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Event oriented analysis of series structural systems

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Abstract

The event oriented analysis of technical objects is in general accomplished by representing them as complete or incomplete systems and subsystems of events. It is argued in the article how the compound engineering systems of events can be partitioned by inclusion-exclusion expansion into individual and common cause modes. The event analysis is based on the random variable model and employs the results of operational modes and effect analysis, of the reliability analysis and of the uncertainty analysis. The system redundancy and robustness are considered as uncertainties, due to the fact that really a number of events are possible, expressed by the entropy concept in probability theory, conditioned by operational and failure modes, respectively. Relative and average uncertainty measures are introduced to facilitate uncertainty interpretations in engineering problems. It is investigated how the sensitivity analysis of reliability measures can be applied to the assessments of system uncertainties. Numerical examples presented in the article illustrate the application of event oriented system analysis to series structural systems with common cause failures. Additionally, system performance presentation and optimization with constraints, as well as potential improvements in system analysis, design and maintenance are investigated. © 2001 Elsevier Science Ltd. All rights reserved.

Keywords: Engineering; Mechanics; System analysis; Structural system; System of events; Event oriented system analysis; Probability; Reliability; Entropy; Information; Uncertainty; Series systems; Redundancy; Robustness; Optimization; Sensitivity analysis; AFORM; SORM; Monte-Carlo simulation

1. Introduction

The goal of an engineering modes analysis applied to a lifetime service of complex objects is to determine all, or at least the most important observable operational and failure modes, as well as their relations [1–4] Semiquantitative and quantitative numerical and simulation methods founded on random variable models can be applied to predict the probabilities of a safe operation or the occurrence of accidents [5,6]. By considering the hierarchical structure of failure modes col-

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Nomenclature

A_i, E_i	Random events in general
$F(\bullet)$	Average number of events of a system of events
$G(\bullet)$	Average probability of a system of events
$H(\bullet)$	Shannon's entropy of a system of events
$H^1(\bullet)$	Renyl's/Shannon's entropy of order one
$h(\bullet)$	Entropy relative to highest value
N, n	Number of systems and subsystems
n_o, n_f	Number of operational and failure subsystems
N_o, N_f	Number of operational and failure events
$p(\bullet)$	Probabilities of random events and sub)systems
$\boldsymbol{P}$	Bimodal probability matrix
$P_f(\mathcal{S})$	Probability of a failure of a system
$R(\mathcal{S})$	System reliability
$\mathrm{RED}(\mathcal{S})$	System redundancy
$\mathrm{ROB}(\mathcal{S})$	System robustness
$\mathrm{red}(\mathcal{S})$	System redundancy relative to the highest value
$\mathrm{rob}(\mathcal{S})$	System robustness relative to the highest value
$\mathcal{S}$	System of events in general
$\mathcal{S}'$	System of the subsystems of events
$\mathcal{O}, \mathcal{F}$	Subsystems of operational and failure events
β	Safety indices
γ	Bimodal dependencies
ρ	Bimodal correlations

lected into subsystems of events, the difficulty in assessing the higher order dependencies among events may be substantially reduced [7]. The service modes and effects analysis, including the relations among common cause events, is an essential step for understanding the behavior of complex engineering systems.

The entropy concept in probability theory expresses the uncertainties of systems of events [8,9]. Entropy of incomplete systems of events, as well as the entropy of mixtures of distributions, are important theoretical extensions [10]. Moreover, the maximal entropy principle [11] is proposed in engineering to derive the form of minimally prejudiced probability distributions of random variables [12]. There are probability distributions leading to the global and unconditional maximum of entropy of unconstrained systems and subsystems of events [13], or the resulting probabilities are as flat or as platykurtic as the constraints allow.

The uncertainty expressed by conditional entropy is useful in understanding problems partitioned into subsystems of events [14]. The event oriented system analysis (EOSA) combines the system reliability and the uncertainty of complete or incomplete engineering systems and subsystems of operational and failure modes [15]. In some engineering problems it may be more appropriate to express the uncertainty either relative to the maximal attainable value [16] or by the average uncertainty of a system [17].

The article considers series systems due to their importance in engineering. Series systems do not possess redundant capacities, but their robustness is always desired, moreover, it is often a requested property. Colloquially, robustness concerns strength and sturdiness. However, there is not a generally agreed definition of structural robustness. The basic comprehension of robustness implies that the product's functional characteristics are not sensitive to noise factors [18]. In the non-probabilistic structural reliability approach, the robustness is viewed as a structural ability to tolerate a large amount of uncertainty before occurrence of a failure [19]. In the theory of structural vulnerability, robustness is the opposite to vulnerability, viewed as a physical ability of the structural form and connectivity, denoted as "well-formedness", in withstanding all failure events, independent of loading action [20]. The term robustness in the article denotes an excessive capability to respond to all demands by a number of adequate failure modes with the maximally attainable uniform probability distribution in order to eliminate the disproportionate consequences from the modes with adverse failure probabilities [21]. The aim is to demonstrate how the system robustness, defined by the conditional entropy of failure modes, can be practically applied in the analysis and design of series structural systems as an additional decisive attribute, independent of the system reliability, weight or cost.

Furthermore, the article investigates how the dispersion of semiprobabilistic and deterministic safety measures can be used in the assessment of system uncertainties with respect to failure modes.

The article also elaborates the sensitivities of uncertainty measures, due to their importance in the structural system analysis, particularly in design procedures or in problems involving optimization.

Finally, the benefits of the application of EOSA to structural problems are investigated in the examples of system performance analysis and optimization of a well-known one-store, one-bay plane frame structure.

2. The mode uncertainty assessment by random variables

Finite number n_{v} of random variables represented by the random vector $\mathbf{X} = (X_1, X_2, ...Xn_{\mathrm{v}})$ defines m individual modes. The limit state functions appropriate to all the modes, $k = 1, 2, ..., m$, are defined as $g_{\mathrm{k}}(\mathbf{X}) = 0$. Since the design variables are considered random, individual operational modes are random events too, denoted as $A^o_{\mathrm{k}} = [g_{\mathrm{k}}(\mathbf{X}) > 0]$.

The reliability of an individual mode is given by the following integration term:

$$p\left(A^{\mathrm{o}}_{\mathrm{k}}\right) = p[g_{\mathrm{k}}(\mathbf{X}) > 0] = \int\int\int_D f(\mathbf{X})\mathrm{d}\mathbf{X} \tag{1}$$

The failure probability of an individual mode is given as a complement to reliability as shown:

$$p\left(A^f_{\mathrm{k}}\right) = 1 - p\left(A^{\mathrm{o}}_{\mathrm{k}}\right) = p[g_{\mathrm{k}}(\mathbf{X}) \leqslant 0] \tag{2}$$

These two complementary events are sometimes jointly denoted as simple alternatives.

The joint probability distribution function $f(\mathbf{X})$ of the random vector $\mathbf{X}$ in (1) comprises the engineering knowledge about statistical uncertainties. The integration domain D is defined by limit state functions and consists of the engineering comprehension and experience about the physical properties of the considered object. The notional probabilities of individual modes can be practically assessed by the advanced first order reliability method (AFORM), the second order reliability methods (SORM) and by the Monte-Carlo simulation (MCS) [5,6].

3. The system uncertainty assessment by random events

Boolean logic based on elementary set algebra can be used to relate the individual modes. An ideal parallel system fails if all of its modes fail, therefore:

$$P_\mathrm{f} = p\left(\bigcap_{\mathrm{all\ k}} A_\mathrm{k}^f\right) = p\left\{\bigcap_{\mathrm{all\ k}} [g_\mathrm{k}(X) \leqslant 0]\right\} \tag{3}$$

An ideal series system fails if any of its modes fails, therefore:

$$P_\mathrm{f} = p\left(\bigcup_{\mathrm{all\ k}} A_\mathrm{k}^f\right) = p\left\{\bigcup_{\mathrm{all\ k}} [g_\mathrm{k}(X) \leqslant 0]\right\} \tag{4}$$

It is commonly held that AFORM and SORM are sufficiently accurate for system reliability assessments. Additionally, MCS can be used for checking the results of non-typical problems.

3.1. Service modes and effects analysis

Methods, such as enumeration, event-tree analysis, fault-tree analysis and failure path approach, make use of the available component operational data to represent the more complex systems or subsystems in terms of possible operational and failure events and predict the probabilities of operation or failure [1–4,22,30].

The series system functions if at least one of the minimal series path structures functions, and the system can be seen as a parallel arrangement of the minimal series path structures and represented as:

$$P_\mathrm{f} = p\left[\bigcap_{m_\mathrm{p}}\left(A_{j_1}^f \cup \ldots \cup A_{j_\mathrm{q}}^f\right)\right] = p\left\{\bigcap_{m_\mathrm{p}}\left[\left(g_{j_1}(\mathbf{X}) \leqslant 0\right) \cup \ldots \cup \left(g_{j_\mathrm{q}}(\mathbf{X}) \leqslant 0\right)\right]\right\} \tag{5}$$

The index set $\{j_1 \ldots j_\mathrm{q}\}$ in (5) represents the minimal path (tie) set and m_p indicates that the interaction is taken by minimal path (tie) sets.

Since the system fails if just one of the minimal parallel cut structures fails, the system can be represented as a series arrangement of the minimal parallel cut structures, as shown:

$$P_{\mathrm{f}} = p\left[\underset{m_{\mathrm{c}}}{\cup}\left(A_{j_1}^{f} \cap \ldots \cap A_{j_{\mathrm{q}}}^{f}\right)\right] = p\left\{\underset{m_{\mathrm{c}}}{\cup}\left[\left(g_{j_1}(\mathbf{X}) \leqslant 0\right) \cap \ldots \cap \left(g_{j_{\mathrm{q}}}(\mathbf{X}) \leqslant 0\right)\right]\right\} \tag{6}$$

The index set $\left(j_1 \ldots j_{\mathrm{q}}\right)$ in (6) represents the minimal cut set and m_{c} indicates that the interaction is taken over the minimal cut sets.

4. Event oriented analysis of series systems

The operation of series systems can be presented by a subsystem $\mathcal{O}$ with only one mode:

$$\mathcal{O} = \begin{pmatrix} E_{N_{\mathrm{o}}=1}^{\mathrm{o}} \\ p\left(E_{N_{\mathrm{o}}=1}^{\mathrm{o}}\right) \end{pmatrix}.$$

The failure of series systems can be presented by a subsystem $\mathcal{F}$ of all the failure modes:

$$\mathcal{F} = \begin{pmatrix} E_1^{f} & E_2^{f} & & \ldots E_{N_{\mathrm{f}}}^{f} \\ p\left(E_1^{f}\right) & p\left(E_2^{f}\right) & \ldots & p\left(E_{N_{\mathrm{f}}}^{f}\right) \end{pmatrix}.$$

The system of events $\mathcal{S}$ with total number of events equal to $N = N_{\mathrm{f}} + 1$, can also be presented as a compound of the subsystem $\mathcal{O}$ of operational modes and the subsystem $\mathcal{F}$ of failure modes, as shown:

$$\mathcal{S} = \begin{pmatrix} E_1^{\mathrm{o}} & E_2^{f} & & \ldots E_{N_{\mathrm{f}}}^{f} \\ p\left(E_1^{\mathrm{o}}\right) & p\left(E_1^{f}\right) & \ldots & p\left(E_{N_{\mathrm{f}}}^{f}\right) \end{pmatrix} = (\mathcal{O} + \mathcal{F}).$$

The reliability of the system $\mathcal{S}$ implies all the outcomes when the system is operating and can be calculated as the probability of the subsystem of operational modes $p(\mathcal{O})$ as:

$$R(\mathcal{S}) = p(\mathcal{O}) = p\left(E_1^{\mathrm{o}}\right) \tag{7}$$

The failure probability of the system $\mathcal{S}$ takes into account all the outcomes when the system fails and can be calculated as the probability of the subsystem of failure modes $p(\mathcal{F})$ as:

$$P_{\mathrm{f}}(\mathcal{S}) = p(\mathcal{F}) = \sum_{i=1}^{N_f} p\left(E_i^{f}\right) \tag{8}$$

A system $\mathcal{S}'$ consists of subsystems of operational and of failure modes, as presented:

$$\mathcal{S}' = (\mathcal{O}, \mathcal{F}) = \begin{pmatrix} \mathcal{O} & \mathcal{F} \\ p(\mathcal{O}) & p(\mathcal{F}) \end{pmatrix}.$$

The probability associated with the system of events $\mathcal{S}$, not necessarily equal to unity, is as follows:

$$p(\mathcal{S}) = p(\mathcal{S}') = p\left(E_1^{\mathrm{o}}\right) + \sum_{i=1}^{N_{\mathrm{f}}} p\left(E_i^f\right) = p(\mathcal{O}) + p(\mathcal{F}) = R(\mathcal{S}) + P_f(\mathcal{S}) \tag{9}$$

4.1. Partitioning of series systems of events

Different approximations are used in system reliability analysis to solve the problems for events neither fully exclusive nor fully inclusive. For fully exclusive events, see Venn diagram in Fig. 1a, which are also denoted as fully independent or uncorrelated failure modes, the subsystem of failure modes is represented as shown:

$$\mathcal{F} = \begin{pmatrix} E_1^f & E_2^f & \dots E_{N_{\mathrm{f}}}^f \\ p\left(E_1^f\right) & p\left(E_2^f\right) & \dots \quad p\left(E_{N_{\mathrm{f}}}^f\right) \end{pmatrix}.$$

For fully inclusive events, see Venn diagram on Fig. 1b, which are also denoted as fully dependent or correlated failure modes, complements of successive failure events in increasing order of failure probabilities can be used as the basic failure events, as presented next:

$$\mathcal{F} = \begin{pmatrix} E_1^f - E_2^f & E_2^f - E_3^f & \dots & E_{N_{\mathrm{f}}}^f \\ p\left(E_1^f\right) - p\left(E_2^f\right) & p\left(E_2^f\right) - p\left(E_3^f\right) & \dots & p\left(E_{N_{\mathrm{f}}}^f\right) \end{pmatrix}.$$

The Cornell's lower and upper probability bounds on union of events are defined on the basis of the fully exclusive and fully inclusive partitioning of series systems, Fig. 1a and b, as follows:

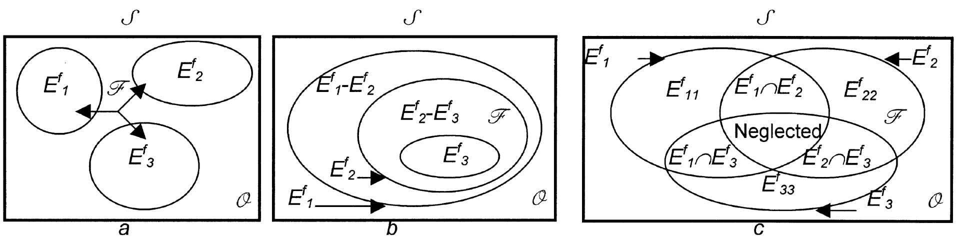

Fig. 1. Venn diagrams — exclusion-inclusion of the union of events up to the second order.

$$\max_{1 \leqslant i \leqslant N_{\mathrm{f}}} p\left(E_i^f\right) \leqslant P_{\mathrm{f}}(\mathcal{S}) \leqslant \sum_{i=1}^{N_{\mathrm{f}}} p\left(E_i^f\right) \tag{10}$$

The joint failure probabilities between the two events E_i and E_j, with known safety indices β_i and β_{j}, are calculated by employing numerical integration of the bivariate normal probability density functions φ, over the range of mode correlation coefficient γ_{ij}, defined as the directional cosine between two limit state tangent hyperplanes, as follows:

$$p\left(E_i^f \cap E_j^f\right) = \Phi(-\beta_i)\cdot\Phi\left(-\beta_j\right) + \int_0^{\gamma_{ij}} \varphi\left(-\beta_i, -\beta_j, \gamma\right)\mathrm{d}\gamma \tag{11}$$

In addition to the correlation between two modes, a degree of dependence [23] is regarded as a more general concept representing a measure of their weighted overlap, as shown:

$$\rho_{E_iE_j} = \frac{p\left(E_i \cap E_j\right)}{\min\left[p(E_i), p\left(E_j\right)\right]} \tag{12}$$

The improvement in reliability and uncertainty analysis by introducing joint probabilities of three or more events appears small in practice. The Ditlevsen's probability bounds in a bimodal representation can be defined more closely of the bound in (10), as:

$$p\left(E_1^f\right) + \sum_{i=2}^{N_{\mathrm{f}}} \max\left\{\left[p\left(E_i^f\right) - \sum_{j=1}^{i-1} p\left(E_i^f \cap E_j^f\right)\right], 0\right\} \leqslant P_f(\mathcal{S}) \leqslant \sum_{i=1}^{N_{\mathrm{f}}} p\left(E_i^f\right) - \sum_{\substack{i=2 \\ j<i}}^{N_{\mathrm{f}}} \max\left[p\left(E_i^f \cap E_j^f\right)\right] \tag{13}$$

It is reasonable to apply the inclusion-exclusion expansion of union of failure events up to the second order. The failure probabilities can be represented in a two dimensional symmetric N_{f} by N_{f} probability matrix as follows:

$$\mathbf{P} = \begin{bmatrix} E_{11}^f & & \text{sym} & \\ E_{12}^f & E_{22}^f & & \\ \dots & E_{23}^f & \dots & \\ E_{1N_{\mathrm{f}}} & E_{2N_{\mathrm{f}}} & \dots & E_{N_{\mathrm{f}}N_{\mathrm{f}}}^f \end{bmatrix}.$$

The expansions for diagonal terms are obtained as complements to individual failures with respect to the joint probabilities with all other events, see Venn diagram on Fig. 1c, as follows.

$$p\left(E^f_{ii}\right) = \max\left\{\left[p\left(E^f_i\right) - \sum_{\substack{j=1 \\ j \neq i}}^{N_\mathrm{f}} p\left(E^f_i \cap E^f_j\right)\right], 0\right\}, \quad i = 1, 2, ..., N_\mathrm{f} \tag{14}$$

The off-diagonal terms of the lower triangle of the probability matrix are simply the intersections of appropriate events, Fig. 1c, and can be calculated according to (11) as follows:

$$p\left(E^f_{ij}\right) = p\left(E^f_i \cap E^f_j\right), \quad i = 1, 2, \ldots, N_\mathrm{f}, \quad j = 1, 2, \ldots, N_\mathrm{f}, \quad j > i \tag{15}$$

A failure set in such a bimodal partitioning can be written as follows:

$$\mathcal{F} = \begin{pmatrix} E^f_{11} & E^f_{22} & \ldots & E^f_{N_\mathrm{f}N_\mathrm{f}} & E^f_1 \cap E^f_2 & E^f_1 \cap E^f_3 & \ldots & E^f_\mathrm{p} \cap E^f_\mathrm{q} \\ p\left(E^f_{11}\right) & p\left(E^f_{22}\right) & \ldots & p\left(E^f_{N_\mathrm{f}N_\mathrm{f}}\right) & p\left(E^f_1 \cap E^f_2\right) & p\left(E^f_1 \cap E^f_3\right) & \ldots & p\left(E^f_\mathrm{p} \cap E^f_\mathrm{q}\right) \end{pmatrix}.$$

In the above notation, p and q represent arbitrary combinations out of all $\binom{N_\mathrm{f}}{2}$ possibilities.

The sum of the elements of the subsystem of failure modes $\mathcal{F}$ represents the lower probability value according to Ditlevsen's bounds and it reads as follows:

$$P_\mathrm{f}(\mathcal{S}) \geqslant \sum_{i=1}^{N_\mathrm{f}} \sum_{j=1}^{N_\mathrm{f}} p\left(E^f_{ij}\right) \tag{16}$$

More generally, for intersections of an arbitrary number of elementary random events m, the principle of inclusion-exclusion, known for more than a hundred years [24], can give $N_\mathrm{f} = 2^m - 1$ disjoint failure modes as shown:

$$\text{For } i = 1, \ldots, m \text{ and } j = 1, \ldots, \binom{m}{i}:$$

$$\begin{aligned} E^f_{i,j} &= \left(A^f_{c_i(1)} \cap \ldots \cap A_{c_i(i)}\right) \cap \left[\left(I - A^f_{l_i(1)}\right) \cap \ldots \cap \left(I - A^f_{l_i(m-i)}\right)\right] = \\ &= \left(A^f_{c_i(1)} \cap \ldots \cap A_{c_i(i)}\right) \cap \left(A^\mathrm{o}_{l_i(1)} \cap \ldots \cap A^0_{l_i(m-i)}\right) \end{aligned} \tag{17}$$

$c_i(1), ..., c_i(i)$ is the jth combination of all i out of m components

$l_i(1), ..., l_i(m-i)$ is a complementary set of indices $l_i(p) \neq c_i(q)$, for all p and q.

Hence, a compound failure set can be disjointed into individual modes when just the considered mode can occur and into a number of intersections with all the other failure modes constituting the failure set, Fig. 1c. Note that some intersections may be practically impossible.

4.2. Evaluation of system and subsystem uncertainties

The entropy of complete or of incomplete systems of events $\mathcal{S}$ and $\mathcal{S}'$ expresses the uncertainties and in general can be determined as the limiting case of the Renyi's entropy of order α, for $\alpha \rightarrow 1$:

$$H_N^1(\mathcal{S}) = H^1(\mathcal{S}) = \left[-\sum_{i=1}^{N} p(E_i) \log p(E_i) \right] / \sum_{i=1}^{N} p(E_i) = \frac{H(\mathcal{S})}{p(\mathcal{S})} \tag{18}$$

$$H_2^1(\mathcal{S}') = H^1(\mathcal{S}') = [p(\mathcal{O}) \cdot \log p(\mathcal{O}) - p(\mathcal{F}) \cdot \log p(\mathcal{F})] / [p(\mathcal{O}) + p(\mathcal{F})] = \frac{H(\mathcal{S}')}{p(\mathcal{S}')} \tag{19}$$

The article uses the notation Renyi's/Shannon's entropy of order one, since for $p(\mathcal{S}) = 1$, the terms (18) and (19) are by definition equal to the Shannon's entropy.

The system $\mathcal{S}$ under the condition that it is operational $\mathcal{O}$, can be presented as follows:

$$\mathcal{S}/\mathcal{O} = \begin{pmatrix} E_1^{\mathrm{o}}/\mathcal{O} & E_2^{\mathrm{o}}/\mathcal{O} & \dots & E_{N_{\mathrm{o}}}^{\mathrm{o}}/\mathcal{O} \\ \dfrac{p(E_1^{\mathrm{o}})}{p(\mathcal{O})} & \dfrac{p(E_2^{\mathrm{o}})}{p(\mathcal{O})} & \dots & \dfrac{p\left(E_{N_{\mathrm{o}}}^{\mathrm{o}}\right)}{p(\mathcal{O})} \end{pmatrix}.$$

The uncertainty of the subsystem of operational modes is considered as a complete conditional distribution with respect to the condition that the system $\mathcal{S}/\mathcal{O}$ is operational with probability $p(\mathcal{O})$. According to the definition of the entropy of operational modes, it follows:

$$H_{N_{\mathrm{o}}}(\mathcal{S}/\mathcal{O}) = -\sum_{i=1}^{N_{\mathrm{o}}} \frac{p(E_i^{\mathrm{o}})}{p(\mathcal{O})} \cdot \log \frac{p(E_i^{\mathrm{o}})}{p(\mathcal{O})} = H_{N_{\mathrm{o}}}^1(\mathcal{O}) + \log p(\mathcal{O}) \tag{20}$$

It can be argued that the conditional entropy of a subsystem of operational modes (20) expresses the redundancy of a system of events viewed as the capacity of a system to continue operations by performing adverse operational modes in case of random component failures [21].

An alternative notation for redundancy is proposed:

$$\mathrm{REDUNDANCY(System/Operational)} = \mathrm{RED}(\mathcal{S}/\mathcal{O}) = \mathrm{RED}(\mathcal{S}) = H_{N_{\mathrm{o}}}(\mathcal{S}/\mathcal{O}) \tag{21}$$

Since the series systems have only one operational mode, the operational uncertainty vanishes:

$$\mathrm{RED}(\mathcal{S}) = H_{N_{\mathrm{o}}=1}(\mathcal{S}/\mathcal{O}) = 0 \tag{22}$$

The system $\mathcal{S}$ under the condition that it is failed $\mathcal{F}$ can be presented as follows:

$$\mathcal{S}/\mathcal{F} = \begin{pmatrix} E_1^f/\mathcal{F} & E_2^f/\mathcal{F} & \dots & E_{N_\mathrm{f}}/\mathcal{F} \\ \dfrac{p\left(E_1^f\right)}{p(\mathcal{F})} & \dfrac{p\left(E_2^f\right)}{p(\mathcal{F})} & \dots & \dfrac{p\left(E_{N_\mathrm{f}}\right)}{p(\mathcal{F})} \end{pmatrix}.$$

The uncertainty of the subsystem of failure modes $\mathcal{F}$ can be expressed as the Shannon's entropy applied only to the systems $\mathcal{S}$ under the condition that it is non-operational $\mathcal{S}/\mathcal{F}$. The subsystem is considered as a complete conditional distribution with respect to the condition that the system in whole is non-operational with probability $p(\mathcal{F})$ and, according to the definition of the entropy of failure modes, it follows:

$$H_{N_\mathrm{f}}(\mathcal{S}/\mathcal{F}) = -\sum_{i=1}^{N_\mathrm{f}} \frac{p\left(E_i^f\right)}{p(\mathcal{F})} \cdot \log \frac{p\left(E_i^f\right)}{p(\mathcal{F})} = H_{N_\mathrm{f}}^1(\mathcal{F}) + \log p(\mathcal{F}) \tag{23}$$

A robust behavior is intuited when the system can provide more adequate failure modes to adverse demands with more uniform failure probabilities, Fig. 2. When the system responds to all demands uniformly, there is a high uncertainty about which of the failure modes could occur. Hence, the system robustness is related only to the failure modes of the system. It can be argued that the conditional entropy (23) expresses the robustness of a system of events regarded as the system's capability to respond uniformly to all possible random failures [21].

An alternative notation for robustness is proposed:

$$\mathrm{ROBUSTNESS(system/fails)} = \mathrm{ROB}(\mathcal{S}/\mathcal{F}) = \mathrm{ROB}(\mathcal{S}) = H_{N_\mathrm{f}}(\mathcal{S}/\mathcal{F}) \tag{24}$$

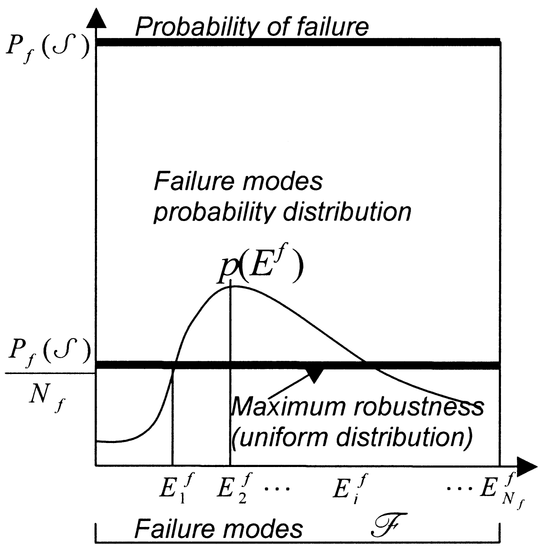

Fig. 2. Distribution of the probabilities of failure modes.

The conditional entropy $H(\mathcal{S}/\mathcal{S}')$ of series system $\mathcal{S}$ with respect to the system of subsystems $\mathcal{S}'$ provides a relation between system failure probability and system robustness, as follows:

$$H_N(\mathcal{S}/\mathcal{S}') = P_f(S)\cdot \mathrm{ROB}(\mathcal{S}) = p(\mathcal{S})\left[H_N^1(\mathcal{S}) - H_n^1(\mathcal{S}')\right] = H_N(\mathcal{S}) - H_n(\mathcal{S}') \tag{25}$$

The failure set can also be partitioned by $n_\mathrm{f} = m$ subsets of failures of different levels of joining, or of a different level of failure seriousness, like fatalities, collapse, serviceability failures or with some other common characteristics of interest, as presented next:

$$\mathcal{F}' = \begin{pmatrix} \mathcal{F}_1 & \mathcal{F}_2 & \ldots & F_{n_\mathrm{f}=m} \\ p(\mathcal{F}_1) & p(\mathcal{F}_2) & \ldots & p(\mathcal{F}n_{\mathrm{f}=m}) \end{pmatrix}.$$

The relation of the robustness of the subsystems to the robustness of the system is expressed as:

$$\sum_{i=1}^{m} p(\mathcal{F}_i)\mathrm{ROB}(\mathcal{F}/\mathcal{F}_i) = p(\mathcal{F})\left[\mathrm{ROB}(\mathcal{S}/\mathcal{F}) - \mathrm{ROB}(\mathcal{S}'/\mathcal{F}') + \log p(\mathcal{F})\right] \tag{26}$$

The following relation between any pair of subsystems having some failure modes in common holds:

$$\begin{aligned} &p(\mathcal{F}_i)\cdot H(\mathcal{S}/\mathcal{F}_i) + p(\mathcal{F}_j)\cdot p(\mathcal{S}/\mathcal{F}_j) - p(\mathcal{F}_i \cap \mathcal{F}_j)\cdot H(\mathcal{S}/\mathcal{F}_i \cap \mathcal{F}_j) \\ &\quad = p(\mathcal{F}_i \cup \mathcal{F}_j)\cdot\left[H(\mathcal{S}/\mathcal{F}_i \cup \mathcal{F}_j) - H(\mathcal{S}'/\mathcal{F}_i \cup \mathcal{F}_j)\right] \end{aligned} \tag{27}$$

The conditional entropy of the system of subsystems having some modes in common is defined as:

$$\begin{aligned} H(\mathcal{S}'/\mathcal{F}_j \cup \mathcal{F}_j) = \frac{1}{p(\mathcal{F}_i \cup \mathcal{F}_j)}\Big[&-p(\mathcal{F}_i)\cdot\log p(\mathcal{F}_i) - p(\mathcal{F}_j)\cdot\log p(\mathcal{F}_j) \\ &+ p(\mathcal{F}_i \cap \mathcal{F}_j)\cdot\log p(\mathcal{F}_j \cap \mathcal{F}_j)\Big] - \log p(\mathcal{F}_i \cup \mathcal{F}_j) \end{aligned} \tag{28}$$

4.3. Relative uncertainty measures

The important feature of entropy is not in the scale of units in which it is measured, but rather it is the meaning of the function. The relative measure of uncertainty will be denoted with small letters $h_{n,N}(\mathcal{S})$, instead of the capitals for entropy by definition $H_N(S)$. The index n emphasizes the number of events in the considered system or subsystem. The index N is the number of events in a reference system or subsystem relative to which the uncertainty is to be expressed. Superscripts "1", when used, emphasize that the entropy is related to Renyi's/Shannon's entropy of order one. The relative measure of uncertainty can be expressed in dimensionless form with respect to any reference system as:

$$h_{n,N}^{1}(\mathcal{S}) = \frac{H_{n}^{1}(\mathcal{S})}{H_{N}^{1}(\mathcal{S})_{\max}} = \frac{H_{n}^{1}(\mathcal{S})}{\log N - \log(p(\mathcal{S}))} = \frac{H_{n}^{1}(\mathcal{S})}{\log[N/p(\mathcal{S})]} \tag{29}$$

The term (29) can be viewed as the application of a logarithm of base $B = N/p(\mathcal{S})$ instead of base $B = 2$ in the entropy calculation.

The value of $h_{n,N}(\mathcal{S})$ represents the fraction of the maximal attainable entropy, equal to the entropy of the system of N equally probable events and it expresses how many times the entropy of the considered system is less than the maximal attainable entropy of the target system. The relative entropy of a subsystem of m_i events with respect to an incomplete system of N events, is defined as:

$$h_{m_i,N}^{1}(\mathcal{S}/\mathcal{S}_i) = \frac{H_{m_i}(\mathcal{S}/\mathcal{S}_i)}{H_{N}^{1}(\mathcal{S})_{\max}} = \frac{H_{m_i}(\mathcal{S}/\mathcal{S}_i)}{\log N - \log p(\mathcal{S})} = \frac{H_{m_i}^{1}(\mathcal{S}_i) + \log p(\mathcal{S}_i)}{\log[N/p(\mathcal{S})]} \tag{30}$$

By substitution of definition (29) and (30) into the relation (26), a useful expression is obtained:

$$\sum_{i=1}^{n} p(\mathcal{S}_i)\cdot \log m_i \cdot h_{m_i,m_i}(\mathcal{S}/\mathcal{S}_i) = p(\mathcal{S})\cdot \left\{\log[N/p(\mathcal{S})]\cdot h_{N,N}^{1}(\mathcal{S}) - \log[n/p(\mathcal{S})]\cdot h_{n,n}^{1}(\mathcal{S}')\right\} \tag{31}$$

The relative redundancy and robustness can be expressed with respect to their maximal values, denoted with small letters, as follows:

$$\mathrm{red}(\mathcal{S}/\mathcal{O}) = \mathrm{red}(\mathcal{S}) = h_{N_o,N}(\mathcal{S}/\mathcal{O}) = \frac{H_{N_o}(\mathcal{S}/\mathcal{O})}{H_{N_o}(\mathcal{S}/\mathcal{O})_{\max}} = \frac{\mathrm{RED}(\mathcal{S})}{\log N_o} \tag{32}$$

$$\mathrm{rob}(\mathcal{S}/\mathcal{F}) = \mathrm{rob}(\mathcal{S}) = h_{N_f,N}(\mathcal{S}/\mathcal{F}) = \frac{H_{N_f}(\mathcal{S}/\mathcal{F})}{H_{N_f}(\mathcal{S}/\mathcal{F})_{\max}} = \frac{\mathrm{ROB}(\mathcal{S})}{\log N_f} \tag{33}$$

The following relation among reliability, failure probability, redundancy and robustness holds for complete and incomplete systems of events:

$$R(\mathcal{S})\cdot \log N_o \cdot \mathrm{red}(\mathcal{S}) + P_f(\mathcal{S})\cdot \log N_f \cdot \mathrm{rob}(\mathcal{S}) = p(\mathcal{S})\cdot \left\{\log[N/p(\mathcal{S})]\cdot h_{N,N}^{1}(\mathcal{S}) - log[n/p(\mathcal{S})]\cdot h_{n,n}^{1}(\mathcal{S}')\right\} \tag{34}$$

4.4. Average uncertainty measures

The Renyi's/Shannon's entropy of order one denoted $H_N^1(\mathcal{S})$ can be written as follows:

$$H_N^1(\mathcal{S}) = -\log G_N(\mathcal{S}) = \log F_N(\mathcal{S}) = \frac{\sum_{i=1}^{N} p(E_i)\log p(E_i)}{p(\mathcal{S})} = \frac{1}{p(\mathcal{S})}\sum_{i=1}^{N} p(E_i)\log\frac{1}{p(E_i)} \tag{35}$$

The average probability of occurrence $G_N(\mathcal{S})$ in (35) [25] and the average number of events $F_N(\mathcal{S})$ in (35) can be defined for either complete or incomplete systems of events as:

$$G_N(\mathcal{S}) = \frac{1}{F_N(\mathcal{S})} = 2^{-H_N^1(\mathcal{S})} = \prod_{i=1}^{N} p(E_i)^{\frac{p(E_i)}{p(\mathcal{S})}} \tag{36}$$

The average probability and the average number of events of a subsystem is related to the probability of the event under the condition that the subsystem itself occurs, and can be written as follows:

$$G_{m_i}(\mathcal{S}/\mathcal{S}_i) = \frac{1}{F_{m_i}(\mathcal{S}/S_i)} = \prod_{j=1}^{m_i}\left[\frac{p(E_{ij})}{p(\mathcal{S}_i)}\right]^{\frac{p(E_{ij})}{p(\mathcal{S}_i)}} = 2^{-H_{m_i}}(\mathcal{S}/\mathcal{S}_i) \tag{37}$$

The relation of the average conditional probabilities of events of subsystems in (37) to the average probability of events of the system defined by (36), can be expressed for both the incomplete and complete systems of events, as the weighted geometric mean of terms, as follows:

$$\prod_{i=1}^{n} G_{m_i}(\mathcal{S}/\mathcal{S}_i)^{p(\mathcal{S}_i)} = \frac{1}{\prod_{i=1}^{n} F_{m_i}(\mathcal{S}/\mathcal{S}_i)^{p(\mathcal{S}_i)}} = \left[\frac{G_N(\mathcal{S})}{G_n(\mathcal{S}')}\right]^{p(\mathcal{S})} = \left[\frac{F_n(\mathcal{S}')}{FN(\mathcal{S})}\right]^{p(\mathcal{S})} \tag{38}$$

The average probability of a complete system of events is maximal and amounts to unity, when one of the events is a "sure" event, i.e. has the probability of one, and all the other probabilities equal zero. The average probability is minimal and amounts to $1/N$ when all the probabilities are equal. The average number of events of a complete system is maximal when all the events are of the same probability and it amounts exactly to N, i.e. the number of events of the basic system. The minimal average number of events is equal to 1, if there is only one "sure" event.

The average probability represents such a probability, which, if considered as equal for all events, gives the same entropy for the average number of events, as it is the entropy of the basic system. The last statement can easily be proven both for complete and incomplete systems as follows:

$$-F_N(\mathcal{S})\cdot G_N(\mathcal{S})\log G_N(\mathcal{S}) = -2^{H_N^1(\mathcal{S})}\cdot 2^{-H_N^1(\mathcal{S})}\log 2^{-H_N^1(\mathcal{S})} = H_N^1(\mathcal{S}) \tag{39}$$

The redundancy and robustness can also be intuited as the average numbers of operational modes and failure modes, $F_{N_f}(\mathcal{S}/\mathcal{F})$, respectively.

4.5. Assessment of uncertainties due to unobservable events

If $p(\mathcal{S}) = p(\mathcal{O}) + p(\mathcal{F}) < 1$, there are some missing or forgotten or undefined or unaccounted for or simply unknown or unobservable events and the system $\mathcal{S}$ is an incomplete one. A system built up of only bimodal joint events, as it is the case in AFORM and in SORM, is clearly not a complete system of events. The joint failure probabilities of three or more modes are neglected and the joint events of higher order may be considered as unobservable events. The number of unobservable events N_u can be found from a service mode analysis, but their probabilities in most cases remain unknown.

The number of unobservable events, N_u, and the probability of their occurrence, $p(\mathcal{U})$, increases the uncertainties, due to the lack of knowledge about all the possible modes. An additional subsystem of missing events can be imagined; let it be denoted as $\mathcal{U}$, with events U_i, $i = 1, 2, ..., N_u$. For a known or assumed distribution of probabilities of missing events, the following relation holds:

$$p(\mathcal{U}) = 1 - p(\mathcal{S}) = 1 - p(\mathcal{O}) - p(\mathcal{F}) = \sum_{i=1}^{N_u} p(U_i) \tag{40}$$

The system $\mathcal{S}$, enlarged with a complementary subsystem $\mathcal{U}$, can now be written as system $\mathcal{S}_{\mathrm{u}}$:

$$\mathcal{S}_u = (\mathcal{S} + \mathcal{U}) = \begin{pmatrix} E_1 & E_2 & \dots & E_N & U_1 & U_2 & \dots & U_{N_u} \\ p(E_1) & p(E_2) & \dots & p(E_N) & p(U_1) & p(U_2) & \dots & p\left(U_{N_u}\right) \end{pmatrix}.$$

Consider the system $\mathcal{S}'_u$ as a system of incomplete subsystems $\mathcal{S}$ and $\mathcal{U}$ of missing events:

$$S'_u = (\mathcal{S}, \mathcal{U}) = \begin{pmatrix} \mathcal{S} & \mathcal{U} \\ p(\mathcal{S}) & p(\mathcal{U}) \end{pmatrix} = \begin{pmatrix} \mathcal{S} & \mathcal{U} \\ p(\mathcal{S}) & 1 - p(\mathcal{S}) \end{pmatrix}.$$

The systems $\mathcal{S}_{\mathrm{u}}$ and $\mathcal{S}'_{\mathrm{u}}$ are by definition complete systems of events due to $p(\mathcal{S}) + p(\mathcal{U}) = 1$. The relation between the complete system $\mathcal{S}_u$ and the incomplete system $\mathcal{S}$ is as follows:

$$H_{N+N_u}(\mathcal{S}_u) = -\sum_{i=1}^{N} p(E_i)\cdot \log p(E_i) - \sum_{i=1}^{N_u} p(U_i)\cdot \log p(U_i) = p(\mathcal{S})\cdot H_N^1(\mathcal{S}) - \sum_{i=1}^{Nu} p(U_i)\cdot \log p(U_i) \tag{41}$$

$$H_2\left(\mathcal{S}'_u\right) = -p(\mathcal{S})\cdot \log p(\mathcal{S}) - [1 - p(\mathcal{S})]\cdot \log[1 - p(\mathcal{S})] \tag{42}$$

The following relation between the incomplete distribution and missing event approach holds:

$$p(\mathcal{S})\cdot H(\mathcal{S}_u/\mathcal{S}) + p(\mathcal{U})\cdot H(\mathcal{S}_u/\mathcal{U}) = H(\mathcal{S}_u) - H\left(\mathcal{S}'_u\right) \tag{43}$$

For a single missing event, $N_u = 1$, and $H(\mathcal{S}_u/\mathcal{U}) = 0$.

Since unobservable events are in question, it is not likely that their probability distribution is known. The simplest assumption about their uniform distribution, i.e. $p(\mathcal{U}_i) = p(U)/N_u$, leads to a maximal increase of system uncertainty which can be calculated for $N_u > 1$ as $H(\mathcal{S}_u/\mathcal{U}) = \log N_u$.

In a more detailed calculation the system $\mathcal{S}$ can be composed of three subsystems $\mathcal{O}$, $\mathcal{F}$ and $\mathcal{U}$ and can be represented as $\mathcal{S}_\mathrm{u} = (\mathcal{O} + \mathcal{F} + \mathcal{U})$ and considered also as the system $\mathcal{S}'_u$ of subsystems $\mathcal{S}'_u = (\mathcal{O}, \mathcal{F}, \mathcal{U})$. The following entropy can be calculated:

$$H\left(S'_u\right) = -p(\mathcal{O})\cdot\log p(\mathcal{O}) - p(\mathcal{F})\cdot\log p(\mathcal{F}) - p(\mathcal{U})\cdot\log p(\mathcal{U}) \tag{44}$$

When the missing events are assumed by their probabilities, the following relation holds:

$$p(\mathcal{O})\cdot H(\mathcal{S}_u/\mathcal{O}) + p(\mathcal{S}_u/\mathcal{F})\cdot H(\mathcal{S}_u/\mathcal{F}) + p(\mathcal{U})\cdot H(\mathcal{S}_u/\mathcal{U}) = H(\mathcal{S}_u) - H\left(\mathcal{S}'_u\right) \tag{45}$$

Note that $H(\mathcal{S}_\mathrm{u}/\mathcal{O}) = H(\mathcal{S}/\mathcal{O})$, $H(\mathcal{S}_\mathrm{u}/\mathcal{F}) = H(\mathcal{S}/\mathcal{F})$ and $p(\mathcal{S}_\mathrm{u}/\mathcal{O}) = p(\mathcal{O})$, $p(\mathcal{S}_\mathrm{u}/\mathcal{F}) = p(\mathcal{F})$.

4.6. Sensitivity analysis with respect to uncertainties

The sensitivity analysis of different reliability measures with respect to sample characteristics, distributional parameters, parameters of limit state functions [5,26] as well as to correlation [27], are well-established efficient, numerical procedures of great importance in structural engineering.

The sensitivity is viewed as the rate of change of the uncertainty measure with respect to selected parameters Ω_k, $k = 1, \ldots, n_\mathrm{p}$, and can be calculated as the partial derivatives.

The chain rule in general form as $\dfrac{\partial\text{uncertainty measure}(\mathcal{S})}{\partial\text{Reliability measure}(\mathcal{S})}\dfrac{\partial\text{Reliability measure}(\mathcal{S})}{\partial\text{Parameter}\,\Omega_k}, k = 1, \ldots n_\mathrm{p}$,

provides specific expressions for sensitivity factors of uncertainty measures for series systems:

$$\frac{\partial H(\mathcal{S})}{\partial\Omega_\mathrm{k}} = \frac{\partial P_f(\mathcal{S})}{\partial\Omega_\mathrm{k}}\log[1 - P_\mathrm{f}(\mathcal{S})] - \sum_{i=1}^{N_\mathrm{f}}\frac{\partial p\left(E_i^f\right)}{\partial\Omega_\mathrm{k}}\log p\left(E_i^f\right) \tag{46}$$

$$\frac{\partial H(\mathcal{S}')}{\partial\Omega_\mathrm{k}} = \frac{\partial P_\mathrm{f}(\mathcal{S})}{\partial\Omega_\mathrm{k}}\log\frac{1 - P_f(\mathcal{S})}{P_\mathrm{f}(\mathcal{S})} \tag{47}$$

$$\frac{\partial \mathrm{ROB}(\mathcal{S})}{\partial\Omega_\mathrm{k}} = \frac{1}{P_\mathrm{f}(\mathcal{S})}\left\{\frac{\partial P_\mathrm{f}(\mathcal{S})}{\partial\Omega_\mathrm{k}}[\log P_\mathrm{f}(\mathcal{S}) - \mathrm{ROB}(\mathcal{S})] - \sum_{i=1}^{N_\mathrm{f}}\frac{\partial p\left(E_i^f\right)}{\partial\Omega_\mathrm{k}}\log p\left(E_i^f\right)\right\} \tag{48}$$

$$\frac{\partial p_\mathrm{f}(\mathcal{S})\mathrm{ROB}(\mathcal{S})}{\partial\Omega_\mathrm{k}} = \frac{\partial\left[H(\mathcal{S}) - H(\mathcal{S}')\right]}{\partial\Omega_\mathrm{k}} = \frac{\partial P_\mathrm{f}(\mathcal{S})}{\partial\Omega_\mathrm{k}}\log P_\mathrm{f}(\mathcal{S}) - \sum_{i=1}^{N_\mathrm{f}}\frac{\partial p\left(E_i^f\right)}{\partial\Omega_\mathrm{k}}\log p\left(E_i^f\right) \tag{49}$$

The calculations of sensitivities of uncertainties require the derivatives of system and component failure probabilities commonly provided by AFORM, SORM and MCS procedures without recalculation of the design model, which are specifically related for series systems as shown:

$$\frac{\partial P_{\mathrm{f}}(\mathcal{S})}{\partial \Omega_{\mathrm{k}}} = \sum_{i=1}^{N_{\mathrm{f}}} \frac{\partial p\left(E_i^f\right)}{\partial \Omega_{\mathrm{k}}} \tag{50}$$

5. A plane frame structure example

For illustration purposes EOSA tackles one-store, one-bay, plane frame structure, Fig. 3, as a typical series system with common cause failure modes. Plastic hinge mechanisms leading to a collapse of the frame are analyzed by elastic–plastic stress–strain relations [28]. All plastic moment capacities M_i, $i = 1, 2, 3, 4, 5$ are log-normally distributed with mean values of 134.9 kNm and standard deviations of 13.49 kNm, ($\mathrm{COV_M} = 0.10$). The horizontal concentrated load F_h is log-normally distributed with mean value of 50 kN and standard deviation of 15 kN, $\mathrm{COV_F} = 0.3$. The vertical concentrated load Fv is log-normally distributed with mean value of 40 kN and standard deviation of 12 kN, $\mathrm{COV_F} = 0.3$. The geometric parameter h = 5 m is considered constant.

The principle of virtual work gives three linear limit state functions, as shown:

$$g_1(M_1, M_2, M_3, M_4, M_5, F_h, F_v) = M_1 + M_2 + M_4 + M_5 - h{\cdot}F_{\mathrm{h}}$$
$$g_2(M_1, M_2, M_3, M_4, M_5, F_h, F_v) = M_1 + 2{\cdot}M_3 + 2{\cdot}M_4 + M_5 - h{\cdot}F_{\mathrm{h}} - h{\cdot}F_{\mathrm{v}}$$
$$g_3(M_1, M_2, M_3, M_4, M_5, F_h, F_v) = M_2 + 2{\cdot}M_3 + 2{\cdot}M_4 - h{\cdot}F_{\mathrm{v}}.$$

Some results of the reliability analysis reported earlier [5] are repeated for comparative purposes.

5.1. System partitioning into the basic modes

Reliability analysis for partitioning of series systems of events uses a range of approximations. The simplest approximations based on the full exclusiveness of events, Fig. 1a, is as shown:

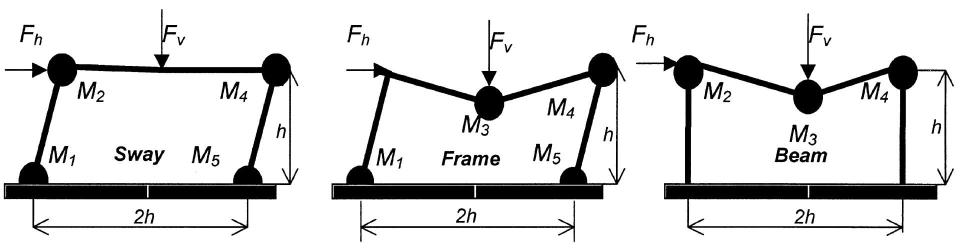

Fig. 3. Plane frame structure with three plastic failure mechanisms.

$$\mathcal{S} = \begin{pmatrix} A_1^f & A_2^f & A_3^f & I - A_1^f \cup A_2^f \cup A_3^f \\ 3.36\cdot 10^{-3} & 1.99\cdot 10^{-3} & 2.91\cdot 10^{-4} & 0.994359 \end{pmatrix}.$$

Another approximation based on full inclusiveness of events, Fig. 1b, is as follows:

$$\mathcal{S} = \begin{pmatrix} A_1^f - A_2^f & A_2^f - A_3^f & A_3^f & I - A_1^f \\ 1.37\cdot 10^{-3} & 1.70\cdot 10^{-3} & 2.91\cdot 10^{-4} & 0.996639 \end{pmatrix}.$$

The fully exclusive and fully inclusive modes corresponds to Cornell's lower and upper failure probability bounds on union of events according to (10), as follows:

$$3.36\cdot 10^{-3} \leqslant P_{\mathrm{f}}(\mathcal{S}) \leqslant 5.64\cdot 10^{-3}.$$

The next approximation is closer to reality but still does not account for all reliability model inaccuracies, Fig. 1c. Supposing that all basic events are pairways mutually independent, the probabilities of all the modes can be expressed as follows:

$$\begin{aligned}
&p(E_1^{\mathrm{o}}) = p(A_1^{\mathrm{o}})\cdot p(A_2^{\mathrm{o}})\cdot p(A_3^{\mathrm{o}}) = 0.99436, \quad O: 3/3 \\
&p\left(E_2^f\right) = p\left(A_1^f\right)\cdot p(A_2^{\mathrm{o}})\cdot p(A_3^{\mathrm{o}}) = 3.35\cdot 10^{-3}, \quad F: 1/3 \\
&p\left(E_3^f\right) = p(A_1^{\mathrm{o}})\cdot p\left(A_2^f\right)\cdot p(A_3^{\mathrm{o}}) = 1.98\cdot 10^{-3}, \quad F: 1/3 \\
&p\left(E_4^f\right) = p(A_1^{\mathrm{o}})\cdot p(A_2^{\mathrm{o}})\cdot p\left(A_3^f\right) = 2.89\cdot 10^{-4}, \quad F: 1/3 \\
&p\left(E_5^f\right) = p\left(A_1^f\right)\cdot p\left(A_2^f\right)\cdot p(A_3^{\mathrm{o}}) = 6.68\cdot 10^{-6}, \quad F: 2/3 \\
&p\left(E_6^f\right) = p\left(A_1^f\right)\cdot p(A_2^{\mathrm{o}})\cdot p\left(A_3^f\right) = 9.76\cdot 10^{-7}, \quad F: 2/3 \\
&p\left(E_7^f\right) = p(A_1^{\mathrm{o}})\cdot p\left(A_2^f\right)\cdot p\left(A_3^f\right) = 5.77\cdot 10^{-7}, \quad F: 2/3 \\
&p\left(E_8^f\right) = p\left(A_1^f\right)\cdot p\left(A_2^f\right)\cdot p\left(A_3^f\right) = 1.95\cdot 10^{-9}, \quad F: 3/3.
\end{aligned}$$

The corresponding system of eight events can be presented as follows:

$$\mathcal{S} = \begin{pmatrix} E_1^{\mathrm{o}} & E_2^f & E_3^f & E_4^f & E_5^f & E_6^f & E_7^f & E_8^f \\ 0.99436 & 3.35\cdot 10^{-3} & 1.98\cdot 10^{-3} & 2.89\cdot 10^{-4} & 6.68\cdot 10^{-6} & 9.76\cdot 10^{-7} & 5.77\cdot 10^{-7} & 1.95\cdot 10^{-9} \end{pmatrix}$$

The failure probability of such a system is equal to:

$$P_{\mathrm{f}}(\mathcal{S}) = \sum_{i=2}^{8} p\left(E_i^f\right) = 5.64 \cdot 10^{-3}.$$

Next, the more accurate joint failure probabilities of any combination of modes for the prototype are considered. AFORM provides only the joint failure probabilities of up to two joint events. After neglecting the intersection of three or more events, the following seven modes for the prototype, Fig. 1c, are encountered:

$$\begin{aligned}
&E_1^{\mathrm{o}} = 1 - E_2^f - E_3^f - E_4^f - E_5^f - E_6^f - E_7^f \\
&E_2^f = E_{1,1}^f = A_1^f - A_1^f \cap A_2^f - A_1^f \cap A_3^f = A_1^f - E_{1,2}^f - E_{1,3}^f \\
&E_3^f = E_{2,2}^f = A_2^f - A_1^f \cap A_2^f - A_2^f \cap A_3^f = A_2^f - E_{1,2}^f - E_{2,3}^f \\
&E_4^f = E_{3,3}^f = A_3^f - A_1^f \cap A_3^f - A_2^f \cap A_3^f = A_3^f - E_{1,3}^f - E_{2,3}^f \\
&E_5^f = E_{1,2}^f = A_1^f \cap A_2^f \\
&E_6^f = E_{1,3}^f = A_1^f \cap A_3^f \\
&E_7^f = E_{2,3}^f = A_2^f \cap A_3^f.
\end{aligned}$$

The failure probability matrix is as follows:

$$\mathbf{P} = \begin{bmatrix} 2.43 \cdot 10^{-3} & \ldots & \mathrm{sym} \\ 9.24 \cdot 10^{-4} & 1.20.10^{-3} & \ldots \\ 1.14 \cdot 10^{-6} & 4.25 \cdot 10^{-5} & 2.47 \cdot 10^{-4} \end{bmatrix}$$

The system of events corresponding to the prototype frame, based on the inclusion-exclusion expansion of up to two joint events with probabilities obtained by an AFORM analysis, is shown next:

$$\mathcal{S} = \begin{pmatrix} E_{1,1}^f & E_{2,2}^f & E_{3,3}^f & E_{1,2}^f & E_{1,3}^f & E_{2,3}^f & E_1^{\mathrm{o}} \\ 2.43 \cdot 10^{-3} & 1.02 \cdot 10^{-3} & 2.47 \cdot 10^{-4} & 9.24 \cdot 10^{-4} & 1.14 \cdot 10^{-6} & 4.25 \cdot 10^{-5} & 0.99534 \end{pmatrix}.$$

The only unaccounted event in this example is $A_1^f \cap A_2^f \cap A_3^f$. The Ditlevsen's lower bound is obtained as the summa of diagonal terms and the elements of the lower triangle of the probability matrix $\mathbf{P}$ according to (16), as follows:

$$P_{\mathrm{f}}(\mathcal{S}) \geqslant \sum_{i=1}^{3} \sum_{j=1}^{3} p\left(E_{ij}^f\right) = 4.66 \cdot 10^{-3}.$$

The Ditlevsen's bounds, according to (16), differ only for $p\left(A_1^f \cap A_3^f\right)$, therefore, let us assume the probability of unobservable modes amounts to $p(\mathcal{U}) = p\left(A_1^f \cap A_3^f\right) = 1.14 \cdot 10^{-6}$.

As there is only one operational mode, there is no redundancy of the system $\mathcal{S}$, i.e. $RED(\mathcal{S})=0$. The prototype's safety indices are: $\beta_1 = 2.71$, $\beta_2 = 2.55$ and $\beta_3 = 3.44$ and $COV_\beta=0.1036$. The system uncertainties are calculated according to (18) and (19) as follows: $H(\mathcal{S}) = 0.0508(2.8073, 0.181, 1.0358)$ bits and $H(\mathcal{S}') = 0.0428(1, 0.0428, 1.0301)$ bits. Note the maximal and relative values, as well as the average number of events in paranthesis. The robustness can be calculated according to (24) as:

$$\mathrm{ROB}(\mathcal{S}) = H_{N_\mathrm{f}}(\mathcal{S}/\mathcal{F}) = -\sum_{i=1}^{6}\frac{p\left(E_i^f\right)}{p(\mathcal{F})}\cdot\log\frac{p\left(E_i^f\right)}{p(\mathcal{F})} = 1.7215 \ \ (2.5849, \ \ 0.6659, \ \ 3.2978) \ \ \text{bits}.$$

The system robustness relative to its maximal attainable value, according to (33) is as shown: $rob(\mathcal{S}) = ROB(\mathcal{S})/\log(6) = 1.7215/2.5849 = 0.6659$.

The conditional entropy of system $\mathcal{S}$ with respect to $\mathcal{S}'$ is calculated according to (25) as follows:

$$H(\mathcal{S}/\mathcal{S}') = P_\mathrm{f}(\mathcal{S})\cdot\mathrm{ROB}(\mathcal{S}) = 0.0080(2.8073, \ \ 0.0028, \ \ 1.0056) \ \ \text{bits}.$$

The average probabilities and average number of events are calculated according to (36, 37) as:

$$F_N(\mathcal{S}) = \frac{1}{G_N(\mathcal{S})} = 1.0358, \quad F_2(\mathcal{S}') = \frac{1}{G_2(\mathcal{S}')} = 1.0301, \quad F_{N_\mathrm{f}}(\mathcal{S}/\mathcal{F}) = \frac{1}{G_{N_\mathrm{f}}(\mathcal{S}/\mathcal{F})} = 3.2978.$$

The results are summarized in Table 1.

5.2. Robustness optimization

Let us suppose that the mean values of random plastic moment capacities M_i, $i = 1, 2, 3, 4, 5$, are free design variables of a robustness optimization problem for the given reliability, stated as follows:

$$\mathrm{Max}\,\mathrm{ROB}(\mathcal{S})\ \text{subjected to}\ \ R(\mathcal{S}) = 0.99534.$$

The results of intensive optimization with respect to plastic moment capacities, using AFORM within a general nonlinear programming procedure, led to the family of design mean values of plastic moment capacities defined by M_1 = in range from 0 to 165 kNm, by M_2 = 115 kNm, M_3 = 35 kNm, M_4 = 290 kNm and by M_5 = in range 165-M_1 kNm. The coefficients of variations of all design variables were constant through the entire optimization process of $COV_{\mathrm{Mi}}=0.1$.

The safety indices of the optimized frame structure are $\beta_1=2.88$, $\beta_2 = 2.84$ and $\beta_3 = 2.99$ and the appropriate coefficient of variation of safety indices is $COV_\beta=0.0218$.

The mode correlation matrix and the matrix of degrees of dependencies for optimized frame with respect to robustness are as follows:

Table 1
Uncertainties for different approximation methods of a plane frame

Approximation methods	N_f	N_o	$p(\mathcal{O})$ Eq. (7)	$p(\mathcal{F})$ Eq. (8)	$H(\mathcal{O}+\mathcal{F})$ bits Eq. (18)	$H(\mathcal{O},\mathcal{F})$ bits Eq. (19)	ROB($\mathcal{S}$) bits Eq. (24)	$F(\mathcal{S}/\mathcal{O})$ bits Eq. (36)	$H(\mathcal{S})-H(\mathcal{S}')$ bits Eq. (25)	COV_β
Exclusive modes	3	1	0.99436	0.00564	0.0570	0.0502	1.1961	2.2912	0.0067	0.1036
	Maximal values				2	1	1.5849	3	2	
	Relative values				0.0285	0.0502	0.7546	1.3093	0.0033	
Inclusive modes	3	1	0.99664	0.00336	0.0369	0.0324	1.3307	2.5152	0.0044	0.1036
	Maximal values				2	1	1.5849	3	2	
	Relative values				0.0184	0.0324	0.8396	1.1927	0.0022	
Independent modes	7	1	0.99437	0.00563	0.0569	0.0501	1.2107	2.3145	0.0068	0.1036
	Maximal values				3	1	2.8073	7	3	
	Relative values				0.0190	0.0501	0.4312	3.0244	0.0022	
Correlated modes	6	1	0.99534	0.00466	0.0508	0.0428	1.7215	3.2978	0.0080	0.1036
	Maximal values				2.8073	1	2.5849	6	2.8073	
	Relative values				0.0181	0.0428	0.6660	1.8193	0.0028	
Optimal robustness (based on AFORM) (also least weight)	6	1	0.99534	0.00466	0.0526	0.0427	2.1229	4.3557	0.0098	0.0218
	Maximal values				2.8073	1	2.5849	6	2.8073	
	Relative values				0.0187	0.0427	0.8212	1.3775	0.0035	
Optimal robustness (checked by MCS)	6	1	0.99516	0.00484	0.0549	0.0442	2.2108	4.6293	0.0108	0.0370
	Maximal values				2.8073	1	2.5849	6	2.8073	
	Relative values				0.0195	0.0442	0.8552	1.2961	0.0038	

$$[\gamma_{ij}] = \begin{bmatrix} 1 & \dots & \mathrm{sym} \\ 0.847 & 1 & \dots \\ 0.039 & 0.550 & 1 \end{bmatrix} \quad [\rho_{ij}] = \begin{bmatrix} 1 & \dots & \mathrm{sym} \\ 0.388 & 1 & \dots \\ 0.003 & 0.111 & 1 \end{bmatrix}$$

The following system of events for the optimized frame is obtained:

$$\mathcal{S} = \begin{pmatrix} E^f_{1,1} & E^f_{2,2} & E^f_{3,3} & E^f_{1,2} & E^f_{1,3} & E^f_{2,3} & E^o_1 \\ 1.21\cdot10^{-3} & 1.32.\cdot10^{-3} & 1.21\cdot10^{-4} & 7.69\cdot10^{-4} & 3.96\cdot10^{-6} & 1.50\cdot10^{-4} & 0.99534 \end{pmatrix}.$$

The Ditlevsen's lower bound of failure probability amounts to the same value as for the prototype:

$$P_f(\mathcal{S}) \geqslant \sum_{i=1}^{3}\sum_{j=1}^{3} p\left(E^f_{ij}\right) = 4.66\cdot10^{-3}.$$

The maximum robustness of the optimized system is, according to (24), $ROB(\mathcal{S}) = 2.1229$ bits and it is significantly greater than for the initial prototype system. The system robustness relative to its maximal attainable value of $rob(\mathcal{S}) = 0.8212$ indicates that the optimized system utilizes a greater part of robustness capacity of the prototype, Table 1. Put succinctly, the plane frame optimized with respect to robustness for the same reliability level of 0.99534 is 1.23 times more robust than the prototype.

5.3. Weight optimization

The lightest frame, assuming proportional cross-sections, with the given reliability, is defined by the following nonlinear programming model:

$$\mathrm{Min}\sum_{i=1}^{5}(M_i)^{\frac{2}{3}} \text{ subjected to } R(\mathcal{S}) = 0.99534.$$

It became evident, on the basis of a number of successive optimizations, that the lightest frame with weight equal to

$$\mathrm{Min}\sum_{i=1}^{5}(M_i)^{\frac{2}{3}} = 108.24$$

belongs to the set of solutions with maximal robustness for the given reliability level, providing, therefore, identical probability distribution.

Moreover, it is evident that the prototype frame, which is certainly a heavy structure, is 22% heavier than the lightest frame, which is in the same time maximally robust. The reduction in weight of the frame structures optimized with respect to robustness is a consequence of a more uniform distribution of failure probabilities pertaining to increasing robustness.

5.4. Checking by Monte-Carlo simulation

The AFORM reliability and uncertainty calculations for the frame optimized with respect to robustness are checked by intensive Monte-Carlo simulation using Latin hypercube sampling plan [29] and the following system of events is obtained:

$$\mathcal{S} = \begin{pmatrix} E^f_{1,1} & E^f_{2,2} & E^f_{3,3} & E^f_{1,2} & E^f_{1,3} & E^f_{2,3} & E^o_1 \\ 0.87\cdot10^{-3} & 1.62.\cdot10^{-3} & 0.89\cdot10^{-3} & 1.04\cdot10^{-3} & 3.25\cdot10^{-6} & 0.42\cdot10^{-3} & 0.99516 \end{pmatrix}.$$

The robustness of the system based on failure probabilities calculated by MCS is, according to (24), $ROB(\mathcal{S}) = 2.2108(2.5849,\ 0.8552,\ 4.6293)$ bits, the generalized safety indices are $\beta_1 = 2.89$, $\beta_2 = 2.74$ and $\beta_3 = 3.00$ and the appropriate coefficient of variation of safety indices is $COV_\beta = 0.0370$.

5.5. System performance optimization

The structural preferences of series systems are the high system reliability and high system robustness. The alternative reliability and uncertainty optimization problems within reasonable bounds on design variables of $50 < M_i < 250$, $i = 1, \ldots, 5$, assuming that the weight of structures are equal, are defined by two criteria, as:

1. max $R(\mathcal{S})$ subjected to : $\frac{1}{5}\sum_{i=1}^{5}\left(\frac{M_i}{134.9}\right)^{\frac{2}{3}}=1$

2. max $\mathrm{ROB}(\mathcal{S})$

The imposed nonlinear constraint in the above defined optimization task substitutes the condition of equal weights of structural components, assuming proportional shapes of cross sectional areas. The results of AFORM calculations within nonlinear programming procedure are presented in Table 2.

The safety indices and systems of events corresponding to the two optimization procedures are presented next in order to illustrate the differences in reliabilities and probability distributions of systems with various robustness, namely,

1. max$R(\mathcal{S})$: $\beta_1=3.20$, $\beta_2=3.42$ and $\beta_3=3.44$ and $\mathrm{COV}_\beta=0.0324$.

$$\mathcal{S}=\begin{pmatrix} E^f_{1,1} & E^f_{2,2} & E^f_{3,3} & E^f_{1,2} & E^f_{1,3} & E^f_{2,3} & E^o_1 \\ 5.25\cdot10^{-4} & 1.47\cdot10^{-4} & 2.78\cdot10^{-4} & 1.58\cdot10^{-4} & 2.71\cdot10^{-7} & 1.01\cdot10^{-5} & 0.99888 \end{pmatrix}.$$

2. maxROB($\mathcal{S}$): $\beta_1=2.76$, $\beta_2=2.72$ and $\beta_3=2.86$ and $\mathrm{COV}_\beta=0.0211$.

$$\mathcal{S}=\begin{pmatrix} E^f_{1,1} & E^f_{2,2} & E^f_{3,3} & E^f_{1,2} & E^f_{1,3} & E^f_{2,3} & E^o_1 \\ 1.75\cdot10^{-3} & 1.92.\cdot10^{-3} & 1.87\cdot10^{-3} & 1.13\cdot10^{-3} & 6.16\cdot10^{-6} & 2.49\cdot10^{-4} & 0.99307 \end{pmatrix}.$$

Note that different measure of dispersion statistics, such as range, mean deviation, variance, standard deviation and coefficient of variation of a set of consistent semiprobabilistic or deterministic safety measures, can be used to assess system robustness heuristically. From an engineering point of view, high robustness can be associated with the small coefficient of variation of safety indices, as it is evident for optimized frames.

Moreover, $COV_\beta=0$ can be attained in the case of equal weight frames, Table 2, employing minimization procedure on COV_β, with safety indices and probabilities as:

3. min$COV\beta$: $\beta_1=2.90$, $\beta_2=2.90$ and $\beta_3=2.90$ and $COV_\beta=0$. The system of events is as shown:

Table 2
Results of different optimization procedures applied to a plane frame structure

Procedure	$R(\mathcal{S})$	ROB($\mathcal{S}$) Eq. (24)	rob($\mathcal{S}$) Eq. (33)	$F_6(\mathcal{S}/\mathcal{F})$ Eq. (36)	$H_7(\mathcal{S})$ Eq. (18)	$h_2(\mathcal{S})$ Eq. (29)	M_1 (Variables)	M_2	M_3	M_4	M_5	COV_β
Prototype	0.99534	1.7215	0.6659	3.2978	0.0508	0.0181	134	134	134	134	134	0.1036
$R_{\max}$	0.99888	1.8590	0.7192	3.6276	0.0146	0.0052	50-250	150	150	150	251-M_1	0.0324
$\mathrm{ROB}_{\max}$	0.99307	2.1334	0.8253	4.3875	0.0745	0.0265	50-250	112	147	50	386-M_1	0.0211
$\mathrm{COV}_{\beta\min}=0$	0.99530	2.0711	0.8012	4.2020	0.0528	0.0188	244	110	116	118	100	0.0000

$$\mathcal{S} = \begin{pmatrix} E^f_{1,1} & E^f_{2,2} & E^f_{3,3} & E^f_{1,2} & E^f_{1,3} & E^f_{2,3} & E^o_1 \\ 1.16 \cdot 10^{-3} & 1.03 \cdot 10^{-3} & 1.69 \cdot 10^{-3} & 6.69 \cdot 10^{-3} & 3.77 \cdot 10^{-6} & 1.43 \cdot 10^{-4} & 0.99530 \end{pmatrix}.$$

Next by repeated optimizations, the manner in which the function $maxROB(S)$ depends on the system reliability, is presented in Fig. 4.

5.6. Robustness of disjoint subsystems at different levels

Various system subdivision schemes in general as k out of n events, can be of special interest [15]. Let us consider the frame optimized with respect to robustness for given reliability.

The subsystem of the failure modes is first partitioned into the subsystem $\mathcal{F}_1$ of individual events and into the subsystem $\mathcal{F}_2$ of common cause modes, presented as follows:

$$S' = \begin{pmatrix} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{O} \\ p(\mathcal{F}_1) & p(\mathcal{F}_2) & p(\mathcal{O}) \end{pmatrix} = \begin{pmatrix} \mathcal{F}_1 & \mathcal{F}_2 & \mathcal{O} \\ 3.74 \cdot 10^{-3} & 9.22 \cdot 10^{-4} & 0.99534 \end{pmatrix} \text{ where}$$

$$\mathcal{F}_1 = \begin{pmatrix} E^f_1 & E^f_{22} & E^f_{33} \\ p\left(E^f_{11}\right) & p\left(E^f_{22}\right) & p\left(E^f_{33}\right) \end{pmatrix} = \begin{pmatrix} \left(E^f_{11}\right) & E^f_{22} & E^f_{33} \\ 1.21 \cdot 10^{-3} & 1.32 \cdot 10^{-3} & 121.10^{-3} \end{pmatrix}$$

$$\mathcal{F}_2 = \begin{pmatrix} E^f_2 & E^f_{13} & E^f_{23} \\ p\left(E^f_{12}\right) & p\left(E^f_{13}\right) & p\left(E^f_{23}\right) \end{pmatrix} = \begin{pmatrix} \left(E^f_{12}\right) & E^f_{13} & E^f_{23} \\ 7.69 \cdot 10^{-4} & 3.96 \cdot 10^{-6} & 1.50 \cdot 10^{-4} \end{pmatrix}$$

The compound system of subsystems of failure modes can be written as:

$$\mathcal{F}' = \begin{pmatrix} \mathcal{F}_1 & \mathcal{F}_2 \\ 3.74 \cdot 10^{-3} & 9.22 \cdot 10^{-4} \end{pmatrix}.$$

The system robustness according to (24) may now be viewed in two components:

$$\mathrm{ROB}(\mathcal{F}/\mathcal{F}_1) = -\frac{p\left(E^f_{11}\right)}{p(\mathcal{F}_1)} \log \frac{p\left(E^f_{11}\right)}{p(\mathcal{F}_1)} - \frac{p\left(E^f_{22}\right)}{p(\mathcal{F}_1)} \log \frac{p\left(E^f_{22}\right)}{p(\mathcal{F}_1)} - \frac{p\left(E^f_{33}\right)}{p(\mathcal{F}_1)} \log \frac{p\left(E^f_{33}\right)}{p(\mathcal{F}_1)}$$
$$= 1.5837 \ \ (1.5849, \ \ 0.9992, \ \ 2.9974) \ \text{bits},$$

$$\mathrm{ROB}(\mathcal{F}/\mathcal{F}_2) = -\frac{p\left(E^f_{12}\right)}{p(\mathcal{F}_2)} \log \frac{p\left(E^f_{12}\right)}{p(\mathcal{F}_2)} - \frac{p\left(E^f_{13}\right)}{p(\mathcal{F}_2)} \log \frac{p\left(E^f_{13}\right)}{p(\mathcal{F}_2)} - \frac{p\left(E^f_{23}\right)}{p(\mathcal{F}_2)} \log \frac{p\left(E^f_{23}\right)}{p(\mathcal{F}_2)}$$
$$= 0.6791 \ \ (1.5849, \ \ 0.4285, \ \ 1.6011) \ \text{bits}.$$

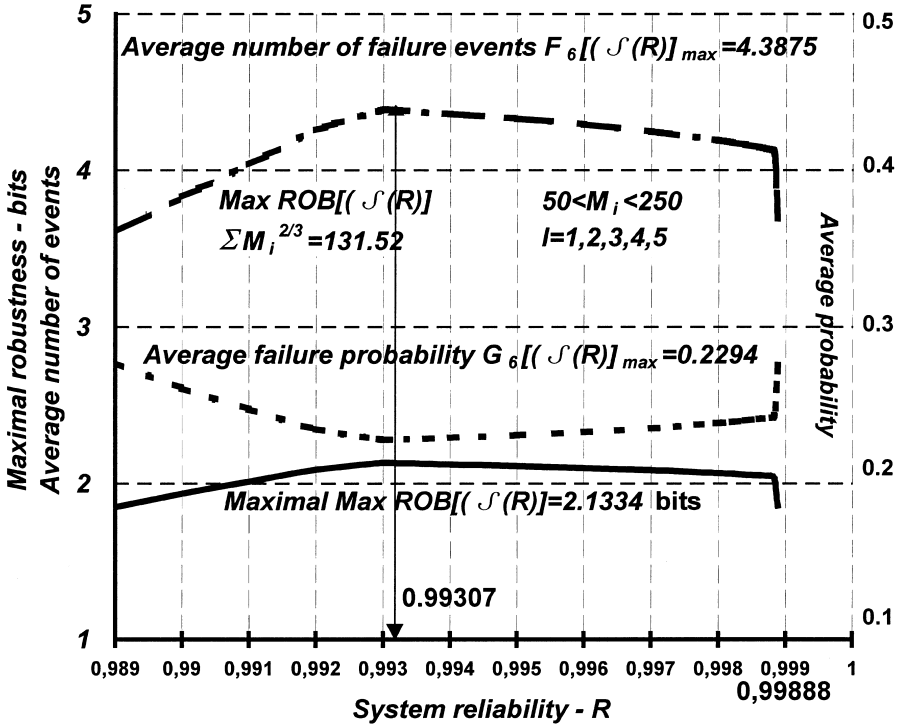

Fig. 4. Plane frame robustness optimization for equal weight structures.

Each component robustness above indicates system endurance with respect to the considered subsystems. The robustness of the system and the system of subsystems can be calculated as follows:

$$\mathrm{ROB}(\mathcal{S}'/\mathcal{F}') = \sum_{i=1}^{2} \frac{p(\mathcal{F}_i)}{p(\mathcal{F})} \log \frac{p(\mathcal{F}_i)}{p(\mathcal{F})} = 0.7174, \quad (1.0000, \quad 0.7174, \quad 1.6442) \text{ bits},$$

$$\mathrm{ROB}(\mathcal{S}/\mathcal{F}) = \sum_{\text{all } E^f} \frac{p\left(E_i^f\right)}{p(\mathcal{F})} \log \frac{p\left(E_i^f\right)}{p(\mathcal{F})} = 2.21124 \ (2.5849, \quad 0.8210, \quad 4.3241) \text{ bits}.$$

According to (26) it is easy to verify the relation among subsystem and system robustness.

5.7. Robustness of the unions of subsystems with common cause failure modes

Let us consider again the frame optimized with respect to robustness for given reliability by partitioning the system of events into the three inclusive subsystems with common cause modes, as follows:

$$\mathcal{F}_1 = \begin{pmatrix} E_{11}^f & E_{12}^f & E_{13}^f \\ p\left(E_{11}^f\right) & p\left(E_{12}^f\right) & p\left(E_{13}^f\right) \end{pmatrix} = \begin{pmatrix} E_{11}^f & E_{12}^f & E_{13}^f \\ 1.21\cdot10^{-3} & 7.69\cdot10^{-4} & 3.96\cdot10^{-6} \end{pmatrix}$$

$$\mathcal{F}_2 = \begin{pmatrix} E_{22}^f & E_{21}^f & E_{23}^f \\ p\left(E_{22}^f\right) & p\left(E_{21}^f\right) & p\left(E_{73}^f\right) \end{pmatrix} = \begin{pmatrix} E_{22}^f & E_{21}^f & E_{23}^f \\ 1.32\cdot10^{-3} & 7.69\cdot10^{-4} & 1.50\cdot10^{-4} \end{pmatrix}$$

$$\mathcal{F}_3 = \begin{pmatrix} E_{33}^f & E_{31}^f & E_{32}^f \\ p\left(E_{33}^f\right) & p\left(E_{31}^f\right) & p\left(E_{32}^f\right) \end{pmatrix} = \begin{pmatrix} E_{33}^f & E_{31}^f & E_{32}^f \\ 1.21\cdot10^{-3} & 3.96\cdot10^{-6} & 1.50\cdot10^{-4} \end{pmatrix}$$

The subsystem of all failure modes can be presented as the union of the subsystems listed above, as $\mathcal{F} = (\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3)$.

The failure probabilities associated with such subsystems consisting of individual modes together with all the common cause failures with respect to all other modes are equal to:

$$P_{\mathrm{f}}(\mathcal{F}_1) = \sum_{i=1}^{3}\left(E_{1i}^f\right) = 1.98\cdot10^{-3}, \quad P_{\mathrm{f}}(F_2) = \sum_{i=1}^{3} p\left(E_{2i}^f\right) = 2.24\cdot10^{-3},$$
$$P_{\mathrm{f}}(\mathcal{F}_3) = \sum_{i=1}^{3} p\left(E_{3i}^f\right) = 1.36\cdot10^{-3}.$$

The robustness of each of the considered subsystems, is calculated as follows:

$$\begin{aligned}
\mathrm{ROB}(\mathcal{S}/\mathcal{F}_1) &= -\frac{p\left(E_{11}^f\right)}{p(\mathcal{F}_1)}\log\frac{p\left(E_{11}^f\right)}{p(\mathcal{F}_1)} - \frac{-p\left(E_{12}^f\right)}{p(\mathcal{F}_1)}\log\frac{p\left(E_{12}^f\right)}{p(\mathcal{F}_1)} - \frac{p\left(E_{13}^f\right)}{p(\mathcal{F}_1)}\log\frac{p\left(E_{13}^f\right)}{p(\mathcal{F}_1)} \\
&= 0.9827\ \ (1.5849,\ 0.6200,\ 1.9762)\ \text{bits}, \\
\mathrm{ROB}(\mathcal{S}/\mathcal{F}_2) &= -\frac{p\left(E_{22}^f\right)}{p(\mathcal{F}_2)}\log\frac{p\left(E_{22}^f\right)}{p(\mathcal{F}_2)} - \frac{p\left(E_{21}^f\right)}{p(\mathcal{F}_2)}\log\frac{p\left(E_{21}^f\right)}{p(\mathcal{F}_2)} - \frac{p\left(E_{23}^f\right)}{p(\mathcal{F}_2)}\log\frac{p\left(E_{23}^f\right)}{p(\mathcal{F}_2)} \\
&= 1.2402\,(1.5849,\ \ 0.7825,\ \ 2.3623)\ \ \text{bits}, \\
\mathrm{ROB}(\mathcal{S}/\mathcal{F}_3) &= -\frac{p\left(E_{33}^f\right)}{p(\mathcal{F}_3)}\log\frac{p\left(E_{33}^f\right)}{p(\mathcal{F}_3)} - \frac{p\left(E_{31}^f\right)}{p(\mathcal{F}_3)}\log\frac{p\left(E_{31}^f\right)}{p(\mathcal{F}_3)} - \frac{p\left(E_{32}^f\right)}{p(\mathcal{F}_3)}\log\frac{p\left(E_{32}^f\right)}{p(\mathcal{F}_3)} \\
&= 0.5280\,(1.5849,\ 0.3331,\ 1.4419)\ \ \text{bits}.
\end{aligned}$$

Let us consider the subsystems built as pairways unions and intersections of inclusive subsystems having common cause failure modes with failure probabilities equal to following amounts:

$$P_{\rm f}(\mathcal{F}_1 \cup \mathcal{F}_2) = p\left(E_{11}^f\right) + p\left(E_{22}^f\right) + p\left(E_{12}^f\right) + p\left(E_{13}^f\right) + p\left(E_{23}^f\right) = 3.45 \cdot 10^{-3},$$

$$P_{\rm f}(\mathcal{F}_1 \cap \mathcal{F}_2) = p\left(E_{12}^f\right) = 7.69 \cdot 10^{-4}$$

$$P_{\rm f}(\mathcal{F}_1 \cup \mathcal{F}_3) = p\left(E_{11}^f\right) + p\left(E_{33}^f\right) + p\left(E_{13}^f\right) + p\left(E_{12}^f\right) + p\left(E_{23}^f\right) = 3.34 \cdot 10^{-3},$$

$$P_{\rm f}(\mathcal{F}_1 \cup \mathcal{F}_3) = p\left(E_{13}^f\right) = 3.96 \cdot 10^{-6}$$

$$P_{\rm f}(\mathcal{F}_2 \cup \mathcal{F}_3) = p\left(E_{22}^f\right) + p\left(E_{33}^f\right) + p\left(E_{23}^f\right) + p\left(E_{12}^f\right) + p\left(E_{13}^f\right) = 3.34 \cdot 10^{-3},$$

$$P_{\rm f}(\mathcal{F}_2 \cap \mathcal{F}_3) = p\left(E_{23}^f\right) = 1.50 \cdot 10^{-4}$$

The robustness of pairways unions of events are calculated as follows:

$$\begin{aligned}
\mathrm{ROB}(\mathcal{S}/\mathcal{F}_1 \cup \mathcal{F}_2) = &-\frac{p\left(E_{11}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} \log \frac{p\left(E_{11}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} - \frac{p\left(E_{22}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} \log \frac{p\left(E_{22}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} \\
&-\frac{p\left(E_{12}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} \log \frac{p\left(E_{12}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} - \frac{p\left(E_{13}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} \log \frac{p\left(E_{13}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} \\
&-\frac{p\left(E_{23}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} \log \frac{p\left(E_{23}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_2)} = 1.7508 \ (2.3219, \ 0.7540, \ 3.3654) \ \text{bits},
\end{aligned}$$

$$\begin{aligned}
\mathrm{ROB}(\mathcal{S}/\mathcal{F}_1 \cup \mathcal{F}_3) = &-\frac{p\left(E_{11}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} \log \frac{p\left(E_{11}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} - \frac{p\left(E_{33}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} \log \frac{p\left(E_{33}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} \\
&-\frac{p\left(E_{13}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} \log \frac{p\left(E_{13}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} - \frac{p\left(E_{12}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} \log \frac{p\left(E_{12}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} \\
&-\frac{p\left(E_{23}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} \log \frac{p\left(E_{23}^f\right)}{p(\mathcal{F}_1 \cup \mathcal{F}_3)} = 1.7615 \ (2.3219, \ 0.7586, \ 3.3905) \ \text{bits},
\end{aligned}$$

$$\begin{aligned}
\mathrm{ROB}(\mathcal{S}/\mathcal{F}_2 \cup \mathcal{F}_3) = &-\frac{p\left(E_{22}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} \log \frac{p\left(E_{22}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} - \frac{p\left(E_{33}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} \log \frac{p\left(E_{33}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} \\
&-\frac{p\left(E_{23}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} \log \frac{p\left(E_{23}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} - \frac{p\left(E_{12}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} \log \frac{p\left(E_{12}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} \\
&-\frac{p\left(E_{13}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} \log \frac{p\left(E_{13}^f\right)}{p(\mathcal{F}_2 \cup \mathcal{F}_3)} = 1.7508 \ (2.3219, \ 0.7540, \ 3.3654) \ \text{bits},
\end{aligned}$$

Since the pairways intersections comprise of only one event, the appropriate robustness vanishes:

$$\mathrm{ROB}(\mathcal{S}/\mathcal{F}_1\cap\mathcal{F}_2)=0,\quad \mathrm{ROB}(\mathcal{S}/\mathcal{F}_1\cap\mathcal{F}_3)=0,\; \mathrm{ROB}(\mathcal{S}/\mathcal{F}_2\cap\mathcal{F}_3)=0.$$

The conditional entropy of the systems of subsystems having some modes in common according to (28), are as shown:

$$H(\mathcal{S}'/\mathcal{F}_1\cup\mathcal{F}_2)=0.3839,\quad H(\mathcal{S}'/\mathcal{F}_1\cup\mathcal{F}_3)=0.9537,\quad H(\mathcal{S}'/\mathcal{F}_2\cup\mathcal{F}_3)=0.7373..$$

The relation (27) between any pair of subsystems can be easily verified by this example.

Finally, let us consider the system of the unions of all three subsystems. The probability of the union of all three events is equal to the robustness of the entire system as calculated earlier:

$$\begin{aligned}P_{\mathrm{f}}(\mathcal{F})=P_f(\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3)&=p\left(E^f_{11}\right)+p\left(E^f_{22}\right)\\&+p\left(E^f_{33}\right)+p\left(E^f_{12}\right)+p\left(E^f_{13}\right)+p\left(E^f_{23}\right)=4.66\cdot10^{-3}.\end{aligned}$$

The robustness of the union of all three events is equal to the robustness of the entire system as calculated above:

$$\mathrm{ROB}(\mathcal{S})=\mathrm{ROB}(\mathcal{S}/\mathcal{F})=\mathrm{ROB}(\mathcal{S}/\mathcal{F}_1\cup\mathcal{F}_2\cup\mathcal{F}_3)=2.1229\;\;(2.5849,\;\;0.8212,\;\;4.3557)\;\;\text{bits}.$$

5.8. Sensitivity calculations

The sensitivity calculations with respect to the variable means and standard deviations are performed on the prototype basis using the analytical procedure given in the article and presented in Table 3.

Table 3
Results of sensitivity analysis applied to the prototype plane frame structure

Sensitivity factors	Equation	M_1	M_2	M_3	M_4	M_5	F_H	F_v units
$\partial H(\mathcal{S})\partial\mu$	(46)	$-7.57\cdot10^{-4}$	$-5.70\cdot10^{-4}$	$-6.84\cdot10^{-4}$	$-1.10\cdot10^{-3}$	$-7.56\cdot10^{-4}$	$+2.75\cdot10^{-3}$	$+1.46\cdot10^{-3}$ bits
$\partial H(\mathcal{S})\partial\sigma$	(46)	$+2.42\cdot10^{-4}$	$+2.00\cdot10^{-4}$	$+3.18\cdot10^{-4}$	$+4.52\cdot10^{-4}$	$+2.36\cdot10^{-4}$	$+1.65\cdot10^{-2}$	$+5.03\cdot10^{-3}$ bits
$\partial H(\mathcal{S}')\partial\mu$	(47)	$-6.30\cdot10^{-4}$	$-4.85\cdot10^{-4}$	$-4.92\cdot10^{-4}$	$-8.74\cdot10^{-4}$	$-6.30\cdot10^{-4}$	$+2.40\cdot10^{-3}$	$+9.00\cdot10^{-4}$ bits
$\partial H(\mathcal{S}')\partial\sigma$	(47)	$+1.93\cdot10^{-4}$	$+1.69\cdot10^{-4}$	$+2.97\cdot10^{-4}$	$+3.49\cdot10^{-4}$	$+1.93\cdot10^{-4}$	$+1.34\cdot10^{-2}$	$+3.84\cdot10^{-3}$ bits
$\partial \mathrm{ROB}(\mathcal{S})\partial\mu$	(48)	$+2.83\cdot10^{-3}$	$+5.15\cdot10^{-3}$	$-1.85\cdot10^{-2}$	$-7.28\cdot10^{-3}$	$+2.96\cdot10^{-3}$	$-4.17\cdot10^{-2}$	$+7.64\cdot10^{-2}$ bits
$\partial \mathrm{ROB}(\mathcal{S})\partial\sigma$	(48)	$+1.35\cdot10^{-3}$	$-1.35\cdot10^{-3}$	$-9.80\cdot10^{-3}$	$+5.30\cdot10^{-3}$	$+9.10\cdot10^{-5}$	$+1.45\cdot10^{-2}$	$+7.20\cdot10^{-2}$ bits
$\partial P_{\mathrm{f}}\mathrm{ROB}(\mathcal{S})\partial\mu$	(49)	$-1.27\cdot10^{-4}$	$-8.41\cdot10^{-5}$	$-1.95\cdot10^{-4}$	$-2.28\cdot10^{-4}$	$-1.26\cdot10^{-4}$	$+3.41\cdot10^{-4}$	$+5.55\cdot10^{-4}$ bits
$\partial P_{\mathrm{f}}\mathrm{ROB}(\mathcal{S})\partial\sigma$	(49)	$+4.92\cdot10^{-5}$	$+3.14\cdot10^{-5}$	$+2.06\cdot10^{-5}$	$+1.02\cdot10^{-4}$	$+4.34\cdot10^{-5}$	$+3.05\cdot10^{-3}$	$+1.19\cdot10^{-3}$ bits
$\partial P_{\mathrm{f}}(\mathcal{S})\partial\mu$	(50)	$-8.14\cdot10^{-5}$	$-6.26\cdot10^{-5}$	$-6.35\cdot10^{-5}$	$-1.13\cdot10^{-4}$	$-8.14\cdot10^{-5}$	$+3.10\cdot10^{-4}$	$+1.16\cdot10^{-4}$
$\partial P_{\mathrm{f}}(\mathcal{S})\partial\sigma$	(50)	$+2.49\cdot10^{-5}$	$+2.18\cdot10^{-5}$	$+3.84\cdot10^{-5}$	$+4.51\cdot10^{-5}$	$+2.49\cdot10^{-5}$	$+1.73\cdot10^{-3}$	$+4.96\cdot10^{-4}$

For example, by using the sensitivity factors in the range of plastic moment capacities from 130 to 140, the variant of the frame with maximal robustness amounting to $ROB(\mathcal{S}) = 1.9058$ is assessed for $M = (140, 140, 130, 130, 140)$. The assessed uncertainties are $H(\mathcal{S}) = 0.0488$, $H(\mathcal{S}') = 0.00437$ and $P_f\ (\mathcal{S}) \bullet ROB(\mathcal{S}) = 0.0835$. For comparison, the appropriate AFORM results are $ROB(\mathcal{S}) = 1.8664$ and $H(\mathcal{S}) = 0.0495$, $H(\mathcal{S}') = 0.00412$ and $P_f(\mathcal{S}) \bullet ROB(\mathcal{S}) = 0.0832$.

6. Conclusion

Complex engineering systems can be subjected to service modes and effects analysis for identification of the events and their relations which can occur at the component, subsystem and system levels. The aim of a probabilistic event oriented analysis is to assess the effects of known operational and failure modes on the overall behavior of a system. In addition to global probabilistic system analysis, the analysis of the hierarchical structure of failure modes presented by subsystems and groups of events of special interests, such as collapse modes and serviceability failures, can be of interest to designers and users of the system.

The article investigates how the difficulties in the application of the EOSA to engineering objects involving complex relations among events can be consistently resolved by employing failure modes analysis accompanied by inclusion-exclusion expansion of union of events and completed by reliability and uncertainty analysis.

The EOSA relates the event space with governing probability and information theory, to the physical aspects of engineering objects by employing the random variable design model, based on technical knowledge and on engineering experience. The reliability in the event space does not appear to be the only decisive attribute effecting the series system performance. The conditional entropy of failure modes defines the system robustness independently of the system reliability as the structural endurance providing uniform distribution of failure probabilities. It is demonstrated how the system robustness for the same reliability level can be increased by redistribution of component characteristics. The dispersion of semiprobabilistic and deterministic safety measures is used to assess heuristically the uncertainty of a number of failure modes.

The application of EOSA provides additional objectives and constraints, defined by reliability and robustness, as independent optimization criteria, leading to more appropriate designs of series structural systems. Sensitivity analysis of system uncertainty measures without additional structural response and reliability evaluations is sufficiently accurate for design uncertainty assessments in a reasonable range of extrapolations, and can be applied as an useful procedure for many engineering considerations.

The EOSA requires more engineering and computational practice than the mere traditional reliability analysis. The relevance of failure modes and their relations depends on whether the circumstances of their occurrence do or do not jeopardize operations, and to which extent they determine the collapse or serviceability failures of the system. The selection of modes, groups and subsystems by their importance depends on the engineering experience and there is a certain freedom of choice within the limits of possibilities.

The event oriented system analysis is feasible if modern powerful computational means and efficient probabilistic numerical and simulation methods are used. Moreover, the conditioning of probabilities and entropy with respect to significant operational and failure modes allows a separate consideration of each particular group or subsystem of relevant events, as well as their relations within complete or incomplete systems of events. Significantly, the application of EOSA to selected parts of systems and subsystems with a manageable number of events is appealing for the evaluation of large-scale systems.

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