles, and especially for electromagnetic radiation, we have $w = 1/3$. This gives a good approximation for the equation of state during roughly the first thousand years of the Universe. This period of time is called the *radiation dominated* epoch.

For non-relativistic matter we may take $w = 0$. This gives a good approximation for the equation of state in the cosmological standard model all the time up until our own time, in the *matter dominated* epoch, which followed the radiation dominated epoch.

For vacuum with a non-zero energy density we have $w = -1$. In the *inflationary models*, which are more speculative extensions of the standard model backwards in time, towards the very beginning of time, it is assumed that the vacuum energy dominated during a certain period, leading to an exponential expansion of the Universe. Such an *inflationary epoch* may explain certain basic properties of our Universe that are not so easy to explain in any other way. The newest observational results in cosmology strongly indicate that there is presently a non negligible contribution to the energy density of the Universe from a cosmological constant, or from some similar source. The effect seen is that the expansion of the Universe accelerates, instead of being braked by the gravitational attraction between galaxies.

Conservation of energy and momentum

In Einstein's gravitational theory the conservation of energy and momentum follows from the field equation, and is expressed in the equation of continuity $T^{\mu\nu}{}_{;\nu} = 0$. From Equation (19.32), together with the identity $g^{\mu\nu}{}_{;\rho} = 0$, follows that

$$T^{\mu\nu}{}_{;\nu} = (\rho_{,\nu}c^2 + P_{,\nu})n^\mu n^\nu + (\rho c^2 + P)n^\mu{}_{;\nu}n^\nu + (\rho c^2 + P)n^\mu n^\nu{}_{;\nu} - P_{,\nu}\, g^{\mu\nu} \ . \tag{19.37}$$

In the FRW coordinates $x^\mu = (ct, r, \theta, \varphi)$ we have that

$$n^\mu{}_{;\nu}n^\nu = n^\mu{}_{;0} = \Gamma^\mu_{\alpha 0}n^\alpha = \Gamma^\mu_{00} = \frac{1}{2}\, g^{\mu\alpha}(2g_{\alpha 0,0} - g_{00,\alpha}) = 0 \ . \tag{19.38}$$

And, defining γ as in Equation (19.59) below, we have that

$$n^\nu{}_{;\nu} = \Gamma^\nu_{\alpha\nu}n^\alpha = \Gamma^\nu_{0\nu} = \gamma_{,0} = \frac{3a_{,0}}{a} \ . \tag{19.39}$$

This gives that

$$T^{\mu\nu}{}_{;\nu} = (\rho_{,0}c^2 + P_{,0})n^\mu + (\rho c^2 + P)n^\mu\, \frac{3a_{,0}}{a} - P_{,0}\, g^{\mu 0} \ . \tag{19.40}$$

Three of the four conservation equations become identities,

$$T^{1\nu}{}_{;\nu} = T^{2\nu}{}_{;\nu} = T^{3\nu}{}_{;\nu} = 0 \ , \tag{19.41}$$

and the only nontrivial one is the conservation law for energy,

$$T^{0\nu}{}_{;\nu} = \rho_{,0}c^2 + (\rho c^2 + P)\,\frac{3a_{,0}}{a} = 0\,. \tag{19.42}$$

It may be written in the following form,

$$\frac{\mathrm{d}(\rho c^2 a^3)}{\mathrm{d}t} + P\,\frac{\mathrm{d}a^3}{\mathrm{d}t} = 0\,. \tag{19.43}$$

Explicit solution for energy density and pressure

Equation (19.43) may be integrated if the equation of state has the form $P = w\rho c^2$ with w constant, then it gives that

$$\rho a^{3(1+w)} = \text{constant}\,. \tag{19.44}$$

In the non-relativistic case $w = 0$, the energy density decreases as $1/a^3$ when the scale factor a increases. The energy within an expanding volume is proportional to $\rho c^2 a^3$, thus it is constant for non-relativistic matter. This conclusion is quite natural, since $P = 0$. The pressure performs no work on the rest of the Universe when the volume expands, and the expansion costs no energy.

In the other extreme, the ultrarelativistic case $w = 1/3$, the energy density decreases as $1/a^4$ when a increases. Thus, the total energy inside an expanding volume decreases as $1/a$, and one natural way to understand the decrease in energy is that the pressure $P = \rho c^2/3$ performs work on the environment. For a gas of photons, such as the cosmic background radiation, this effect may seem surprising, since the only way the photons interact with themselves and with the rest of the Universe is through gravitation. The number of photons is conserved, and when the total energy decreases, it must be because every photon loses energy. The energy per photon is inversely proportional to a. The photons are "running" in all directions at the speed of light c, and they are losing energy because they are "running uphill", regardless of the direction! Since the energy of a photon is also inversely proportional to its wave length, the energy loss due to expansion can be understood in another, very natural way. The effect is simply that the wave length of the electromagnetic radiation is stretched by the same factor as the linear scale of the whole Universe.

We see that the energy density decreases faster for radiation, that is, for photons and other particles with relativistic velocities, than it does for non-relativistic matter. In addition, relativistic particles with nonzero rest mass will become non-relativistic after losing most of their energy. Therefore non-relativistic matter will sooner or later come to dominate, even though the very early Universe is radiation dominated.

The third case, $w = -1$, is interesting because then the energy density is constant, and the energy increases proportionally to the volume. Remember that the cosmological constant is supposed to be constant. The energy increases because the pressure $P = -\rho c^2$ is *negative*, it is not pressure, but rather *stretch*, like in a rubber band.

Temperature

Equation (19.43) is the general relativistic conservation law for energy, but it may also be interpreted as a version of the *first law of thermodynamics*,

$$\mathrm{d}E + P\,\mathrm{d}\mathcal{V} = 0\,. \tag{19.45}$$

Here the quantity $\mathcal{V} = \mathcal{V}_0\, a^3$, with $\mathcal{V}_0$ constant, is the time dependent volume of a given three dimensional region in the expanding Universe, whereas $E = \rho c^2 \mathcal{V}$ is the energy within the region. If $P = 0$, then E is constant. If $P > 0$, then the energy E decreases while the volume $\mathcal{V}$ increases, because of the work performed on the rest of the Universe by the pressure forces.

More generally, the first law of thermodynamics reads

$$\mathrm{d}E + P\,\mathrm{d}\mathcal{V} = \mathrm{d}Q\ , \tag{19.46}$$

where $\mathrm{d}Q$ is the heat flowing into the volume. Equation (19.43), which is identical to Equation (19.45), states that $\mathrm{d}Q = 0$. There is no heat flow in the Friedmann universe, indeed there can not be any heat flow in an isotropic universe, with no preferred direction in which heat could flow.

For a quasistatic process, in which the system is all the time in thermodynamic equilibrium, the first law of thermodynamics holds in the following form,

$$\mathrm{d}E + P\,\mathrm{d}\mathcal{V} = T\,\mathrm{d}\mathcal{S} + \sum_i \mu_i\,\mathrm{d}N_i\ . \tag{19.47}$$

Here T is the temperature, $\mathcal{S}$ is the entropy inside the volume. The volume $\mathcal{V}$ contains N_i particles of a given species i, we allow these particle numbers to change, and the chemical potential of which is μ_i.

It will often be the case that $\mu_i = 0$ for most or for all particle species i, either exactly, or to a good approximation. Then the entropy is constant, $\mathrm{d}\mathcal{S} = 0$, in the Friedmann universe. In other words, the Universe expands *reversibly*. However, the conclusion that the entropy is constant, holds only if the expansion is slow enough that thermodynamic equilibrium is maintained all the time. If the expansion is faster than the processes that produce equilibrium, then the entropy may very well increase.

When we speak about temperature, entropy, and chemical potential, we assume thermodynamic equilibrium. For a gas of particles in thermodynamic equilibrium, and with negligable interaction energy, the ocupation probability of a one particle state of momentum $\boldsymbol{p}$ is

$$f(\boldsymbol{p}) = \frac{1}{\mathrm{e}^{\beta(E-\mu)} \mp 1}\ , \tag{19.48}$$

with $-$ for bosons and $+$ for fermions. Here μ is the chemical potential, whereas β is the inverse temperature,

$$\beta = \frac{1}{k_B T}\ , \tag{19.49}$$

and k_B is Boltzmann's constant. If the particle has mass m, its energy is

$$E = E(\boldsymbol{p}) = c\,\sqrt{m^2c^2 + |\boldsymbol{p}|^2}\ , \tag{19.50}$$

and its velocity is

$$\boldsymbol{v} = \frac{c^2\boldsymbol{p}}{E(\boldsymbol{p})} = \frac{c\boldsymbol{p}}{\sqrt{m^2c^2 + |\boldsymbol{p}|^2}}\ . \tag{19.51}$$

This means that the number density n, the energy density ρc^2, and the pressure P are given by the following integrals, where $h = 2\pi\hbar$ is Planck's constant,

$$\begin{aligned} n &= \frac{g}{h^3} \int \mathrm{d}^3\boldsymbol{p}\, f(\boldsymbol{p}) \;, \\ \rho c^2 &= \frac{g}{h^3} \int \mathrm{d}^3\boldsymbol{p}\, f(\boldsymbol{p})\, E(\boldsymbol{p}) \;, \\ P &= \frac{g}{h^3} \int \mathrm{d}^3\boldsymbol{p}\, f(\boldsymbol{p})\, \frac{\boldsymbol{p}\cdot\boldsymbol{v}}{3} \;. \end{aligned} \tag{19.52}$$

The factor g is the degeneracy due to internal degrees of freedom, such as spin. We have, for example, $g = 2$ for photons, since a photon has two possible polarization states, whereas $g = 2s + 1$ for a particle of mass $m > 0$ and spin $s = 0, 1/2, 1, 3/2, \ldots$. See Appendix C for a brief recapitulation of statistical mechanics.

For photons we have $m = 0$ and $\mu = 0$. More generally, when the temperature is high, that is, when $k_B T$ is much larger than both mc^2 and μ, it is a good approximation to take $m = 0$ and $\mu = 0$. Then the integrals can be computed exactly, and we get for a relativistic boson gas that

$$\begin{aligned} n &= n_b = g\, \frac{\zeta(3)(k_B T)^3}{\pi^2(\hbar c)^3} \;, \\ \rho c^2 &= \rho_b\, c^2 = g\, \frac{\pi^2(k_B T)^4}{30(\hbar c)^3} \;. \end{aligned} \tag{19.53}$$

Here $\zeta(3) = 1.202057\ldots$ is Riemann's zeta function of 3. For a relativistic fermion gas we have that

$$\begin{aligned} n &= n_f = \frac{3}{4}\, n_b \;, \\ \rho &= \rho_f = \frac{7}{8}\, \rho_b \;. \end{aligned} \tag{19.54}$$

In both cases we have $P = \rho c^2/3$.

Thus, in a radiation dominated universe in thermodynamic equilibrium we may write quite generally the energy density as

$$\rho c^2 = g_* \, \frac{\pi^2(k_B T)^4}{30(\hbar c)^3} \;, \tag{19.55}$$

when we introduce a numerical factor g_* which is the *effective number of degrees of freedom*,

$$g_* = g_b + \frac{7}{8}\, g_f \;. \tag{19.56}$$

g_b is the total number of degrees of freedom for all kinds of bosons that are relativistic at the given temperature, whereas g_f is the number of degrees of freedom for all kinds ofrelativistic fermions, with a factor of $7/8$ which is the reduction in energy density for fermions relative to bosons.

Note that the energy density of a relativistic gas is given uniquely by the effective number of degrees of freedom g_* and by the temperature T. It is proportional to g_* and to T^4. Since the energy density is also inversely proportional to a^4, it means that the temperature is simply inversely proportional to the scale factor a, as long as g_* is constant.

However, the effective number of degrees of freedom g_* is not constant, it will sometimes be reduced during the expansion of the Univers. A reduction takes place every time the temperature has dropped so much that k_BT becomes of the same order of magnitude as the rest energy mc^2 of a givien type of particlec. Then these particles will "freeze out" and become non-relativistic. If there happened to be originally as many antiparticles as particles of this species, then the antiparticles and particles may mutually annihilate, so that there will remain only a very small net number, maybe none at all, of them. When g_* is reduced, it implies that the temperature drops slightly less than it would have done if g_* had remained constant.

Even though there are the same number of particles and antiparticles of a given species, they do not necessarily annihilate each other. An example of this are the neutrinos, which interact so weakly that they simply "decouple" from the rest of the Universe at a given moment, when the density has fallen sufficiently low. Therefore the Universe is almost certainly filled by a "sea" of neutrinos, and just as many antineutrinos, which are relics of the Big Bang, and which do not interact with each other or with the rest of the Universe, except gravitationally. The number of particles in this sea of neutrinos has to be approximately the same as the number of photons in the cosmic microwave radiation, that is, around 10^9 to 10^{10} times the number of protons, neutrons, and electrons. In spite of the enormous number of cosmic neutrinos filling the Universe, it has so far not been possible to observere them.

19.4 The Ricci tensor

We will now use Equation (6.55) to compute the Ricci tensor $R_{\mu\nu}$ of the FRW metric

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -a^2/(1-kr^2) & 0 & 0 \\ 0 & 0 & -a^2r^2 & 0 \\ 0 & 0 & 0 & -a^2r^2\sin^2\theta \end{pmatrix} . \tag{19.57}$$

Our coordinates are $x^\mu = (ct, r, \theta, \varphi)$, and $a = a(t) = a(x^0)$ is the universal scale factor of the three dimensional metric. We write $\dot{a} = \mathrm{d}a/\mathrm{d}t = ca_{,0}$ and $\ddot{a} = \mathrm{d}^2a/\mathrm{d}t^2 = c^2a_{,00}$. The determinant is

$$g = \det(g_{\mu\nu}) = -\frac{a^6r^4\sin^2\theta}{1-kr^2} , \tag{19.58}$$

and we define

$$\gamma = \frac{1}{2}\ln|g| = 3\ln a + 2\ln r - \frac{1}{2}\ln(1-kr^2) + \ln\sin\theta . \tag{19.59}$$

The derivatives of γ are

$$\gamma_{,0} = \frac{3a_{,0}}{a} , \quad \gamma_{,1} = \frac{2}{r} + \frac{kr}{1-kr^2} = \frac{2-kr^2}{r(1-kr^2)} , \quad \gamma_{,2} = \cot\theta , \quad \gamma_{,3} = 0 . \tag{19.60}$$

For a vector field with constant components A^μ we have, by Equation (6.55),

$$R_{\lambda\nu}A^\lambda A^\nu = R^{(\mathrm{I})} + R^{(\mathrm{IIa})} + R^{(\mathrm{IIb})} + R^{(\mathrm{III})} , \tag{19.61}$$

with

$$\begin{aligned} R^{(\mathrm{I})} &= B^{\kappa}{}_{,\kappa} + \gamma_{,\kappa}\, B^{\kappa} \,, \\ R^{(\mathrm{IIa})} &= \frac{1}{4}\, g^{\mu\sigma} g^{\kappa\rho} (A_{\kappa,\sigma} - A_{\sigma,\kappa})(A_{\rho,\mu} - A_{\mu,\rho}) \,, \\ R^{(\mathrm{IIb})} &= -\frac{1}{4}\, g^{\mu\sigma} g^{\kappa\rho} (A^{\nu} g_{\kappa\sigma,\nu})(A^{\lambda} g_{\rho\mu,\lambda}) \,, \\ R^{(\mathrm{III})} &= -\gamma_{,\lambda\nu}\, A^{\lambda} A^{\nu} \,, \end{aligned} \tag{19.62}$$

and with

$$B^{\kappa} = g^{\kappa\rho} A^{\nu} \left(A_{\rho,\nu} - \frac{1}{2}\, A_{\nu,\rho} \right). \tag{19.63}$$

First, and most simply, we have that

$$R^{(\mathrm{III})} = 3 \left(-\frac{a_{,00}}{a} + \frac{(a_{,0})^2}{a^2} \right) (A^0)^2 + \frac{2 - 5kr^2 + k^2 r^4}{r^2 (1 - kr^2)^2}\, (A^1)^2 + \frac{1}{\sin^2\theta}\, (A^2)^2 \,. \tag{19.64}$$

Next,

$$A_0 = A^0 \,, \quad A_1 = -\frac{a^2}{1 - kr^2}\, A^1 \quad A_2 = -a^2 r^2\, A^2 \,, \quad A_3 = -a^2 r^2 \sin^2\theta\, A^3 \,. \tag{19.65}$$

The partial derivatives of A_μ that do not vanish identically, are

$$\begin{array}{ll} A_{1,0} = -(2aa_{,0}/(1 - kr^2))\, A^1 \,, & A_{1,1} = -(2ka^2 r/(1 - kr^2)^2)\, A^1 \,, \\ A_{2,0} = -2aa_{,0}\, r^2\, A^2 \,, & A_{2,1} = -2a^2 r\, A^2 \,, \\ A_{3,0} = -2aa_{,0}\, r^2 \sin^2\theta\, A^3 \,, & A_{3,1} = -2a^2 r \sin^2\theta\, A^3 \,, \\ A_{3,2} = -2a^2 r^2 \sin\theta\cos\theta\, A^3 \,. & \end{array} \tag{19.66}$$

This gives that

$$\begin{aligned} B^0 &= -\frac{1}{2}\, (A^1 A_{1,0} + A^2 A_{2,0} + A^3 A_{3,0}) \\ &= aa_{,0} \left(\frac{1}{1 - kr^2}\, (A^1)^2 + r^2 (A^2)^2 + r^2 \sin^2\theta\, (A^3)^2 \right), \\ B^1 &= -\frac{1 - kr^2}{a^2} \left(A^0 A_{1,0} + \frac{1}{2}\, A^1 A_{1,1} - \frac{1}{2}\, A^2 A_{2,1} - \frac{1}{2}\, A^3 A_{3,1} \right) \\ &= \frac{2a_{,0}}{a}\, A^0 A^1 + \frac{kr}{1 - kr^2}\, (A^1)^2 - r(1 - kr^2) \left((A^2)^2 + \sin^2\theta\, (A^3)^2 \right) \,, \\ B^2 &= -\frac{1}{a^2 r^2} \left(A^0 A_{2,0} + A^1 A_{2,1} - \frac{1}{2}\, A^3 A_{3,2} \right) \\ &= \frac{2a_{,0}}{a}\, A^0 A^2 + \frac{2}{r}\, A^1 A^2 - \sin\theta\cos\theta\, (A^3)^2 \,. \end{aligned} \tag{19.67}$$

We do not compute B^3, which does not contribute, since $x^3 = \varphi$ is a cyclic coordinate. We have that

$$\begin{aligned} R^{(\mathrm{I})} &= \left(\frac{aa_{,00} + 4(a_{,0})^2}{1 - kr^2} + \frac{3k}{(1 - kr^2)^2} \right) (A^1)^2 + (3\sin^2\theta - 2)\, (A^3)^2 \\ &\quad + \left(r^2 (aa_{,00} + 4(a_{,0})^2) - 3 + 4kr^2 \right) \left((A^2)^2 + \sin^2\theta\, (A^3)^2 \right) \\ &\quad + \frac{2a_{,0}}{a}\, \frac{2 - kr^2}{r(1 - kr^2)}\, A^0 A^1 + \frac{2a_{,0}}{a}\, \cot\theta\, A^0 A^2 + \frac{2}{r}\, \cot\theta\, A^1 A^2 \,. \end{aligned} \tag{19.68}$$

Furthermore,

$$\begin{aligned} R^{(\mathrm{IIa})} &= -\frac{1-kr^2}{2a^2}\,(A_{1,0})^2 - \frac{1}{2a^2r^2}\,(A_{2,0})^2 - \frac{1}{2a^2r^2\sin^2\theta}\,(A_{3,0})^2 \\ &\quad + \frac{1-kr^2}{2a^4r^2}\,(A_{2,1})^2 + \frac{1-kr^2}{2a^4r^2\sin^2\theta}\,(A_{3,1})^2 + \frac{1}{2a^4r^4\sin^2\theta}\,(A_{3,2})^2 \\ &= -\frac{2(a_{,0})^2}{1-kr^2}\,(A^1)^2 + 2\left(-r^2(a_{,0})^2 + 1 - kr^2\right)\left((A^2)^2 + \sin^2\theta\,(A^3)^2\right) \\ &\quad + 2\cos^2\theta\,(A^3)^2\,. \end{aligned} \tag{19.69}$$

The metric tensor is diagonal, and for the diagonal elements we have that

$$\begin{aligned} A^\nu g_{00,\nu} &= 0\,, \\ A^\nu g_{11,\nu} &= -\frac{2aa_{,0}}{1-kr^2}\,A^0 - \frac{2ka^2r}{(1-kr^2)^2}\,A^1\,, \\ A^\nu g_{22,\nu} &= -2aa_{,0}r^2\,A^0 - 2a^2r\,A^1\,, \\ A^\nu g_{33,\nu} &= -2aa_{,0}r^2\sin^2\theta\,A^0 - 2a^2r\sin^2\theta\,A^1 - 2a^2r^2\sin\theta\cos\theta\,A^2\,. \end{aligned} \tag{19.70}$$

It follows that

$$\begin{aligned} R^{(\mathrm{IIb})} &= -\frac{(1-kr^2)^2}{4a^4}\,(A^\nu g_{11,\nu})^2 - \frac{1}{4a^4r^4}\,(A^\nu g_{22,\nu})^2 - \frac{1}{4a^4r^4\sin^4\theta}\,(A^\nu g_{33,\nu})^2 \\ &= -\frac{3(a_{,0})^2}{a^2}\,(A^0)^2 - \frac{2-4kr^2+3k^2r^4}{r^2(1-kr^2)^2}\,(A^1)^2 - \cot^2\theta\,(A^2)^2 \\ &\quad -\frac{2a_{,0}}{a}\,\frac{2-kr^2}{r(1-kr^2)}\,A^0A^1 - \frac{2a_{,0}}{a}\,\cot\theta\,A^0A^2 - \frac{2}{r}\,\cot\theta\,A^1A^2\,. \end{aligned} \tag{19.71}$$

The sum of all the four contributions is

$$\begin{aligned} R_{\lambda\nu}A^\lambda A^\nu &= -\frac{3a_{,00}}{a}\,(A^0)^2 \\ &\quad + (aa_{,00} + 2(a_{,0})^2 + 2k)\left(\frac{1}{1-kr^2}\,(A^1)^2 + r^2\,(A^2)^2 + r^2\sin^2\theta\,(A^3)^2\right). \end{aligned} \tag{19.72}$$

We may read the components of the Ricci tensor out of this formula, and we see that it is diagonal, like the metric. The scalar curvature is

$$\begin{aligned} R = g^{\lambda\nu}R_{\lambda\nu} &= R_{00} - \frac{1-kr^2}{a^2}\,R_{11} - \frac{1}{a^2r^2}\,R_{22} - \frac{1}{a^2r^2\sin^2\theta}\,R_{33} \\ &= -6\left(\frac{a_{,00}}{a} + \frac{(a_{,0})^2+k}{a^2}\right). \end{aligned} \tag{19.73}$$

The Einstein tensor $G_{\lambda\nu}$ is also diagonal, and its diagonal elements are

$$\begin{aligned} G_{00} &= R_{00} - \frac{1}{2}\,R = 3\,\frac{(a_{,0})^2+k}{a^2}\,, \\ G_{11} &= R_{11} + \frac{1}{2}\,\frac{a^2}{1-kr^2}\,R = -\left(\frac{2a_{,00}}{a} + \frac{(a_{,0})^2+k}{a^2}\right)\frac{a^2}{1-kr^2}\,, \\ G_{22} &= R_{22} + \frac{1}{2}\,a^2r^2R = -\left(\frac{2a_{,00}}{a} + \frac{(a_{,0})^2+k}{a^2}\right)a^2r^2\,, \\ G_{33} &= G_{22}\sin^2\theta\,. \end{aligned} \tag{19.74}$$

We see that

$$G_{ij} = \left(\frac{2a_{,00}}{a} + \frac{(a_{,0})^2 + k}{a^2} \right) g_{ij} \quad \text{for} \quad i,j = 1,2,3 \; . \tag{19.75}$$

19.5 The gravitational equation

After this long calculation we arrive at a surprisingly simple field equation. Or maybe the simplicity is not so surprising, since we started out by assuming maximal symmetry.

To be general, let us include the cosmological constant Λ. Of the 10 equations $G_{\mu\nu} = \Lambda g_{\mu\nu} + \kappa T_{\mu\nu}$ we get only two independent equations, e.g., with $\mu = \nu = 0$ and $\mu = \nu = 1$,

$$3\,\frac{\dot{a}^2 + kc^2}{a^2} = \Lambda c^2 + 8\pi G\rho \; , \tag{19.76}$$

$$\frac{2\ddot{a}}{a} + \frac{\dot{a}^2 + kc^2}{a^2} = \Lambda c^2 - \frac{8\pi G P}{c^2} \; . \tag{19.77}$$

They are actually not quite independent, because both left hand sides contain the same time independent scale factor a and its time derivatives $\dot{a} = ca_{,0}$ and $\ddot{a} = c^2 a_{,00}$. If both equations are to be satisfied simultaneously, there is a consistency condition to be satisfied, that

$$8\pi G \left(\frac{\mathrm{d}(\rho c^2 a^3)}{\mathrm{d}t} + P\,\frac{\mathrm{d}a^3}{\mathrm{d}t} \right) + \frac{\mathrm{d}\Lambda}{\mathrm{d}t}\, c^4 a^3 = 0 \; . \tag{19.78}$$

We assume that Λ is constant, and then this is just Equation (19.43), which is the conservation law for energy, $T^{0\nu}{}_{;\nu} = 0$.

The Equations (19.76) and (19.77) bear the name of Friedmann, who first derived them in 1923. They contain 6 quantities that are all observable, at least in principle. The two left hand sides contain three independent quantities describing the gravitational field, we may take them to be, first, the Hubble parameter H from Equation (19.2), which we now redefine as the relative rate of expansion,

$$H = \frac{\dot{a}}{a} = \frac{ca_{,0}}{a} \; , \tag{19.79}$$

second, the time derivative of H, often expressed in terms of the dimensionsless *deceleration parameter* $q = a\ddot{a}/\dot{a}^2$ as

$$\dot{H} = \frac{\ddot{a}}{a} - H^2 = -(q+1)H^2 \; , \tag{19.80}$$

and, third, the curvature scalar of the three dimensional space,

$$R^{(3)} = \frac{6k}{a^2} \; . \tag{19.81}$$

The two right hand sides of Friedmann's equations contain three other quantities acting as sources of the field, the cosmological constant Λ, the energy density ρc^2, and the pressure P. Thus, the equations can in principle be tested against observations, even though it is hard to measure all relevant quantities with high presision.

Critical density

The most precise way to determine the value of the three dimensional curvature scalar $R^{(3)} = 6k/a^2$ is perhaps to assume the validity of Equation (19.76), and rewrite it as

$$\frac{6kc^2}{a^2} = -6H^2 + 2\Lambda c^2 + 16\pi G\rho \; . \tag{19.82}$$

We may express all three terms on the right hand side of this equation in terms of equivalent mass densities. First, the *critical density*, defined as

$$\rho_c = \frac{3H^2}{8\pi G} = 1.87 \times 10^{-29}\, h^2\, \mathrm{kg/dm^3} = 1.68 \times 10^{-12}\, h^2\, \mathrm{J}/(c^2\, \mathrm{dm^3}) \; , \tag{19.83}$$

where $h \approx 0.7$ from observations. And, second, the mass density corresponding to the vacuum energy,

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} \; , \tag{19.84}$$

which we assume to be time independent. By these definitions, we have that

$$\frac{6kc^2}{a^2} = 16\pi G(\rho + \rho_\Lambda - \rho_c) \; . \tag{19.85}$$

If the density $\rho + \rho_\Lambda$ is precisely equal to the critical density ρ_c, then $k = 0$, meaning that the three dimensional space is flat. If the density is higher, then $k > 0$, meaning that the three dimensional space is positively curved, like the surface of a sphere, and consequently should have a finite three dimensional volume. If the density is lower, then $k < 0$, and the three dimensional space has negative curvature, like a hyperbolic surface. Thus, according to the Friedmann equations we may check whether the three dimensional space is closed, like the surface of a sphere, or open, like a flat space or a hyperboloid, by checking whether the energy density is larger than or smaller thann a critical value determined by the Hubble parameter H.

Let the dimensionless parameter Ω be the ratio between the density $\rho+\rho_\Lambda$ and the critical density ρ_c, then we may rewrite Equation (19.85) in yet another way,

$$\Omega = \frac{\rho + \rho_\Lambda}{\rho_c} = \frac{1}{1 - \dfrac{3kc^2}{8\pi G(\rho + \rho_\Lambda)a^2}} \; . \tag{19.86}$$

We see that in the case $k = 0$, that is, when the three dimensional space is flat, we have $\Omega = 1$ identically, and the density remains critical at all times.

In the case $k = -1$, when the three dimensional space is hyperbolic, then $\Omega < 1$, and the density is "undercritical" at all times. We arrive at this conclusion by making the rather inncocent assumption that $\rho + \rho_\Lambda > 0$.

In the caase $k = 1$, when the three dimensional space is closed, then $\Omega > 1$, and the density is "overcritical" at all times. Again we assume that $\rho + \rho_\Lambda > 0$. We see furthermore from Equation (19.86) that $\Omega \to \infty$ for a finite value of the density, when

$$(\rho + \rho_\Lambda)a^2 = \frac{3c^2}{8\pi G} \; . \tag{19.87}$$

Consequently, this is a lower limit for the quantity $(\rho + \rho_\Lambda)a^2$. That $\Omega = \infty$, must mean that $\rho_c = 0$, meaning in turn that $H = 0$. If the Universe ever reaches this lower limit, it will stop expanding, and begin to contract instead. Thus, a universe in which the energy density is higher than the critical density, is finite not only in space, but will be finite also in time, unless the vacuum energy density is so high that $(\rho + \rho_\Lambda)a^2$ stays forever larger than $3c^2/(8\pi G)$.

In the limit $a \to 0$, any physically reasonable equation of state will give that $\rho a^2 \to \infty$, at the same time as ρ_Λ is constant and can be neglected. For example, if we assume the equation of state $P = w\rho c^2$, implying that ρ varies with a as in Equation (19.44), we have that $\rho a^2 \propto a^{-1-3w}$, and the inequality $w > -1/3$ is sufficient to ensure that $\rho a^2 \to \infty$ as $a \to 0$. Hence we may safely conclude from Equation (19.86) that $\Omega \to 1$ as $a \to 0$.

In the opposite limit $a \to \infty$ we may have either $\Omega \to 0$ or $\Omega \to 1$, depending on whether the cosmological constant Λ is zero or nonzero. As we have seen, in the case $k = 1$ it may happen that the scale factor a reaches a maximum value, at which $(\rho + \rho_\Lambda)a^2 = 3c^2/(8\pi G)$, and afterwards the Universe will recontract and collapse completely. Let us assume that this does not happen, and that a will actually grow arbitrarily large. Normally, we will have that $\rho a^2 \to 0$ when $a \to \infty$. Hence, if $\Lambda \neq 0$, the vacuum contribution $\rho_\Lambda a^2$ will go to infinity, and will sooner or later dominate over ρa^2, implying that $\Omega \to 1$. If $\Lambda = 0$, on the other hand, and if $k = 0$ or $k = -1$ so that $a \to \infty$ is possible, then we see that $\Omega \to 0$ in the limit.

Why is the density so nearly critical?

Since the observed value of Ω is close to unity at present, and since $\Omega \to 1$ in the limit $a \to 0$, the Universe must have started with a density incredibly close to the critical density. Or, stated differently, an incredible fine tuning of the initial values for the expansion velocity H and the energy density ρc^2 are required to make the Universe expand as much as it really has done, during several billion years, without the energy density becoming very small relative to the critical density.

Such a fine tuning of the initial value for the parameter Ω is hard to imagine, without some mechanism that is responsible. One mechanism that has been proposed is inflation, that is, an exponential expansion of the Universe, driven by a positive value of the cosmological constant, or by some physical field having the same effect.

Newtonian cosmology

The cosmological constant Λ belongs to the general theory of relativity, and is a foreign element in Newton's gravitational theory, even though it represents nothing more strange than a constant density of vacuum energy. If we take $\Lambda = 0$ in Equation (19.82), the resulting equation may be derived from Newton's theory, without any reference to the general theory of relativity. It states simply that the non-relativistic energy of a small "test particle" is constant.

We assume that the Universe is perfectly homogeneous and isotropic, and we imagine a spherical region of radius r, not too large. In the expansion of the Universe, the radius r increases and the mass density ρ decreases. The mass inside the volume is

$$M = \frac{4\pi\rho r^3}{3} \,. \tag{19.88}$$

Consider a small "test mass" m at rest relative to its close environment, on the surface of the spherical volume. Thus, the test mass follows the expansion of the Universe, in such a way that it has a velocity $\dot{r} = Hr$ relative to the centre of the sphere. According to Newton, the masses M and m will attract each other, and the non-relativistic equation of motion states that the energy of the test mass,

$$E = \frac{1}{2}\, m\dot{r}^2 - \frac{GMm}{r}\ , \tag{19.89}$$

is constant. Introducing the Hubble parameter $H = \dot{r}/r$ and the mass density ρ we get from this non-relativistic energy equation that

$$\frac{2E}{mc^2r^2} = \frac{H^2}{c^2} - \frac{8\pi G\rho}{3c^2}\ . \tag{19.90}$$

In order to identify this equation with Equation (19.82), under the assumption that $\Lambda = 0$, we only have to identify

$$\frac{2E}{mc^2r^2} = -\frac{k}{a^2}\ . \tag{19.91}$$

That is, we have to define the scale factor a so that

$$k = -\frac{2Ea^2}{mc^2r^2} = \begin{cases} 0\ , \\ \pm 1\ . \end{cases} \tag{19.92}$$

The non-relativistic reasoning is fully justified as long as the radius r is small enough that the velocity $\dot{r} = Hr$ is non-relativistic. The part of the Universe lying outside the spherical volume has no influence on the motion, as long as the Universe is perfectly rotationally symmetric. This is in fact a well known consequence of Newton's gravitational theory, and is an example of Birkhoff's theorem in the general theory of relativity.

Newton's theory has also something to say about whether the Universe is going to continue expanding indefinitely, or whether the expansion is going to stop and be folowed by a collapse. The decisive question is whether the velocity $\dot{r}$ is larger or smaller than the escape velocity, or equivalently, whether the non-relativistic energy E is positive or negative. For a given relative expansion velocity $H = \dot{r}/r$ there exists a critical mass density $\rho = \rho_c$ such that $E = 0$. If the density is smaller, then $E > 0$, and the Universe will expand forever. If the density is larger, then $E < 0$, and the Universe will expand only for a finite time, before turning around and collapsing.

The critical mass density ρ_c determined from Newton's gravitational equation is precisely the same as we found above, from Einstein's gravitational equation, in the case $\Lambda = 0$. Indeed, it has to be, since we are well within the range of validity of Newton's theory as long as we consider the gravitational forces over distances that are small on cosmic scales.

Acceleration

One consequence of the Friedmann Equations (19.76) and (19.77) is that

$$\frac{\ddot{a}}{a} = \frac{\Lambda c^2}{3} - \frac{4\pi G}{3c^2}\,(\rho c^2 + 3P)\ . \tag{19.93}$$

The right hand side of this equation determines whether the expansion is braked, or rather accelerated, by the gravitational forces. What Einstein and Friedmann have to say on this point, may come as a surprise. In fact, a positive pressure increases the gravitational attraction, so much that it actually *brakes* the expansion. What is needed in order to produce an explosion is, strangely enough, a *negative* pressure P, or a positive cosmological constant Λ, which also represents a negative pressure.

This effect is probably counterintuitive, but may nevertheless be understood from the above non-relativistic reasoning, with the additional input from the special theory of relativity that energy and mass are equivalent. Remember that Equation (19.93) is an equation of motion derived from the Equations (19.76) and (19.77), but it can also be derived from Equation (19.76) together with the conservation law for energy, Equation (19.43).

We have just seen that Equation (19.76) may be derived by a non-relativistic argument, when $\Lambda = 0$. Thus, the key to understanding the effect of the pressure must lie in the energy law, Equation (19.43). The point is that when the imaginary spherical shell of radius r expands or contract, the pressure P performs a work that changes the total energy within the shell. Hence, the gravitational mass M in Equation (19.88) is not constant, but varies with the radius r. According to the law of energy conservation,

$$\mathrm{d}(\rho c^2 r^3) + P\,\mathrm{d}(r^3) = 0\ , \tag{19.94}$$

implying that

$$\frac{\mathrm{d}M}{\mathrm{d}r} = \frac{4\pi}{3}\frac{\mathrm{d}(\rho r^3)}{\mathrm{d}r} = -\frac{4\pi P r^2}{c^2}\ . \tag{19.95}$$

This extra r dependence of the gravitational potential $-GM/r$ strengthens the attraction when $P > 0$. The same effect has a certain importance in compact stars, that is, in white dwarf stars and neutron stars.

19.6 Solutions of the gravitational equation

In order to solve the Friedmann Equations (19.76) and (19.77) and find explisitly how the scale factor a varies with time, we have to eliminate both the energy density ρc^2 and the pressure P from the two equations. In order to eliminate two variables from two equations, and still have one equation left, we need an extra relation between them, in this case what we need is the equation of state.

We have seen above that an equation of state of the form $P = w\rho c^2$, with w a constant numerical factor, describes three of the most interesting special cases. This equation of state, together with the law of energy conservation gives a relation between the density ρ and the scale factor a, Equation (19.44), which we may insert into Equation (19.76). First we write Equation (19.76) on the form

$$\dot{a} = \pm\sqrt{-kc^2 + \frac{\Lambda c^2 a^2}{3} + \frac{8\pi G\rho a^2}{3c^2}}\ . \tag{19.96}$$

As is well known, at present we have $\dot{a} > 0$, so that the sign of the square root at present is plus. Knowing ρ as a function of a, we have here an equation which will give us the time dependence of a.

If $k = 0$ or $k = -1$, meaning that the three dimensional space is open, either flat or hyperbolic, and if furthermore $\Lambda \geq 0$, then all three terms under the root sign are positive, so that $\dot{a}$ can never become zero and switch sign.

If, on the other hand, $k = 1$, and the three dimensional space is closed, if the cosmological constant Λ is either zero or not too much positive, and if $w > -1/3$ implying that $\rho a^2 \to 0$ when $a \to \infty$, then there exists a maximal possible value of a. When the Universe has expanded to this size, the expansion velocity has become zero, and at that time the Universe will start collapsing. In the case $\Lambda = 0$, the Universe will be closed in time (have a finite lifetime) if and only if it is closed in space.

In some special cases, Equation (19.96) has elementary analytic solutions. One such case is when $k = 0$ and $\Lambda = 0$, so that the energy density equals the critical value, $\rho = \rho_c$. Then the equation takes the form

$$\dot{a}\, a^{\frac{1+3w}{2}} = A \; . \tag{19.97}$$

Here A is a constant, entering as an integration constant from the energy conservation law, Equation (19.43). The solution of this equation, with the natural choice for the zero point of the time t, is

$$a(t) = (Bt)^{\frac{2}{3(1+w)}} \qquad \text{with} \qquad B = \frac{3(1+w)}{2}\, A \; . \tag{19.98}$$

It follows that the Hubble parameter is

$$H = \frac{\dot{a}}{a} = \frac{2}{3(1+w)t} \; , \tag{19.99}$$

and that the age of the Universe is proportional to the Hubble time $t_H = 1/H$,

$$t = \frac{2}{3(1+w)}\, t_H \; . \tag{19.100}$$

For relativistic particles, for example photons, we have $w = 1/3$, and

$$a(t) = \sqrt{Bt} \; . \tag{19.101}$$

This was, in the standard model, the time dependenc of the scale factor a during the first few thousand years, while the Universe was radiation dominated.

For non-relativistic particles we have $w = 0$, and

$$a(t) = (Bt)^{\frac{2}{3}} \; . \tag{19.102}$$

The conditions for this relation to hold, seem to be fulfilled here and now, at least approximately. Firstly, the Universe is at present matter dominated, and has been so ever since the radiation dominated epoch ended. Secondly, the observed density is of the same order of magnitude as the critical one, and the cosmological constant is observed to be small, although it may be at present actually larger than the density of visible and invisible non-relativistic matter. We see that the age of the matter dominated Universe is 2/3 of the Hubble time.

The De Sitter Universe

Assume that $k = 0$ and $\rho = 0$, but that $\Lambda > 0$. If the cosmological constant dominates over the contributions from the curvature of the three dimensional space as well as from matter and radiation, means that $w = -1$ in the equation of state $P = w\rho c^2$. Equation (19.97) then takes the form

$$H = \frac{\dot{a}}{a} = A = \text{constant} \,, \tag{19.103}$$

and its solution is an exponential expansion,

$$a(t) = a_0 \, \mathrm{e}^{Ht} \,. \tag{19.104}$$

19.7 Nucleosynthesis

Hydrogen is the raw material from which heavier elements are produced inside stars. Heavier elements were produced also in the early Universe, under rather different conditions. One important difference is that the neutrons needed for building up atomic nuclei do not exist beforehand in a star, but have to be made from protons in the nuclear reactions, whereas free neutrons and protons were both present before nuclei were produced in nuclear reactions in the early Universe.

Neutrons and protons had to coexist, because they can be transmuted into each other in interactions with electrons, positrons, electron neutrinos and antielectron neutrinos,

$$\begin{aligned} \mathrm{e}^- + \mathrm{p} &\leftrightarrow \nu_\mathrm{e} + \mathrm{n} \,, \\ \overline{\nu}_\mathrm{e} + \mathrm{p} &\leftrightarrow \mathrm{e}^+ + \mathrm{n} \,, \\ \mathrm{n} &\leftrightarrow \mathrm{p} + \mathrm{e}^- + \overline{\nu}_\mathrm{e} \,. \end{aligned} \tag{19.105}$$

Although these processes are relatively slow, since they take place by the weak interaction, they could be in equilibrium at the high particle densities existing at that time.

The production of atomic nuclei in the early Universe started with protons and neutrons combining into deuterium nuclei (deuterons, d),

$$\mathrm{n} + \mathrm{p} \leftrightarrow \mathrm{d} + \gamma \,. \tag{19.106}$$

The photon is necessary in the reaction in order to conserve energy and momentum. Next, the deuterons may fuse together to form helium 4 nuclei (α particles),

$$\mathrm{d} + \mathrm{d} \leftrightarrow \alpha + \gamma \,. \tag{19.107}$$

These fusion processes involve both the strong interaction (the nuclear forces holding atomic nuclei together) and the electromagnetic interaction. Other processes from deuterons to helium are more effecient, because they involve only the strong interaction, without photons. They take the detour via ^{3}H (tritium, t), or ^{3}He,

$$\begin{aligned} \mathrm{d} + \mathrm{d} &\leftrightarrow \mathrm{t} + \mathrm{p} \,, \\ \mathrm{t} + \mathrm{d} &\leftrightarrow \alpha + \mathrm{n} \,, \\ \mathrm{d} + \mathrm{d} &\leftrightarrow {}^3\mathrm{He} + \mathrm{n} \,, \\ {}^3\mathrm{He} + \mathrm{d} &\leftrightarrow \alpha + \mathrm{p} \,. \end{aligned} \tag{19.108}$$

It is less important for the fusion processes that the particle densities are high, since they take place by strong and electromagnetic interactions. However, all fusion processes, with the exception of neutron absorption, require high temperature, because they involve two positively charged particles. The electrical repulsion between equal charges represents a "Coulomb barrier" which the particles need high kinetic energy to either "climb over" or "tunnel through" before they may react.

The energy scale is given by the mass differences (we use nuclear masses and not atomic masses),

$$\begin{aligned} m_n - m_p &= 1.2933\,\mathrm{MeV}/c^2\ , \\ m_n + m_p - m_d &= 2.2245\,\mathrm{MeV}/c^2\ , \\ 2m_d - m_\alpha &= 23.8466\,\mathrm{MeV}/c^2\ . \end{aligned} \tag{19.109}$$

Thus, the relevant temperature range is where the thermal energy per particle is a few MeV. That $k_B T = 1\,\mathrm{MeV}$ means that $T = 1.2\,10^{10}\,\mathrm{K}$.

Helium nuclei are very stable, since the mass difference $2m_d - m_\alpha$ is large. If we cool down a "hot soup" of protons and neutrons, a number of protons and as many neutrons will react and form helium nuclei. If the cooling is sufficiently slow that the soup is all the time in thermodynamic equilibrium, then in the end all nucleons will end up in helium nuclei, or in even heavier nuclei. The final products would be iron nuclei, in which the energy per nucleon is minimal. If some protons, or some neutrons, should happen to be left over to begin with, because they find no partners, then some of them would be transmuted, in such a way that all would end up in nuclei, with approximately the same number of protons and neutrons.

That could easily have been the fate of our Universe, in which case the stars, including our own Sun, would have had no hydrogen to burn in order to shine for billions of years. One important reason that it did not happen, was that the Universe expanded fast enough to get out of equilibrium. But this is not the whole story, the theory for the synthesis of light elements in the Big Bang is a story with more than one unexpected twist. The fact that it can be quantitatively verified is perhaps the greatest triumph of the present standard cosmology.

Matter in the Universe does not consist of 100 % helium or 100 % iron, but of roughly 75 % hydrogen and 25 % helium, as measured in kilogram. Thus, for every helium nucleus, containing four nucleons, two protons and two neutrons, there are about 12 single protons. Out of 16 nucleons, there are 14 protons and 2 neutrons. For every neutron in the Universe, there are on the average 7 protons. This imbalance in the numbers of protons and neutrons has its explanation in the fact that the neutron is slightly heavier than the proton.

In thermodynamic equilibrium, the ratio between the number densities of neutrons and protons will be given roughly by the exponential Boltzmann factor

$$\frac{n_\mathrm{n}}{n_\mathrm{p}} = \mathrm{e}^{\frac{-\Delta E}{k_B T}}\ , \tag{19.110}$$

in which the energy difference ΔE corresponds to the mass difference between a neutron and a proton,

$$\Delta E = (m_n - m_p)c^2 = 1.3\,\mathrm{MeV}\ . \tag{19.111}$$

Hence, the ratio between the numbers of neutrons and protons was very close to one, until the temperature dropped enough that the thermal energy $k_B T$ approached 1 MeV. After that

happened, the ratio would drop rather fast with the decreasing temperature, as long as the density was high enough, and the expansion slow enough, that the processes (19.105) were still in equilibrium.

It so happened, by a remarkable coincidence, that when k_BT was slightly below 1 MeV, and the neutron to proton ratio was around 1/5, the ratio "froze" to this value, because the transmutation processes became too slow, relative to the rate of cooling by expansion, and could no longer maintain the temperature dependent equilibrium ratio. After the neutron to proton ratio had frozen, it dropped slowly from 1/5 to about 1/7, before all the neutrons had time to get bound in stable nuclei. The ratio continued to drop mainly because a free neutron is an unstable particle, with a mean lifetime of 887 s.

It seems therefore that the skew ratio between the numbers of neutrons and protons in our Universe is the accidental result of a great coincidence, that the weak interaction has just the right strength. Had it been weaker, the ratio would have frozen earlier, very close to one. And had it been stronger, again there would have been hardly any free protons left, because a free proton would have been quickly transformed into a neutron and absorbed in a nucleus. But this is not yet the whole story of Big Bang nucleosynthesis.

One might expect that the production of deuterons would start immediately when k_BT dropped below 2.2 MeV, which is the binding energy of the deuteron. However, at this temperature the ratio between neutrons and protons was still close to one. Apparently, the deuteron production did not start until k_BT was well below 1 MeV, and again a rather special explanation is called for. Gamow's explanation in 1948 was that the entropy per nucleon in the early Universe was extremely high, and that this prevented the nucleons from being caught up inside nuclei.

The carriers of entropy were the photons, and the photon gas had so much entropy because the number of photons per nucleon was extremely high, somewhere between 10^9 and 10^{10}. As a result, the equilibrium conditions for the process (19.106) prohibited the production of deuterons until the temperature had dropped well below 2.2 MeV. Stated in a different, equivalent way, even after the average energy per photon was below 1 MeV, there were still sufficiently many photons with energy above 2.2 MeV that they could break up the deuterons that were forming.

Since the nucleosynthesis took place outside of equilibrium, detailed and complicated computations are needed for a quantitative description. But the main features can be understood without too much effort. The effective number of degrees of freedom, g_*, at a temperature of 1 MeV, or 10^{10} K, is found by the following enumeration. The number of bosonic degrees of freedom is $g_b = 2$, because photons are the only relativistic bosons. A photon has two possible polarization directions, that is, two internal degrees of freedom. The number of fermion degrees of freedom is $g_f = 10$, because the relativistic fermions are electrons and positrons, each with two internal degrees of freedom, since they have spin 1/2, plus three types of neutrinos, the electron, muon, and tau neutrino, each with its own antiparticle. Every neutrino or antineutrino has one internal spin degree of freedom, even though they have spin 1/2, because there exist, as far as we know, only left handed neutrinos and right handed antineutrinos. This gives altogether

$$g_* = g_b + \frac{7}{8}\, g_f = \frac{43}{4}\,. \tag{19.112}$$

Hence, the energy density should be

$$\rho c^2 = g_* \frac{\pi^2 (k_B T)^4}{30(\hbar c)^3} = 4.3\,10^{37}\,\mathrm{MeV/m^3}\ , \tag{19.113}$$

equivalent to a mass density of

$$\rho = 7.6\,10^7\,\mathrm{kg/m^3}\ , \tag{19.114}$$

about the density of iron at atmospheric temperature and pressure.

This should be close to the actual energy density, although it is strictly speaking a lower limit, since we have not included the non-relativistic particles, that is, the protons and neutrons, either free or bound inside atomic nuclei. Every baryon (proton or neutron) has an energy of 940 MeV, compared to an average energy of 1 MeV for each of the lighter particles at this temperature. However, the baryon number was very much lower than the number of photons, electrons, positrons, neutrinos and antineutrinos. Even though they have not been observed, there might also exist right handed neutrinos and left handed antineutrinos that should have been counted.

Knowing that the density was very nearly critical, we may compute the Hubble parameter as

$$H = \sqrt{\frac{8\pi G\rho}{3}} = 0.21/s\ . \tag{19.115}$$

The Hubble time $t_H = 1/H = 4.8\,\mathrm{s}$ gives a rough time scale: those processes that required longer times, at the given density, would not have time to reach equilibrium. The age of the Universe was half of the Hubble time during the radiation dominated epoch, that is, it was 2.4 s when $k_B T = 1\,\mathrm{MeV}$.

Problems

1. One possible generalization of the Friedmann–Robertson–Walker metric, Equation (19.28), is to allow the three dimensional curvature parameter k to vary with time.
   Show that if you take k to be time dependent, the equation $G_{01} = \Lambda g_{01} + \kappa T_{01} = 0$ implies that k is constant.

Appendix A

A simple Maple program for differential geometry

The computer algebra program Maple may be of great help in the often very lengthy computations in differential geometry. Here is an example of a Maple program which computes:

– the Ricci tensor $R_{\mu\nu}$ (`R[i,j]` in the program);
– the scalar curvature R (`Rscalar`);
– the Einstein tensor $G_{\mu\nu}$ (`G[i,j]`).

for a given metric. Notation:

– the dimension (number of coordinates) is `ndim`;
– the coordinates are `x[1],..,x[ndim]`;
– the differential of `x[i]` is `dx[i]`;
– the metric tensor with lower indices, $g_{\mu\nu}$, is `gmat[i,j]`;
– the metric tensor with upper indices, $g^{\mu\nu}$, is `gmatinv[i,j]`.

The required input data are:

– the dimension `ndim`;
– the line element on the form `ds2 = sum(i,j=1..ndim) gmat[i,j]*dx[i]*dx[j]`;
– the signature of the metric, `gsign` ($= +1$ or -1);
– the square root of the absolute value of the determinant of the metric tensor, `srgdet`.

The output data are:

– The Ricci tensor on the forme `Ricci = sum(i,j=1..ndim) R[i,j] A[i] A[j]`,
  where `A[1],..,A[ndim]` are the components of a contravariant vector
  (arbitrary, except that they do not depend on the coordinates);
– the scalar curvature `Rscalar`;
– the Einstein tensor on the form `Einstein = sum(i,j=1..ndim) G[i,j] A[i] A[j]`.

Here is the Maple program.
Read in the (old) Maple program package for linear algebra.

```
> with(linalg):
```

Define a procedure `gmatdef` that computes the matrix `gmat` (the metric tensor) from the given line element `ds2`.

```
> gmatdef := proc() global ds2; local a,b,c,d,i,j;
>   a := [];
>   for i from 1 to ndim do
>     b := [];  c := diff(ds2,dx[i])/2;
>     for j from 1 to ndim do
>       d := diff(c,dx[j]);  b := [op(b),d];
>     od;
>     a := [op(a),b];
>   od;
>   eval(matrix(a));
> end:
```

Here are some examples of input data.
Choose one of them, or make your own.
Example 1:

```
> ndim := 2:
> gsign := 1:
> ds2 := (dx[1]^2/(1-kk*x[1]^2))+x[1]^2*dx[2]^2:
> srgdet := x[1]/sqrt(1-kk*x[1]^2):
```

Example 2: a general two dimensional metric with negative signature (in a special coordinate system).

```
> ndim := 2:
> gsign := -1:
> ds2 := 2*exp(ff(x[1],x[2]))*dx[1]*dx[2]:
> srgdet := exp(ff(x[1],x[2])):
```

Example 3: a general two dimensional metric with positive signature (in a special coordinate system).

```
> ndim := 2:
> gsign := 1:
> ds2 := exp(ff(x[1],x[2]))*(dx[1]^2+dx[2]^2):
> srgdet := exp(ff(x[1],x[2])):
```

Example 4: a homogeneous three dimensional space.

```
> ndim := 3:
> gsign := 1:
> ds2 := aa^2*((dx[1]^2/(1-kk*x[1]^2))+x[1]^2*(dx[2]^2+sin(x[2])^2*dx[3]^2));
> srgdet := aa^3*x[1]^2*sin(x[2])/sqrt(1-kk*x[1]^2);
```

$$ds2 := aa^2 \left(\frac{{dx_1}^2}{1 - kk\, {x_1}^2} + {x_1}^2 \left({dx_2}^2 + \sin(x_2)^2 \; {dx_3}^2 \right) \right)$$

$$srgdet := \frac{aa^3 \, {x_1}^2 \sin(x_2)}{\sqrt{1 - kk\, {x_1}^2}}$$

Example 5: the Friedmann–Robertson–Walker metric.

```
> ndim := 4:
> gsign := -1:
> ds2 := dx[1]^2-a(x[1])^2*((dx[2]^2/(1-kk*x[2]^2))
>        +x[2]^2*(dx[3]^2+sin(x[3])^2*dx[4]^2)):
> srgdet := a(x[1])^3*x[2]^2*sin(x[3])/sqrt(1-kk*x[2]^2):
```

Here we start the computations.
We compute the metric tensor, `gmat`.

```
> gmat := gmatdef();
```

$$gmat := \begin{bmatrix} \frac{aa^2}{1-kk\,x_1{}^2} & 0 & 0 \\ 0 & aa^2\,x_1{}^2 & 0 \\ 0 & 0 & aa^2\,x_1{}^2\sin(x_2)^2 \end{bmatrix}$$

We compute the inverse metric tensor, `gmatinv`.

```
> gmatinv := inverse(gmat);
```

$$gmatinv := \begin{bmatrix} -\frac{-1+kk\,x_1{}^2}{aa^2} & 0 & 0 \\ 0 & \frac{1}{aa^2\,x_1{}^2} & 0 \\ 0 & 0 & \frac{1}{\sin(x_2)^2\;aa^2\,x_1{}^2} \end{bmatrix}$$

This is a check, the result ought to be 0.

```
> simplify(det(gmat)-gsign*srgdet^2);
```

$$0$$

We compute part III of the Ricci tensor (see Chapter 6).

```
> xxx := expand(ln(srgdet));
> yyy := 0:
> for i from 1 to ndim do
>   xxxx:=diff(xxx,x[i]); gam[i]:=xxxx; yyy:=yyy+A[i]*xxxx; od:
> zzz := 0:
> for i from 1 to ndim do zzz:=zzz+A[i]*diff(yyy,x[i]) od:
> RIII := -expand(zzz);
```

$$xxx := 3\ln(\,aa\,) + 2\ln(x_1) + \ln(\sin(x_2)) - \frac{1}{2}\ln\left(1-kk\,x_1{}^2\right)$$

$$RIII := 2\,\frac{A_1{}^2}{x_1{}^2} - \frac{A_1{}^2\,kk}{1-kk\,x_1{}^2} - 2\,\frac{A_1{}^2\,kk^2\,x_1{}^2}{(1-kk\,x_1{}^2)^2} + A_2{}^2 + \frac{A_2{}^2\cos(x_2)^2}{\sin(x_2)^2}$$

We compute part I of the Ricci tensor.

```
> for i from 1 to ndim do B[i]:=sum(gmat[i,iq]*A[iq],iq=1..ndim) od:
> AA := sum(A[iq]*B[iq],iq=1..ndim);
> for i from 1 to ndim do
> for j from 1 to ndim do C[i,j]:=diff(B[i],x[j]) od; od;
```

$$AA := \frac{A_1{}^2\,aa^2}{1-kk\,x_1{}^2} + A_2{}^2\,aa^2\,x_1{}^2 + A_3{}^2\,aa^2\,x_1{}^2\sin(x_2)^2$$

```
> for i from 1 to ndim do
>   xxxa[i] := sum(A[iq]*(C[i,iq]-C[iq,i]/2),iq=1..ndim) od:
> for i from 1 to ndim do
>   xxxb[i] := sum(gmatinv[i,iq]*xxxa[iq],iq=1..ndim) od:
> xxx := 0:
> for i from 1 to ndim do xxx := xxx+diff(xxxb[i],x[i])+gam[i]*xxxb[i] od:
> RI := xxx;
```

$$RI := -2\,\frac{kk\,x_1\left(\dfrac{{A_1}^2\,aa^2\,kk\,x_1}{\left(1-kk\,{x_1}^2\right)^2}-{A_2}^2\,aa^2\,x_1-{A_3}^2\,aa^2\,x_1\sin(x_2)^2\right)}{aa^2}-$$
$$\left(-1+kk\,{x_1}^2\right)$$
$$\left(4\,\frac{{A_1}^2\,aa^2\,kk^2\,{x_1}^2}{\left(1-kk\,{x_1}^2\right)^3}+\frac{{A_1}^2\,aa^2\,kk}{\left(1-kk\,{x_1}^2\right)^2}-{A_2}^2\,aa^2-{A_3}^2\,aa^2\sin(x_2)^2\right)\Big/aa^2$$
$$-\left(2\,\frac{1}{x_1}+\frac{kk\,x_1}{1-kk\,{x_1}^2}\right)\left(-1+kk\,{x_1}^2\right)$$
$$\left(\frac{{A_1}^2\,aa^2\,kk\,x_1}{\left(1-kk\,{x_1}^2\right)^2}-{A_2}^2\,aa^2\,x_1-{A_3}^2\,aa^2\,x_1\sin(x_2)^2\right)\Big/aa^2$$
$$+\frac{-{A_3}^2\,aa^2\,{x_1}^2\cos(x_2)^2+{A_3}^2\,aa^2\,{x_1}^2\sin(x_2)^2}{aa^2\,{x_1}^2}$$
$$+\frac{\cos(x_2)\left(2\,A_1\,aa^2\,x_1\,A_2-{A_3}^2\,aa^2\,{x_1}^2\sin(x_2)\cos(x_2)\right)}{\sin(x_2)\,aa^2\,{x_1}^2}$$

We compute part II of the Ricci tensor.

```
> for i from 1 to ndim do
>   DD[i,i] := 0;
>   xxx := 0;
>   for k from 1 to ndim do xxx := xxx+A[k]*diff(gmat[i,i],x[k]) od;
>   EE[i,i] := xxx;
>   for j from i+1 to ndim do
>     DD[i,j] := C[i,j]-C[j,i]; DD[j,i] := -DD[i,j];
>     xxx := 0;
>     for k from 1 to ndim do xxx := xxx+A[k]*diff(gmat[i,j],x[k]) od;
>     EE[i,j] := xxx; EE[j,i] := xxx;
>   od
> od:
> RII := sum(sum(sum(sum(
> gmatinv[iq,jq]*gmatinv[kq,lq]*(DD[iq,kq]*DD[jq,lq]-EE[iq,kq]*EE[jq,lq])/4
> ,iq=1..ndim),jq=1..ndim),kq=1..ndim),lq=1..ndim);
```

$$RII := -\frac{\left(-1+kk\,{x_1}^2\right)^2\,{A_1}^2\,kk^2\,{x_1}^2}{\left(1-kk\,{x_1}^2\right)^4}-2\left(-1+kk\,{x_1}^2\right)\,{A_2}^2$$
$$-2\left(-1+kk\,{x_1}^2\right)\,{A_3}^2\sin(x_2)^2-\frac{{A_1}^2}{{x_1}^2}+2\,{A_3}^2\cos(x_2)^2$$
$$-\frac{1}{4}\,\frac{\left(2\,A_1\,aa^2\,x_1\sin(x_2)^2+2\,A_2\,aa^2\,{x_1}^2\sin(x_2)\cos(x_2)\right)^2}{\sin(x_2)^4\,aa^4\,{x_1}^4}$$

We sum and simplify.

```
> xxx := RI+RII+RIII: zzz := 0: www := 0:
> for i from 1 to ndim do yyy := diff(xxx,A[i]);
>   uuu := factor(diff(yyy,A[i]))/2;
>   zzz := zzz+uuu*A[i]^2;
>   www := www+uuu*gmatinv[i,i];
>   for j from i+1 to ndim do
>     uuu := factor(diff(yyy,A[j]));
>     zzz := zzz+uuu*A[i]*A[j];
>     www := www+uuu*gmatinv[i,j]
```

```
>   od
> od:
> Ricci := zzz;
> Rscalar := factor(www);
> xxx := Ricci-(1/2)*AA*Rscalar:
> zzz := 0:
> for i from 1 to ndim do yyy := diff(xxx,A[i]);
>   uuu := factor(diff(yyy,A[i]))/2;
>   zzz := zzz+uuu*A[i]^2;
>   for j from i+1 to ndim do
>     uuu := factor(diff(yyy,A[j]));
>     zzz := zzz+uuu*A[i]*A[j]
>   od
> od:
> Einstein := zzz;
```

$$Ricci := -2\,\frac{kk\,{A_1}^2}{-1+kk\,{x_1}^2}+2\,kk\,{x_1}^2\,{A_2}^2+2\,kk\,{x_1}^2\,{A_3}^2\sin\left(x_2\right)^2$$

$$Rscalar := 6\,\frac{kk}{aa^2}$$

$$Einstein := \frac{kk\,{A_1}^2}{-1+kk\,{x_1}^2}-kk\,{x_1}^2\,{A_2}^2-kk\,{x_1}^2\,{A_3}^2\sin\left(x_2\right)^2$$

Appendix B

Group theory

This appendix gives a very brief summary of some central concepts of group theory, which is the mathematical theory of symmetry transformations.

B.1 The group axioms

A *group* in the mathematical sense is a set G of group elements, with a mathematical operation of *group multiplication* which assigns to any two group elements f and g a third element fg. The number of elements in G is denoted by $|G|$, it is the *order* of the group.

Three simple axioms define what it means to call G a group:

(i) G contains a unit element e such that $eg = g$ for all $g \in G$.

(ii) Every $g \in G$ has an inverse, g^{-1}, such that $g^{-1}g = e$.

(iii) The group multiplication is *associative*: $f(gh) = (fg)h$ for all $f, g, h \in G$.

If in addition the group multiplication is *commutative*, that is, if $fg = gf$ for all $f, g \in G$, then we have a *commutative* group, also called an *Abelian* group. The non-Abelian groups are usually the most interesting cases.

That the group multiplication is associative, means that it is unnecessary to write parentheses in a product of more than two factors. In particular, powers are uniquely defined: $g^2 = gg$, $g^3 = gg^2 = g^2g$, $g^4 = gg^3 = g^2g^2 = g^3g$, and so on.

It follows from the axioms that $e^2 = e$. The equation $g^2 = g$ has $g = e$ as its only solution, because it implies that

$$g = eg = (g^{-1}g)g = g^{-1}g^2 = g^{-1}g = e \; . \tag{B.1}$$

Thus, a group can not contain more than one unit element. It follows that $gg^{-1} = e$, since

$$(gg^{-1})^2 = (gg^{-1})(gg^{-1}) = g((g^{-1}g)g^{-1}) = g(eg^{-1}) = gg^{-1} \; . \tag{B.2}$$

Furthermore,

$$ge = g(g^{-1}g) = (gg^{-1})g = eg = g \; . \tag{B.3}$$

The inverse of any $g \in G$ is unique, because if $fg = e$, then

$$f = fe = f(gg^{-1}) = (fg)g^{-1} = eg^{-1} = g^{-1} \; . \tag{B.4}$$

Since the inverse is unique, and $gg^{-1} = e$, it follows that $(g^{-1})^{-1} = g$. It also follows that $(fg)^{-1} = g^{-1}f^{-1}$, since $g^{-1}f^{-1}fg = g^{-1}eg = e$.

Transformation groups

G is a *transformation group* if the group elements of G are transformations over some set X, and if the group multiplication is composition of functions. To be more precise, any two elements $f, g \in G$ are functions, $f : X \to X$ and $g : X \to X$, and the product $h = fg$ is the function $h : X \to X$ defined such that $h(x) = f(g(x))$ for all $x \in X$.

The unit element of a transformation group is the identity transformation, often denoted by I,

$$e(x) = I(x) = x \quad \text{for all} \quad x \in X \; . \tag{B.5}$$

Every function $f \in G$ has to be *invertible*, that is, for every $y \in X$ the equation $f(x) = y$ must have a unique solution for $x \in X$. Then the inverse function f^{-1} is defined such that

$$f^{-1}(y) = x \quad \text{when} \quad f(x) = y \; . \tag{B.6}$$

The usefulness of the group axioms is due especially to the fact that functional composition is automatically associative: given any three functions $h : X \to Y$, $g : Y \to Z$ and $f : X \to U$, the two composite functions $a = f(gh) : X \to U$ and $b = (fg)h : X \to U$ are identical because they are defined in the same way,

$$a(x) = b(x) = f(g(h(x))) \quad \text{for all} \quad x \in X \; . \tag{B.7}$$

Every group may be regarded as a transformation group. In fact, a group G is a transformation group acting on itself by group multiplication: let $f \in G$ transform $g \in G$ into fg. The transformation $g \mapsto fg$ may be called a *left translation*. The group product $f_1 f_2$ corresponds to the transformation $g \mapsto f_1 f_2 g$, which we obtain by transforming first by f_2 and afterwards by f_1, that is, $g \mapsto f_2 g \mapsto f_1 f_2 g$.

An alternative is to let $f \in G$ act on G by the *right translation* $g \mapsto g f^{-1}$. Here we need to use f^{-1} instead of f, in order that $g \mapsto g(f_1 f_2)^{-1}$ becomes the composite function $g \mapsto g {f_2}^{-1} \mapsto g {f_2}^{-1} {f_1}^{-1}$.

Subgroups

A subset $H \subset G$ is a *subgroup* of G if it is a group with the same group multiplication as in G. Two trivial exemples are $\{e\}$, which is a subgroup of order one, and G itself, which we define to be a subgroup.

A less trivial example is

$$H = \{\, g^n \mid n = 0, \pm 1, \pm 2, \dots \,\} \; , \tag{B.8}$$

for an arbitrary element $g \in G$. We define $g^0 = e$, and $g^{-n} = (g^{-1})^n$ for $n > 0$, and we may show in general that $g^m \, g^n = g^n \, g^m = g^{m+n}$. H is then what we call a *cyclic* subgroup of G, *generated* by g. We say that g is a cyclic element of G, and that G is cyclic, if the cyclic subgroup generated by g, is all of G. A cyclic group is Abelian.

A subgroup H is by definition closed under multiplication: $fg \in H$ whenever $f \in H$ and $g \in H$. Furthermore, the unit element of H has to be the same as in G, because it is a solution of the equation $g^2 = g$, and $g = e$ is the only solution in G. Since inverses are unique, it also follows that H has to be closed under inversion: $g^{-1} \in H$ for every $g \in H$.

Conversely, it is easy to prove that a subset $H \subset G$ is a subgroup if it is closed under multiplication and inversion. Thus, this double condition is both necessary and sufficient. Note that if H is finite, closure under multiplication implies closure under inversion.

Cosets

Given a subgroup $H \subset G$ and an element $g \in G$. We define a *left coset* of H as

$$gH = \{\, gh \mid h \in H \,\}\,, \tag{B.9}$$

and a *right coset* of H as

$$Hg = \{\, hg \mid h \in H \,\}\,. \tag{B.10}$$

The set of all left cosets we call G/H, and the set of all right cosets we call $H\backslash G$.

H itself is a coset, both left and right, since $H = gH = Hg$ for all $g \in H$. Since $e \in H$, and $g = ge = eg$, we always have $g \in gH$ and $g \in Hg$. In particular, $g \in H$ if either $gH = H$ or $Hg = H$.

We have that $gH = fH$ if $f \in gH$, and $Hg = Hf$ if $f \in Hg$. Hence, two left cosets gH and hH are identical, $gH = hH = fH$, if they have at least one common element f, and the same is true for two right cosets Hg and Hh.

It follows that a subgroup $H \subset G$ splits G into disjoint left cosets, and into disjoint right cosets. The number of elements in every coset equals $|H|$, the order of H. If G is of finite order, this means that the order of H must be a factor of the order of G, that is, the ratio $|G|/|H|$ must be an integer.

This result is very interesting if for example $|G|$ is a prime number p. In that case there can exist no subgroup of G apart from the trivial subgroup $\{e\}$, of order one, and the whole group G, of order p. The cyclic subgroup generated by any element $g \neq e$ must be all of G.

A group of prime order must be cyclic, and every element $g \neq e$ must be cyclic.

Homogeneous spaces

Let G be a group with H as a subgroup, and let $X = G/H$. Then G acts as a transformation group over X, such that every $f \in G$ transforms a coset $x = gH$ into $f(x) = fgH$. X is a *homogeneous space* for the group G, that is, G acts *transitively* on X, which means that for two arbitrary cosets $x = gH$ and $y = hH$ there always exists an element $f \in G$ such that $y = f(x)$. The proof is simple: choose $f = hg^{-1}$, then $fg = h$ and $fgH = hH$.

H is the *fixed point group* of the single point $x_0 = H \in X$, that is,

$$H = \{\, g \in G \mid g(x_0) = x_0 \,\}\,. \tag{B.11}$$

Every homogeneous space of a transformation group G may be identified with the quotient set G/H, where H is the fixed point group of an arbitrary point in X. Hence $|X|$, the number of points in a homogeneous space X, must be a divisor of $|G|$, the order of G, when G is finite.

Conjugation classes

If $f, g \in G$ then $h = fgf^{-1}$ is the *conjugate* of g by f. Conjugation preserves the group multiplication,

$$(fg_1f^{-1})(fg_2f^{-1}) = f(g_1g_2)f^{-1}\,. \tag{B.12}$$

If $f = e$ then $g = fgf^{-1}$. In general, $g = fgf^{-1}$ if and only if f and g commute, $gf = fg$. Hence, conjugation is a trivial operation in an Abelian group.

Every element $g \in G$ belongs to a *conjugation class*

$$C(g) = \{\, fgf^{-1} \mid f \in G \,\} \,. \tag{B.13}$$

Two conjugation classes in G are either identical, or they have no elements in common.

A conjugation class in G is a homogeneous space for G, hence its number of elements must be a divisor of $|G|$, when G is finite.

Invariant subgroups

In an Abelian group G there is no difference between left and right cosets: $gH = Hg$ for every subgroup H and every element $g \in G$. Also in a non-Abelian group G it is possible for a subgroup H to have the property that $gH = Hg$ for all $g \in G$. In general we say that a subgroup $H \subset G$ is *invariant*, or *normal*, if the left cosets of H are identical to the right cosets.

That $gH = Hg$ means that for every $h_1 \in H$ there is an $h_2 \in H$ such that $gh_1 = h_2 g$, that is, $h_2 = gh_1 g^{-1}$. Thus, an invariant subgroup of G is invariant under conjugation by an arbitrary element $g \in G$.

We may multiply subsets $A, B \subset G$ by defining

$$AB = \{\, ab \mid a \in A,\ \ b \in B \,\} \,. \tag{B.14}$$

We have for example that $HH = H$ for every subgroup $H \subset G$.

This multiplication turns G/H into a group, which we may call a *quotient group*, whenever H is an invariant subgroup. The essential point is that the product of two cosets $gH = Hg$ and $hH = Hh$ is a coset when H is invariant:

$$(gH)(hH) = g(Hh)H = g(hH)H = (gh)(HH) = (gh)H \,. \tag{B.15}$$

Homomorphism and isomorphism

Let G and H be two groups. A function $\rho : G \to H$ preserving the group multiplication, such that $\rho(fg) = \rho(f)\,\rho(g)$ for all $f, g \in G$, is called a *homomorphism*. The *image* of ρ,

$$\rho(G) = \{\, \rho(g) \mid g \in G \,\} \,, \tag{B.16}$$

is then a subgroup of H, while the *kernel* of ρ,

$$K = \rho^{-1}(e_H) = \{\, g \in G \mid \rho(g) = e_H \,\} \,, \tag{B.17}$$

is a subgroup of G. We write e_G and e_H in order to distinguish the unit elements of G and H. It is necessary that $\rho(e_G) = e_H$, because $(\rho(e_G))^2 = \rho(e_G)$.

The homomorphism $\rho : G \to H$ is an *isomorphism* if $\rho(G) = H$ and $\rho^{-1}(e_H) = \{e_G\}$. In this case ρ is an invertible function, and the function $\rho^{-1} : H \to G$ is also an isomorphism. Two groups G and H are *isomorphic* when there exists an isomorphism between them. In a sense two isomorphic groups G and H are the same group: the two multiplication tables are identical, except that the group elements have different names.

An *automorphism* of a group G is an isomorphism of the group with itself. Conjugation by an element $f \in G$, that is, the function $\rho : g \mapsto fgf^{-1}$, is an example of an automorphism.

This special type of automorphism is called an *inner* automorphism, and all the rest are then *outer* automorphisms.

All the automorphisms of a group G form a group, this is the symmetry group of the group G. The inner automorphisms form an invariant subgroup of the automorphism group.

The kernel K of a homomorphism $\rho : G \to H$ is not only a subgroup of G, it is an invariant subgroup. Hence G/K is a group. The homomorphism ρ from G to H is an isomorphism between $\rho(G)$ and the quotient group G/K.

B.2 Examples

The symmetric group

Given two sets X and Y. That a function $f : X \to Y$ is invertible means that there exists an inverse function $f^{-1} : Y \to X$ such that $f^{-1}(f(x)) = x$ for all $x \in X$ and $f(f^{-1}(y)) = y$ for all $y \in Y$.

A *permutation* of the elements in a set X is an invertible function $f : X \to X$. All the permutations over X together form a group, the *symmetric group* S_X. Every transformation group over X is a subgroup of S_X. In particular, every group G is (isomorphic to) a subgroup of S_G.

S_X and S_Y are isomorphic groups if there exists an invertible function $f : X \to Y$. Then to every permutation $g \in S_X$ corresponds a permutation $fgf^{-1} \in S_Y$, and the mapping $g \mapsto fgf^{-1}$ is an isomorphism.

Matrix groups. The General Linear group

Matrix multiplication is associative, and many of the groups we meet in physics are matrix groups, consisting of $n \times n$ matrices for some $n = 1, 2, 3, \ldots$. The matrix elements may be real or complex. The identity matrix

$$I = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} \tag{B.18}$$

is the unit element of a matrix group, $IA = AI$ for all $n \times n$ matrices A.

Every matrix A in a matrix group has to be invertible, that is, the inverse matrix A^{-1} has to exist, or equivalently, we must have $(\det A) \neq 0$. Every matrix group of real or complex $n \times n$ matrices is a subgroup of the *General Linear* group $\mathrm{GL}(n, \mathbf{C})$, consisting of all complex $n \times n$ matrices with nonzero determinant.

A subgroup of $\mathrm{GL}(n, \mathbf{C})$ is for example $\mathrm{GL}(n, \mathbf{R})$, consisting of all invertible real $n \times n$ matrices. Two other subgroups are the *Special Linear* groups $\mathrm{SL}(n, \mathbf{C})$ and $\mathrm{SL}(n, \mathbf{R})$, consisting of all complex, or real, $n \times n$ matrices having determinant equal to 1. $\mathrm{SL}(n, \mathbf{R})$ is a subgroup of $\mathrm{GL}(n, \mathbf{R})$ as well as of $\mathrm{SL}(n, \mathbf{C})$.

The Orthogonal group

Yet other subgroups of $\mathrm{GL}(n, \mathbf{C})$ are the *Orthogonal* groups $\mathrm{O}(n, \mathbf{C})$ and $\mathrm{O}(n) = \mathrm{O}(n, \mathbf{R})$, consisting of all orthogonal $n \times n$ matrices, complex or real. A matrix A is orthogonal when

it is invertible and

$$A^{\mathrm{T}} = A^{-1} \,. \tag{B.19}$$

A^{T} is the transpose of A (a transposition is a reflection about the matrix diagonal). When A is a finite dimensional matrix we may take the determinant of the above equation and obtain the equation $\det A = (\det A)^{-1}$, which has the two solutions $\det A = \pm 1$. The *Special Orthogonal* groups $\mathrm{SO}(n, \mathbf{C})$ and $\mathrm{SO}(n) = \mathrm{SO}(n, \mathbf{R})$ are subgroups of $\mathrm{O}(n, \mathbf{C})$ and $\mathrm{O}(n) = \mathrm{O}(n, \mathbf{R})$ consisting of those matrices having determinant $+1$.

The condition $A^{\mathrm{T}} = A^{-1}$, or $A^{\mathrm{T}}A = I$, may be generalized to

$$A^{\mathrm{T}}GA = G \,, \tag{B.20}$$

where G is an invertible matrix, a *metric*. With $G = I$ this defines again the orthogonal groups $\mathrm{O}(n, \mathbf{C})$ and $\mathrm{O}(n, \mathbf{R})$. If G is diagonal and has m times $+1$ and $n-m$ times -1 on its diagonal, this defines the pseudo-orthogonal groups $\mathrm{O}(m, n-m, \mathbf{C})$ and $\mathrm{O}(m, n-m) = \mathrm{O}(m, n-m, \mathbf{R})$. Note that the groups $\mathrm{O}(m, n-m, \mathbf{C})$ and $\mathrm{O}(n-m, m, \mathbf{C})$ are isomorphic (why?). A subgroup of $\mathrm{O}(m, n-m)$ is again $\mathrm{SO}(m, n-m)$, for example.

The group $\mathrm{O}(3)$, as an example, consists of all rotations and reflections in our familiar three dimensional space, whereas $\mathrm{SO}(3)$ consists of the rotations alone, without the reflections. $\mathrm{O}(1,3)$ is the group of all Lorentz transformations, including the reflections P, T and PT, whereas $\mathrm{SO}(1,3)$ includes the space and time inversion PT, but not the space inversion P or the time reversion T.

The Symplectic group

The condition $A^{\mathrm{T}}GA = G$ defines the *Symplectic* groups $\mathrm{Sp}(2n, \mathbf{C})$ and $\mathrm{Sp}(2n, \mathbf{R})$ if G is an antisymmetric and invertible $(2n) \times (2n)$ matrix, for example,

$$G = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \tag{B.21}$$

where we take the zero matrix 0 and the identity matrix I to be $n \times n$ matrices.

The Unitary group

The *Unitary* group $\mathrm{U}(n)$ consists of all unitary $n \times n$ matrices. A complex matrix A is unitary if

$$A^{\dagger} = A^{-1} \,. \tag{B.22}$$

Here $A^{\dagger}$ is the Hermitean conjugate of A, that is, the transposed and complex conjugated matrix. With finite dimensional matrices we may take the determinant of the equation $A^{\dagger} = A^{-1}$ and obtain the equation $(\det A)^* = (\det A)^{-1}$, with the solution $|\det A| = 1$, or equivalently $\det A = \mathrm{e}^{\mathrm{i}\alpha}$ with α real. The *Special Unitary* group $\mathrm{U}(n)$ consists of those unitary matrices having determinant $+1$. The unitary groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ may be generalized to $\mathrm{U}(m, n-m)$ and $\mathrm{SU}(m, n-m)$, in a similar way as with the orthogonal groups.

B.3 Lie groups

All matrix groups mentioned above are examples of *Lie groups*. A Lie group is a group in which the group elements depend on one or more continuous parameters, in such a way that the group operations, multiplication and inversion, are continuous and differentiable.

It is natural here to confine our discussion to the *connected* Lie groups. In a Lie group G we may start with the unit element e and vary all parameters continuously. In that way we may reach all elements of G, if G is connected. Otherwise, we generate a subset $G_0 \subset G$, and G_0 is a connected Lie group, which is an invariant subgroup of G. How to extend the subgroup G_0 so as to obtain the whole group G is then a problem in general group theory which is not of special interest in the theory of Lie groups.

The fundamental result in the theory of Lie groups may be formulated as follows:

> *The group multiplication in a connected Lie group is (in essence) uniquely determined by the group multiplication in an arbitrarily small neighbourhood of the unit element, and this again is determined by the Lie algebra of the group.*

We will now see, in rough outline, how this result comes about, what a Lie algebra is, and what the connection is between the Lie group and its Lie algebra.

To be slightly more specific let us consider a connected Lie group G where the group elements are real matrices of a given dimension $n \times n$, and where the group multiplication is matrix multiplication. This is anyway no serious restriction, since essentially all Lie groups may be represented as matrix groups (some Lie groups can not be faithfully represented as groups of finite dimensional matrices). The unit element of G is the identity matrix I. The inverse matrix A^{-1} has to exist for every $A \in G$, thus we must have $(\det A) \neq 0$.

One parameter groups

Looking at small neighbourhoods about the identity I means in the limit looking at *infinitesimal* neighbourhoods. A group element infinitesimally close to I is of the form

$$A(\alpha) = I + \alpha X \ , \tag{B.23}$$

where X is a finite matrix and α is an infinitesimal parameter. Then X is what we call a *generator*, or *infinitesimal generator*, of the matrix group. By multiplying together infinitely many matrices infinitesimally close to I we may generate the whole group.

We may formulate this somewhat vague idea more precisely by means of the exponential function, which may be defined by either one of the two expressions

$$\mathrm{e}^X = \lim_{k\to\infty} \left(I + \frac{X}{k}\right)^k = I + X + \frac{X^2}{2} + \cdots + \frac{X^k}{k!} + \cdots \ . \tag{B.24}$$

What happens in the limit $k \to \infty$ is, loosely speaking, that X/k becomes infinitesimal, and we get a product of infinitely many factors all infinitesimally different from I. The infinite series is always convergent, for any matrix X. The exponential function of matrices is a direct generalization of the exponential function of numbers (a number is just a 1×1 matrix).

Thus, the generalization of Equation (B.23), where α is infinitesimal, is

$$A(\alpha) = \mathrm{e}^{\alpha X} \ , \tag{B.25}$$

where α takes finite values. In this case the usual rule for the exponential function holds, that

$$A(\alpha)\,A(\beta) = A(\alpha+\beta)\ . \tag{B.26}$$

Note that $A(0) = I$, and that $(A(\alpha))^{-1} = A(-\alpha)$, because $A(-\alpha)\,A(\alpha) = I$. This means that the elements $A(\alpha) \in G$ form a subgroup of G. Every generator X of the Lie group defines in this way its own *one parameter subgroup*. Such one parameter subgroups in a Lie group correspond to cyclic subgroups in a finite group.

As usual, the logarithm is the inverse of the exponential function,

$$\mathrm{e}^{\ln X} = \ln\Bigl(\mathrm{e}^{X}\Bigr) = X\ . \tag{B.27}$$

Its series expansion is the usual one,

$$\ln(I+X) = X - \frac{X^2}{2} + \frac{X^3}{3} - \cdots + (-1)^{k+1}\,\frac{X^k}{k} + \cdots\ . \tag{B.28}$$

However, this series converges only when the matrix X is "small". More precisely, it converges if all the eigenvalues of X have absolute values smaller than 1, and it diverges otherwise.

It follows from the definition of the exponential function that

$$\det\Bigl(\mathrm{e}^{X}\Bigr) = \lim_{k\to\infty}\left(\det\left(I+\frac{X}{k}\right)\right)^k = \lim_{k\to\infty}\left(1+\frac{\operatorname{Tr} X}{k}\right)^k = \mathrm{e}^{\operatorname{Tr} X}\ . \tag{B.29}$$

$\operatorname{Tr} X$ is the sum of the diagonal elements of the matrix X. We use the following relation,

$$\begin{aligned}\det(I+\epsilon X) &= \begin{vmatrix} 1+\epsilon X_{11} & \epsilon X_{12} & \dots & \epsilon X_{1n} \\ \epsilon X_{21} & 1+\epsilon X_{22} & \dots & \epsilon X_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \epsilon X_{n1} & \epsilon X_{n2} & \dots & 1+\epsilon X_{nn} \end{vmatrix} \\ &= 1 + \epsilon \operatorname{Tr} X + (\text{terms of order } \epsilon^2)\ .\end{aligned} \tag{B.30}$$

The Campbell–Baker–Hausdorff theorem

The commutator between two matrices X and Y is defined as

$$[X,Y] = XY - YX\ . \tag{B.31}$$

The usual rule that $\mathrm{e}^X\,\mathrm{e}^Y = \mathrm{e}^{X+Y}$ holds without restrictions if X and Y commute, that is, if $[X,Y]=0$. The generalization to the case $[X,Y]\neq 0$ is the so called *Campbell–Baker–Hausdorff theorem*, that $\mathrm{e}^X\,\mathrm{e}^Y = \mathrm{e}^Z$, where

$$Z = X + Y + \frac{1}{2}\,[X,Y] + (\text{commutators of commutators})\ . \tag{B.32}$$

The theorem does not hold for completely arbitrary exponents X and Y, because it may happen that the series for Z is infinite and divergent, but it always holds when X and Y are "sufficiently small". An important assumption in the proof, even though the formula is somewhat more generally valid, is that all the eigenvalues of the matrix

$$U = \mathrm{e}^X\,\mathrm{e}^Y - I \tag{B.33}$$

should be smaller than 1, so that we may use the series expansion to compute $Z = \ln(I+U)$. This calculation is straightforward in principle, but it is not very easy to prove the Campbell–Baker–Hausdorff theorem in such a way, by brute force. Try it!

Here is an alternative proof, using the formula

$$\mathrm{e}^{\beta X}\, Y\, \mathrm{e}^{-\beta X} = Y + \beta\,[X,Y] + \frac{\beta^2}{2}\,[X,[X,Y]] + \cdots + \frac{\beta^k}{k!}\,[X,[X,\ldots[X,Y]\ldots]] + \cdots \,. \tag{B.34}$$

This formula is trivial for $\beta = 0$, and it holds in general because it solves the following first order differential equation for $V(\beta) = \mathrm{e}^{\beta X}\, Y\, \mathrm{e}^{-\beta X}$,

$$\frac{\mathrm{d}V}{\mathrm{d}\beta} = \frac{\mathrm{d}(\mathrm{e}^{\beta X})}{\mathrm{d}\beta}\, Y\, \mathrm{e}^{-\beta X} + \mathrm{e}^{\beta X}\, Y\, \frac{\mathrm{d}(\mathrm{e}^{-\beta X})}{\mathrm{d}\beta} = XV - VX = [X,V]\,. \tag{B.35}$$

Given the matrices X and Y, let us define $W(\alpha)$ such that $W(0) = Y$ and $\mathrm{e}^{W(\alpha)} = \mathrm{e}^{\alpha X}\,\mathrm{e}^{Y}$. Differentiation of this equation and multiplication from the right by $\mathrm{e}^{-W} = \mathrm{e}^{-Y}\,\mathrm{e}^{-\alpha X}$ gives that

$$\frac{\mathrm{d}(\mathrm{e}^{W})}{\mathrm{d}\alpha}\,\mathrm{e}^{-W} = X\,. \tag{B.36}$$

It is tempting to set the left hand side of this equation simply equal to $W' = \mathrm{d}W/\mathrm{d}\alpha$, but that is correct only under the assumption that W and W' commute. To find the correct formula in general we have to go back to the definition of the exponential function. We have that

$$\frac{\mathrm{d}(\mathrm{e}^{W})}{\mathrm{d}\alpha} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(I + \frac{W}{n}\right)^{k-1} \frac{W'}{n} \left(I + \frac{W}{n}\right)^{n-k} = \int_0^1 \mathrm{d}\beta\; \mathrm{e}^{\beta W}\, W'\, \mathrm{e}^{(1-\beta)W}\,, \tag{B.37}$$

and hence,

$$\begin{aligned} X \;&=\; \frac{\mathrm{d}(\mathrm{e}^{W})}{\mathrm{d}\alpha}\,\mathrm{e}^{-W} = \int_0^1 \mathrm{d}\beta\; \mathrm{e}^{\beta W}\, W'\, \mathrm{e}^{-\beta W} \\ &=\; W' + \frac{1}{2}\,[W,W'] + \cdots + \frac{1}{(k+1)!}\,[W,[W,\ldots[W,W']\ldots]] + \cdots\,. \end{aligned} \tag{B.38}$$

We need the inverse formula,

$$W' = X - \frac{1}{2}\,[W,X] + \frac{1}{12}\,[W,[W,X]] + \cdots + \frac{B_k}{k!}\,[W,[W,\ldots[W,X]\ldots]] + \cdots\,, \tag{B.39}$$

where B_k is the k-th Bernoulli number,

$$\begin{aligned} &B_0 \;=\; 1\,, \quad B_1 = -\frac{1}{2}\,, \quad B_2 = \frac{1}{6}\,, \quad B_4 = -\frac{1}{30}\,, \quad B_6 = \frac{1}{42}\,, \\ &B_3 \;=\; B_5 = \ldots = B_{2j+1} = \ldots = 0\,. \end{aligned} \tag{B.40}$$

Observe that the coefficients of the series in Equation (B.38) are the same as in the power series

$$\frac{\mathrm{e}^{x}-1}{x} = 1 + \frac{1}{2}\,x + \frac{1}{6}\,x^2 + \cdots + \frac{1}{(k+1)!}\,x^k + \cdots\,. \tag{B.41}$$

We may then prove Equation (B.39) rather directly using Equation (B.38), together with the definition of the Bernoulli numbers,

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!}\, x^k \ . \tag{B.42}$$

Equation (B.39) is a non-linear differential equation for $W(\alpha)$. The initial value $W(0) = Y$, together with the form of the equation, with commutators, implies the Campbell–Baker–Hausdorff theorem, which we wanted to prove.

In order to see this in a little more detail, we may calculate the third order terms in the Campbell–Baker–Hausdorff formula for $Z = W(1)$, saying that X and Y are of first order and $[X, Y]$ of second order. Differentiation of Equation (B.39) gives that

$$W'' = -\frac{1}{2}\,[W', X] + \frac{1}{12}\,[W', [W, X]] + \frac{1}{12}\,[W, [W', X]] + \cdots \ . \tag{B.43}$$

Inserting $\alpha = 0$ and calculating to third order we get that

$$\begin{aligned} W'(0) \ &= \ X - \frac{1}{2}\,[Y, X] + \frac{1}{12}\,[Y, [Y, X]] + \cdots \ , \\ W''(0) \ &= \ \frac{1}{4}\,[[Y, X], X] + \frac{1}{12}\,[X, [Y, X]] + \cdots = \frac{1}{6}\,[X, [X, Y]] + \cdots \ . \end{aligned} \tag{B.44}$$

Hence we have, to third order,

$$\begin{aligned} Z \ &= \ W(0) + W'(0) + \frac{1}{2}\,W''(0) + \cdots \\ &= \ X + Y + \frac{1}{2}\,[X, Y] + \frac{1}{12}\,[X, [X, Y]] + \frac{1}{12}\,[Y, [Y, X]] + \cdots \ . \end{aligned} \tag{B.45}$$

B.4 Lie algebras

The infinitesimal generators of a Lie group form a real vector space, this means that an arbitrary linear combination $\alpha X + \beta Y$ of two generators X and Y, with arbitrary real coefficients α and β, is a generator. To prove this we consider $A = \mathrm{e}^{\epsilon\alpha X}\,\mathrm{e}^{\epsilon\beta Y}$, where ϵ is a real number. Being a product of two group elements, A itself is a groupelement, and in the limit when ϵ is infinitesimal, we have that

$$A = I + \epsilon\,(\alpha X + \beta Y) \ . \tag{B.46}$$

The commutator $[X, Y]$ is also a generator. To prove this we consider

$$A = \mathrm{e}^{\delta X}\,\mathrm{e}^{\delta Y}\,\mathrm{e}^{-\delta X}\,\mathrm{e}^{-\delta Y} \ , \tag{B.47}$$

which is product of four group elements, and therefore a group element itself. In the limit when $\epsilon = \delta^2$ is infinitesimal it follows from the Campbell–Baker–Hausdorff theorem that

$$A = I + \epsilon\,[X, Y] \ . \tag{B.48}$$

The vector space of infinitesimal generators, with the commutator product, is the *Lie algebra* of the group. The essential contents of the Campbell–Baker–Hausdorff theorem is

that the group multiplication in a small, but *finite*, region around the unit element of a Lie group is completely determined by the Lie algebra.

Note that the fundamental algebraic structure of a Lie algebra is the commutator product $[X, Y]$. The properties characterising the commutator product in a Lie algebra are *bilinearity*:

$$\begin{aligned} [X, \alpha Y + \beta Z] &= \alpha[X, Y] + \beta[X, Z] \ , \\ [\alpha X + \beta Y, Z] &= \alpha[X, Y] + \beta[Y, Z] \ , \end{aligned} \tag{B.49}$$

antisymmetry:

$$[X, Y] = -[Y, X] \ , \tag{B.50}$$

and the *Jacobi identity*:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0 \ . \tag{B.51}$$

All of these properties follow directly from the definition $[X, Y] = XY - YX$. For example, the Jacobi identity follows because the usual matrix product XY is associative, so that

$$[X, [Y, Z]] = XYZ - XZY - YZX + ZYX \ . \tag{B.52}$$

However, it is quite possible to forget about the definition $[X, Y] = XY - YX$ and take instead the three basic properties of the commutator product, the bilinearity, the antisymmetry and the Jacobi identity, as axioms defining a Lie algebra. We may very well have an abstract Lie algebra, with a commutator product $[X, Y]$, where the elements X and Y are no longer matrices, and where the associative product XY is not even defined.

B.5 Examples

The Lie algebras of the general linear groups $\mathrm{GL}(n, \mathbf{R})$ and $\mathrm{GL}(n, \mathbf{C})$ are usually called $\mathrm{gl}(n, \mathbf{R})$ and $\mathrm{gl}(n, \mathbf{C})$.

One basis for $\mathrm{gl}(n, \mathbf{R})$ consists of the n^2 matrices E^j_k, for $j, k = 1, 2, \ldots, n$, with matrix elements

$$(E^j_k)_{rs} = \delta^j_r \delta_{ks} \ . \tag{B.53}$$

Multiplication of two such basis matrices gives either 0 or a third basis matrix,

$$E^j_k \, E^l_m = \delta^l_k \, E^j_m \ . \tag{B.54}$$

This gives the following commutation relations, defining the Lie algebra $\mathrm{gl}(n, \mathbf{R})$,

$$[E^j_k, E^l_m] = \delta^l_k \, E^j_m - \delta^j_m \, E^l_k \ . \tag{B.55}$$

The Lie algebra $\mathrm{gl}(n, \mathbf{C})$ is twice as large as $\mathrm{gl}(n, \mathbf{R})$, since it also contains basis matrices

$$F^j_k = \mathrm{i} E^j_k \ . \tag{B.56}$$

We then get the following new commutation relations,

$$\begin{aligned} [E^j_k, F^l_m] &= \delta^l_k \, F^j_m - \delta^j_m \, F^l_k \ , \\ [F^j_k, F^l_m] &= -\delta^l_k \, E^j_m + \delta^j_m \, E^l_k \ . \end{aligned} \tag{B.57}$$

Appendix C

Statistical mechanics

Here we summarize briefly some of the statistical mechanics used in cosmology.

Given a physical system consisting of different types of particles, with N_i particles of type i. Let $a = 1, 2, \ldots$ enumerate the quantum mechanical states of the system, with variable numbers of particles of each type. In state number a the energy is E_a, and the number of particles of type i is N_{ia}.

The particles are identified by certain additive quantum numbers $Q_1, Q_2, \ldots$, which we choose to call "charges", for example the electric charge q, baryon number B and lepton number L. We assume that all these charges are conserved. A particle of type i has the values $Q_{1i}, Q_{2i}, \ldots$ of the charges, and the total charge Q_J in the state a is

$$Q_{Ja} = \sum_i N_{ia} Q_{Ji} \ . \tag{C.1}$$

We imagine an ensemble of $\mathcal{N}$ such systems, which may exchange both energy and particles, in such a way that the average energy E is fixed, and also the avarage values $Q_1, Q_2, \ldots$ of the various conserved charges. Assume that $\mathcal{N}_a$ of the $\mathcal{N}$ systems are in the state a. Thus, we have that

$$\begin{aligned} \sum_a \mathcal{N}_a &= \mathcal{N} \ , \\ \sum_a \mathcal{N}_a E_a &= \mathcal{N} E \ , \\ \sum_a \mathcal{N}_a Q_{Ja} &= \mathcal{N} Q_J \ , \qquad (J = 1, 2, \ldots) \ . \end{aligned} \tag{C.2}$$

The probability of this distribution of the systems in the ensemble is proportional to the multinomial coefficient

$$\mathcal{W} = \frac{\mathcal{N}!}{\prod_a \mathcal{N}_a!} \ . \tag{C.3}$$

We assume that $\mathcal{N}$ is such a large number that the numbers $\mathcal{N}_a$ separately are large, and hence that we may use the following approximation,

$$\ln \mathcal{W} = \mathcal{N} \ln \mathcal{N} - \mathcal{N} - \sum_a (\mathcal{N}_a \ln \mathcal{N}_a - \mathcal{N}_a) = -\sum_a \mathcal{N}_a (\ln \mathcal{N}_a - \ln \mathcal{N}) \ . \tag{C.4}$$

The last equality follows because $\sum_a \mathcal{N}_a = \mathcal{N}$. We may introduce the dimensionless entropy, defined as

$$\mathcal{S} = \frac{\ln \mathcal{W}}{\mathcal{N}} = -\sum_a \mathcal{P}_a \ln \mathcal{P}_a \ , \tag{C.5}$$

where $\mathcal{P}_a = \mathcal{N}_a/\mathcal{N}$ is the probability of finding a system in the state a if it is picked at random from the ensemble.

We find the most probable distribution by maximizing $\mathcal{W}$, or equivalently the entropy $\mathcal{S}$, under the constraints given in Equation (C.2). We introduce Lagrange multipliers α, β, and γ_J, and solve the equations

$$\frac{\partial}{\partial \mathcal{P}_a}\left(\mathcal{S} - \alpha\left(\sum_a \mathcal{P}_a - 1\right) - \beta\left(\sum_a \mathcal{P}_a E_a - E\right) - \sum_J \gamma_J \left(\sum_a \mathcal{P}_a Q_{Ja} - Q_J\right)\right) = 0 \ . \tag{C.6}$$

The solution is $\mathcal{P}_a = P_a$, with the definition

$$P_a = \mathrm{e}^{-1-\alpha-\beta E_a - \sum_J \gamma_J Q_J} \ . \tag{C.7}$$

P_a is the probability of finding a system in the state a if it is picked at random from an ensemble in equilibrium. The Lagrange multiplier α is chosen so that $\sum_a P_a = 1$. We define the grand canonical partition function as

$$\Xi = \mathrm{e}^{1+\alpha} = \sum_a \mathrm{e}^{-\beta E_a - \sum_J \gamma_J Q_J} \ . \tag{C.8}$$

The Lagrange multiplier β is the inverse of the absolute temperature T,

$$\beta = \frac{1}{k_B T} \ , \tag{C.9}$$

where k_B is Boltzmann's constant. We may define the chemical potential μ_J for the charge Q_J by the relation $\beta\mu_J = -\gamma_J$, and we define the chemical potential for particles of type i as

$$\mu_i = \sum_J \mu_J Q_{Ji} \ . \tag{C.10}$$

Then we may write

$$P_a = \mathrm{e}^{-1-\alpha-\beta(E_a - \sum_i \mu_i N_{ia})} \ . \tag{C.11}$$

Equation (C.10) has interesting consequences. First, we know that every particle type has its own antiparticle with the opposite sign of all charge quantum numbers, and the equation tells us that the chemical potential is of equal magnitude but opposite sign for particle and antiparticle. In particular, the chemical potential must be zero for neutral particles, that is, particles having all charges equal to zero and hence being their own antiparticles. The photon is the most important example of a neutral particle. The neutrino is not a neutral particle, even though it is electrically neutral, because it has lepton number 1.

Second, since all the charge quantum numbers are conserved, the equation tells us that in equilibrium the chemical potential is conserved in all particle reactions (remember that the chemical potentials characterize an equilibrium state, and are undefined outside of equilibrium). For example, an electron (e^-) and a proton (p) may be converted to an electron neutrino (ν_e) and a neutron (n), and vice versa, according to the reaction equation

$$e^- + p \leftrightarrow \nu_e + n \, . \tag{C.12}$$

The equilibrium condition for these two opposite processes is that the chemical potentials satisfy the relation

$$\mu_{e^-} + \mu_p = \mu_{\nu_e} + \mu_n \, . \tag{C.13}$$

All termodynamical quantities may be computed from the partition function

$$\Xi = \Xi(\beta, \mu_1, \mu_2, \ldots) = e^{1+\alpha} = \sum_a e^{-\beta(E_a - \sum_i \mu_i N_{ia})} \, . \tag{C.14}$$

For example the average particle numbers and the average energy of one system,

$$\begin{aligned} \langle N_i \rangle &= \sum_a P_a N_{ia} = \frac{1}{\beta} \frac{\partial}{\partial \mu_i} \ln \Xi \, , \\ \langle E \rangle &= \sum_a P_a E_a = -\frac{\partial}{\partial \beta} \ln \Xi + \sum_i \mu_i \langle N_i \rangle \, . \end{aligned} \tag{C.15}$$

Ideal gas

In an ideal gas the interactions between particles can be neglected. A state a of such a system is specified by all the occupation numbers $N_{i\alpha}$, where the first index i identifies a particle type, and the second index α identifies a one particle state for particles of type i. The one particle energies are $E_{i\alpha}$. Thus, in the state a the number of particles of type i is

$$N_{ia} = \sum_\alpha N_{i\alpha} \, , \tag{C.16}$$

and the energy is

$$E_a = \sum_i \sum_\alpha N_{i\alpha} E_{i\alpha} \, . \tag{C.17}$$

This means that each exponential function in Equation (C.14) factorizes into one particle factors,

$$e^{-\beta(E_a - \sum_i \mu_i N_{ia})} = \prod_i \prod_\alpha e^{-\beta N_{i\alpha}(E_{i\alpha} - \mu_i)} \, . \tag{C.18}$$

The sum over states in Equation (C.14) is a multiple sum over occupation numbers $N_{i\alpha}$, one sum for each index pair $i\alpha$, and the partition function Ξ factorizes into a product

$$\Xi = \prod_i \prod_\alpha \Xi_{i\alpha} \, , \tag{C.19}$$

with

$$\Xi_{i\alpha} = \sum_n \mathrm{e}^{-\beta n(E_{i\alpha}-\mu_i)} \; . \tag{C.20}$$

Since the particles do not interact we may deal with the different particle types one at a time. In order to simplify the discussion we will assume from now on that there is only one type of particles, so that we may leave out the particle index i.

If the particles are fermions, then

$$\Xi_\alpha = 1 + \mathrm{e}^{-\beta(E_\alpha-\mu)} \; , \tag{C.21}$$

because the particle number n in the one particle state α can only be either 0 or 1. For bosons, the number of particles in one and the same state is unlimited, so that

$$\Xi_\alpha = \frac{1}{1-\mathrm{e}^{-\beta(E_\alpha-\mu)}} \; . \tag{C.22}$$

This gives the logarithm of the partition function for an ideal gas of bosons or fermions as a sum over one particle states,

$$\ln \Xi = \mp \sum_\alpha \ln\Big(1 \mp \mathrm{e}^{-\beta(E_\alpha-\mu)}\Big) \, . \tag{C.23}$$

The signs are $-$ for bosons and $+$ for fermions. The average particle number and energy are then given as

$$\begin{aligned} \langle N \rangle \; &= \; \sum_\alpha \frac{1}{\mathrm{e}^{\beta(E_\alpha-\mu)} \mp 1} \; , \\ \langle E \rangle \; &= \; \sum_\alpha \frac{E_\alpha}{\mathrm{e}^{\beta(E_\alpha-\mu)} \mp 1} \; . \end{aligned} \tag{C.24}$$

In a large volume $\mathcal{V}$ the sum over α may be approximated by an integral over the three dimensional momentum $\boldsymbol{p}$, such that

$$\ln \Xi = \mp \frac{g\mathcal{V}}{h^3} \int \mathrm{d}^3\boldsymbol{p} \, \ln\Big(1 \mp \mathrm{e}^{-\beta(E(\boldsymbol{p})-\mu)}\Big) \, . \tag{C.25}$$

The degeneracy factor g is the number of inner degrees of freedom of a particle. The relativistic energy of a particle of mass m and momentum $\boldsymbol{p}$ is

$$E = E(\boldsymbol{p}) = c\sqrt{m^2c^2 + |\boldsymbol{p}|^2} \; . \tag{C.26}$$

This gives the following expressions for the particle number density and energy density,

$$\begin{aligned} n \; &= \; \frac{\langle N \rangle}{\mathcal{V}} = \frac{g}{h^3} \int \mathrm{d}^3\boldsymbol{p} \, \frac{1}{\mathrm{e}^{\beta(E(\boldsymbol{p})-\mu)} \mp 1} \; , \\ \rho c^2 \; &= \; \frac{\langle E \rangle}{\mathcal{V}} = \frac{g}{h^3} \int \mathrm{d}^3\boldsymbol{p} \, \frac{E(\boldsymbol{p})}{\mathrm{e}^{\beta(E(\boldsymbol{p})-\mu)} \mp 1} \; . \end{aligned} \tag{C.27}$$

The integrals may be computed exactly in the special case when both the mass and the chemical potential are zero. That $m = 0$ means that we have a *relativistic* gas, in which all the particles move with the speed of light. Then we have, for an ideal gas of bosons, that

$$\begin{aligned} n &= n_b = g\,\frac{\zeta(3)(k_BT)^3}{\pi^2(\hbar c)^3}\;, \\ \rho c^2 &= \rho_b c^2 = g\,\frac{\pi^2(k_BT)^4}{30(\hbar c)^3}\;. \end{aligned} \tag{C.28}$$

The constant $\zeta(3) = 1.202057\ldots$ is the Riemann zeta function of 3. For an ideal gas of relativistic fermions we have that

$$\begin{aligned} n &= n_f = \frac{3}{4}\,n_b\;, \\ \rho c^2 &= \rho_f c^2 = \frac{7}{8}\,\rho_b c^2\;. \end{aligned} \tag{C.29}$$

The ratio between fermions and bosons is simply a factor of 3/4 in the number density and a different factor of 7/8 in the energy density, independent of the temperature T.

Thus, the energy density of a relativistic gas consisting of several types of particles may be written as

$$\rho c^2 = g_*\,\frac{\pi^2(k_BT)^4}{30(\hbar c)^3}\;,$$

where g_* is the *effective number of degrees of freedom*,

$$g_* = \sum_b g_b + \frac{7}{8}\sum_f g_f\;. \tag{C.30}$$

Here $\sum_b$ is a sum over all types of particles that are bosons, whereas $\sum_f$ is a sum over all types of fermions, with a factor 7/8 which is the ratio between the energy densities of fermions and bosons. The energy density is proportional to g_* and to T^4.

Setting $m = 0$ and $\mu = 0$ will always be a good approximation when the temperature is high, or more precisely when $m << k_BT$ and $|\mu| << k_BT$. This simple formula for the energy density is therefore very interesting and useful, not least in cosmology.

Pressure

The general formula for pressure,

$$P = \frac{1}{\beta\mathcal{V}}\,\ln\Xi\;, \tag{C.31}$$

gives for an ideal gas of bosons or fermions that

$$P = \mp\frac{g}{\beta h^3}\int \mathrm{d}^3\boldsymbol{p}\,\ln\!\left(1 \mp \mathrm{e}^{-\beta(E-\mu)}\right) = \mp\frac{4\pi g}{\beta h^3}\int_0^\infty \mathrm{d}p\,p^2\ln\!\left(1 \mp \mathrm{e}^{-\beta(E-\mu)}\right). \tag{C.32}$$

We may change the integration variable here from $p = |\boldsymbol{p}|$ to $E = c\sqrt{m^2c^2 + p^2}$. Since $E^2 = c^2(m^2c^2 + p^2)$ it follows that

$$E\;\mathrm{d}E = c^2 p\;\mathrm{d}p\;, \tag{C.33}$$

and,

$$P = \mp \frac{4\pi g}{\beta c^2 h^3} \int_{mc^2}^{\infty} \mathrm{d}E \, pE \ln\left(1 \mp \mathrm{e}^{-\beta(E-\mu)}\right) . \tag{C.34}$$

Equation (C.33) also gives that

$$pE = c^2 p^2 \frac{\mathrm{d}p}{\mathrm{d}E} = \frac{c^2}{3} \frac{\mathrm{d}(p^3)}{\mathrm{d}E} . \tag{C.35}$$

By partial integration we find that

$$P = \frac{4\pi g}{3h^3} \int_{mc^2}^{\infty} \mathrm{d}E \, \frac{p^3}{\mathrm{e}^{\beta(E-\mu)} \mp 1} . \tag{C.36}$$

This integral may also be written as

$$P = \frac{4\pi g c^2}{h^3} \int_0^{\infty} \mathrm{d}p \, \frac{p^4}{3E} \frac{1}{\mathrm{e}^{\beta(E-\mu)} \mp 1} = \frac{g}{h^3} \int \mathrm{d}^3\boldsymbol{p} \, \frac{\boldsymbol{p} \cdot \boldsymbol{v}}{3} \frac{1}{\mathrm{e}^{\beta(E-\mu)} \mp 1} , \tag{C.37}$$

where $\boldsymbol{v}$ is the particle speed,

$$\boldsymbol{v} = \frac{c^2 \boldsymbol{p}}{E} = \frac{c\boldsymbol{p}}{\sqrt{m^2 c^2 + |\boldsymbol{p}|^2}} . \tag{C.38}$$

We see that in the case of a relativistic ideal gas, with $m = 0$ and $\boldsymbol{p} \cdot \boldsymbol{v} = pc = E$, the relation $P = \rho c^2/3$ holds independent of the temperature and the chemical potential.

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