2 Basic Mathematical Concepts

2.1 INTRODUCTION

Although the origin of the word *mathematics* can be traced back to the ancient Greek word *mathema*, which essentially means "science, knowledge, or learning," the history of our current number symbols goes back to around 250 BC to the stone columns erected by the Scythian emperor Asoka of India [1]. Over the centuries, mathematics has developed into various specialized areas, including probability and statistics, applied mathematics, and pure mathematics and is successfully being applied to solve various types of science- and engineering-related problems.

The application of the mathematical concepts in science and engineering ranges from solving interplanetary problems to designing and maintaining engineering equipment in the industrial sector. More specifically, over the past few decades various mathematical concepts, particularly probability distributions and stochastic processes (i.e., Markov modeling), have been used to study various types of problems concerning human reliability and error.

For example, in the late 1960s and early 1970s various statistical distributions were used to represent times to human error [2–4]. Furthermore, in the early 1980s the Markov method was employed to perform human reliability-related analysis of redundant systems [5–7].

This chapter presents various introductory mathematical concepts considered useful for conducting human reliability and error analysis in engineering maintenance.

2.2 BOOLEAN ALGEBRA LAWS AND PROBABILITY PROPERTIES

Boolean algebra is named after an English mathematician, George Boole (1813–1864), who developed it in 1854 [8, 9]. As Boolean algebra plays an important role in human reliability and error-related studies, five of its laws are as follows [10–11]:

Associative Law:

$$(A+B)+C=A+(B+C) \tag{2.1}$$

where A is an arbitrary event or set, B is an arbitrary event or set, C is an arbitrary event or set, and + denotes the union of sets.

$$(A \bullet B) \bullet C=A \bullet (B \bullet C) \tag{2.2}$$

where the dot (•) denotes the intersections of sets. When Equation (2.2) is written without the dot, it still conveys the same meaning.

Commutative Law:

$$A+B=B+A \tag{2.3}$$

$$A\,B=B\,A \tag{2.4}$$

Distributive Law:

$$A(B+C)=AB+AC \tag{2.5}$$

$$(A+B)(A+C)=A+BC \tag{2.6}$$

Idempotent Law:

$$A+A=A \tag{2.7}$$

$$AA=A \tag{2.8}$$

Absorption Law:

$$A(A+B)=A \tag{2.9}$$

$$A+(AB)=A \tag{2.10}$$

Probability is the study of random or nondeterministic experiments and it had its real beginnings in the early part of the seventeenth century as a result of investigations of various games of chance by people such as Pierre Fermat (1601–1665) and Blaise Pascal (1623–1662) [12]. The basic properties of probability are presented below [12–15].

- The probability of occurrence of event, say X, is

$$O\leq P(X)\leq 1 \tag{2.11}$$

- The probability of the sample space S is

$$P(S)=1 \tag{2.12}$$

- The probability of the negation of the sample space S (i.e., $\bar{S}$) is

$$P(\bar{S})=0 \tag{2.13}$$

- The probability of occurrence and nonoccurrence of an event, say X, is

$$P(X) + P(\bar{X}) = 1 \tag{2.14}$$

where $P(X)$ is the occurrence probability of event X and $P(\bar{X})$ is the nonoccurrence probability of event X.

- The probability of an intersection of K independent events is

$$P(Y_1 Y_2 Y_3 \ldots Y_K) = P(Y_1)\,P(Y_2)\,P(Y_3)\ldots P(Y_K) \tag{2.15}$$

where Y_j is the jth event for $j = 1, 2, 3, \ldots, K$ and $P(Y_j)$ is the occurrence probability of event Y_j for $j = 1, 2, 3, \ldots, K$.

- The probability of the union of K independent events is

$$P(X_1 + X_2 + \cdots + X_K) = 1 - \prod_{j=1}^{K} (1 - P(X_j)) \tag{2.16}$$

where X_j is the jth event; for $j = 1, 2, \ldots, K$ and $P(X_j)$ is the occurrence probability of event X_j; for $j = 1, 2, \ldots, K$.

For $K = 2$, Equation (2.16) reduces to

$$P(X_1 + X_2) = P(X_1) + P(X_2) - P(X_1)\,P(X_2) \tag{2.17}$$

- The probability of the union of K mutually exclusive events is

$$P(X_1 + X_2 + \cdots + X_K) = \sum_{j=1}^{K} P(X_j) \tag{2.18}$$

Example 2.1

A maintenance worker is performing a maintenance task composed of two independent steps: X and Y. The task will be accomplished correctly only if both the steps are performed correctly. The probabilities of performing steps X and Y correctly by the maintenance worker are 0.9 and 0.8, respectively. Calculate the probability of accomplishing the task correctly by the maintenance worker.

By substituting the specified data into Equation (2.15), we get

$$\begin{aligned} P(XY) &= P(X)\,P(Y) \\ &= (0.9)(0.8) \\ &= 0.72 \end{aligned}$$

where $X = Y_1$ and $Y = Y_2$. Thus, the probability of accomplishing the task correctly by the maintenance worker is 0.72.

Example 2.2

In Example 2.1, by using Equations (2.14) and (2.17) calculate the probability that the task will not be accomplished successfully by the maintenance worker.

Thus, by using Equation (2.14) and the specified data values, we obtain

$$\begin{aligned} P(\bar{X}) &= 1 - P(X) \\ &= 1 - 0.9 \\ &= 0.1 \end{aligned}$$

and

$$\begin{aligned} P(\bar{Y}) &= 1 - P(Y) \\ &= 1 - 0.8 \\ &= 0.2 \end{aligned}$$

where $P(\bar{X})$ is the probability of not accomplishing step X correctly by the maintenance worker and $P(\bar{Y})$ is the probability of not accomplishing step Y correctly by the maintenance worker.

Using Equation (2.17) and the above calculated values, we get

$$\begin{aligned} P(\bar{X} + \bar{Y}) &= P(\bar{X}) + P(\bar{Y}) - P(\bar{X})P(\bar{Y}) \\ &= 0.1 + 0.2 - (0.1)(0.2) \\ &= 0.28 \end{aligned}$$

where $\bar{X} = X_1$ and $\bar{Y} = X_2$, and $P(\bar{X} + \bar{Y})$ is the probability of not performing steps X or Y correctly. Thus, the probability that the task will not be accomplished successfully by the maintenance worker is 0.28.

2.3 USEFUL DEFINITIONS

This section presents mathematical definitions that are considered useful for performing human reliability and error analysis in engineering maintenance.

2.3.1 Probability

This is expressed as [14]

$$P(Y) = \lim_{m \to \infty} \left(\frac{M}{m} \right) \tag{2.19}$$

where $P(Y)$ is the probability of occurrence of event Y and M is the total number of times that Y occurs in the m repeated experiments.

2.3.2 Cumulative Distribution Function Type I

For continuous random variables, this is defined by [14]

$$F(t) = \int_{-\infty}^{t} f(x)\,dx \tag{2.20}$$

where $f(t)$ is the probability density function (in human reliability work it is also known as the human error density function), t is the time-continuous random variable, and $F(t)$ is the cumulative distribution function.

2.3.3 Probability Density Function Type I

For continuous random variables, using Equation (2.20) this is expressed as follows:

$$\begin{aligned} \frac{d\ F(t)}{dt} &= \frac{d\left[\int_{-\infty}^{t} f(x)\,dx\right]}{dt} \\ &= f(t) \end{aligned} \tag{2.21}$$

2.3.4 Probability Density Function Type II

For a single-dimension discrete random variable, say X, the discrete probability density function of the random variable X is represented by $f(x_j)$ if the following conditions apply [12]:

$$f(x_j) \geq 0, \quad for\ \ all\ \ x_j \in R_x\ (range\ \ space), \tag{2.22}$$

and

$$\sum_{\substack{all \\ x_j}} f(x_j) = 1 \tag{2.23}$$

2.3.5 Cumulative Distribution Function Type II

For discrete random variables, the cumulative distribution function is expressed by [12]

$$F(x) = \sum_{x_j \leq x} f(x_j) \tag{2.24}$$

where $F(x)$ is the cumulative distribution function and its value is always $0 \leq F(x) \leq 1$.

2.3.6 Reliability Function

For continuous random variables, this is expressed by

$$R(t) = 1 - F(t) = 1 - \int_{-\infty}^{t} f(x)\,dx \tag{2.25}$$

where $f(x)$ is the failure/human error density function and $R(t)$ is the reliability function.

2.3.7 Hazard Rate Function

It is also known as the time-dependent failure/error rate function and is defined by

$$\lambda(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{R(t)} \tag{2.26}$$

where $\lambda(t)$ is the hazard rate function or the time-dependent failure/error rate function.

2.3.8 Expected Value Type I

The expected value, $E(t)$, of a continuous random variable is expressed by [12, 14]

$$E(t) = \mu = \int_{-\infty}^{\infty} t\, f(t)\,dt \tag{2.27}$$

where μ is the mean value. It is to be noted that in human reliability work, μ is called *mean time to human error*, and f(t) *human error density function.*

2.3.9 Expected Value Type II

The expected value, $E(x)$, of a discrete random variable x is defined by [12, 14]

$$E(x) = \sum_{j=1}^{k} x_j\, f(x_j) \tag{2.28}$$

where k is the number of discrete values of the random variable x.

2.3.10 Laplace Transform

The Laplace transform of the function $f(t)$ is defined by

$$F(s) = \int_{0}^{\infty} f(t)\, e^{-St}\,dt \tag{2.29}$$

where s is the Laplace transform variable, t is the time variable, and $F(s)$ is the Laplace transform of $f(t)$.

Example 2.3

Find the Laplace transform of the following function:

$$f(t) = C \tag{2.30}$$

where C is a constant.

Using the above function in Equation (2.29) yields

$$\begin{aligned} F(s) &= \int_0^\infty C e^{-St} dt \\ &= \frac{C e^{-St}}{-s}\Big|_0^\infty \\ &= \frac{C}{s} \end{aligned} \tag{2.31}$$

Example 2.4

Find the Laplace transform of the following function:

$$f(t) = e^{-\alpha t} \tag{2.32}$$

where α is a constant. In human reliability work, it is known as the human error rate.

By substituting Equation (2.32) into Equation (2.29) we get

$$\begin{aligned} F(s) &= \int_0^\infty e^{-\alpha t} e^{-St} dt \\ &= \frac{e^{-(S+\alpha)t}}{-(s+\alpha)}\Big|_0^\infty \\ &= \frac{1}{s+\alpha} \end{aligned} \tag{2.33}$$

Table 2.1 presents Laplace transforms of some commonly occurring functions in human reliability-related analysis [16, 17].

2.3.11 Laplace Transform: Final-Value Theorem

If the following limits exist, then the final-value theorem may be expressed as

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} [sF(s)] \tag{2.34}$$

TABLE 2.1
Laplace Transforms of Some Frequently Occurring Functions in Human Reliability-Related Analysis

No.	***F*(*t*)**	***F*(*s*)**
1	C, a constant	C/s
2	t^m, for $m = 0, 1, 2, 3, \ldots$	$m!/s^{m+1}$
3	$e^{-\alpha t}$	$1/(s+\alpha)$
4	$te^{-\alpha t}$	$1/(s+\alpha)^2$
5	$\dfrac{df(t)}{dt}$	$sF(s)-f(0)$
6	$tf(t)$	$-\dfrac{dF(s)}{ds}$
7	$\int_0^t f(t)\,dt$	$F(s)/s$
8	$\alpha f_1(t) + \beta f_2(t)$	$\alpha F_1(s) + \beta F_2(s)$
9	$t^{m-1}/(m-1)!$	$\dfrac{1}{s^m}$, $m = 1, 2, 3, \ldots$

2.4 PROBABILITY DISTRIBUTIONS

In human reliability-related analysis various types of discrete and continuous random variable probability distributions are used. Some examples of these distributions are binomial, Poisson, exponential, and normal distribution. This section presents probability distributions that are considered useful for application in performing human reliability and error analysis in engineering maintenance [18].

2.4.1 Poisson Distribution

The Poisson distribution is a discrete random variable distribution and is named after Simeon Poisson (1781–1840) [1]. The distribution is used in situations when one is concerned with the occurrence of a number of events that are of the same kind. The occurrence of an event is denoted as a point on a time scale, and in human reliability work an event denotes a human error. The distribution density function is expressed by

$$f(K) = \frac{(\alpha t)^K e^{-\alpha t}}{K!}, \quad \text{for } K = 0, 1, 2, 3, \ldots \tag{2.35}$$

where t is time and α is the constant arrival or error rate.

The cumulative distribution function, F, is

$$F = \sum_{j=0}^{K} [(\alpha t)^j e^{-\alpha t}]/j! \tag{2.36}$$

The distribution mean is given by [15, 18]

$$\mu_p = \alpha t \tag{2.37}$$

where μ_p is the mean of the Poisson distribution.

2.4.2 Binomial Distribution

This is another discrete random variable distribution. The distribution is also known as the Bernoulli distribution, after Jakob Bernoulli (1654–1705), its originator [1]. The distribution is used in situations when one is interested in the probability of outcome such as the total number of errors/failures in a sequence of, say K, trials. The distribution is based on the condition that each trial has two possible outcomes (e.g., success and failure), and each trial's probability remains constant.

The distribution probability density function, $f(x)$, is defined by

$$f(x) = \binom{K}{j} p^x q^{K-x}, \quad \text{for } x = 0, 1, 2, 3, \ldots, K. \tag{2.38}$$

where

$$\binom{K}{j} = \frac{K!}{j!(K-j)!}$$

x is the total number of failures/errors in K trials, q is the probability of failure of a single trial, and p is the probability of success of a single trial.

The cumulative distribution function is given by

$$F(x) = \sum_{j=0}^{x} \binom{K}{j} p^j q^{K-j} \tag{2.39}$$

where $F(x)$ is the probability of x or less failures (errors) in K trials.

The distribution mean is given by [18]

$$\mu_b = Kp \tag{2.40}$$

where μ_b is the mean of the binomial distribution.

2.4.3 Geometric Distribution

This discrete random variable distribution is based on the same assumptions as the binomial distribution, except that the number of trials is not fixed. More specifically, all trials are independent and identical and each can result in one of the two possible outcomes (i.e., a success or a failure (error)). The distribution probability density function, $f(x)$, is defined by [13, 19]

$$f(x) = pq^{x-1}, \quad \text{for } x = 1, 2, 3, \ldots \tag{2.41}$$

The cumulative distribution function is given by

$$F(x) = \begin{cases} 0, & x \langle 1 \\ \sum_{x_j \le [x]} pq^{x_j - 1}, & x \ge 1 \end{cases} \tag{2.42}$$

The distribution mean is given by

$$\mu_g = \frac{1}{p} \tag{2.43}$$

where μ_g is the mean of the geometric distribution.

2.4.4 Exponential Distribution

This is probably the most widely used continuous random variable probability distribution in performing reliability studies, because many engineering parts exhibit constant failure rate during their useful life period [20].

The distribution probability density function is defined by

$$f(t) = \lambda e^{-\lambda t} \quad t \ge 0, \lambda \rangle 0 \tag{2.44}$$

where $f(t)$ is the probability density function (in reliability work, it is also called failure density function or error density function), λ is the distribution parameter (in human reliability work, it is known as the constant human error rate), and t is time.

Using Equations (2.20) and (2.44), we obtain the following expression for the cumulative distribution function:

$$\begin{aligned} F(t) &= \int_0^t \lambda e^{-\lambda t}\, dt \\ &= 1 - e^{-\lambda t} \end{aligned} \tag{2.45}$$

By substituting Equation (2.44) into Equation (2.27), we get the following expression for the distribution expected or mean value:

$$\begin{aligned} E(t) = \mu &= \int_0^\infty t\lambda e^{-\lambda t}\, dt \\ &= \frac{1}{\lambda} \end{aligned} \tag{2.46}$$

Example 2.5

Assume that the constant error rate of maintenance personnel in performing a certain maintenance task is 0.009 errors/hour. Calculate the probability that the maintenance personnel will make an error during an 8-hour mission.

By substituting the specified data values into Equation (2.45), we get

$$F(8) = 1 - e^{-(0.009)(8)} = 0.0695$$

Thus, the probability that the maintenance personnel will make an error during the specified time period is 0.0695.

2.4.5 Normal Distribution

This is a widely used continuous random variable probability distribution and sometimes it is also called the Gaussian distribution after Carl Friedrich Gauss (1777–1855), the German mathematician. The probability density function of the distribution is defined by

$$f(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(t-\mu)^2}{2\sigma^2}\right], \; -\infty \langle t \langle +\infty \quad (2.47)$$

where μ and σ are the distribution parameters (i.e., mean and standard deviation, respectively).

Substituting Equation (2.47) into Equation (2.20), we obtain the following cumulative distribution function:

$$F(t) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx \quad (2.48)$$

Using Equation (2.47) in Equation (2.27), we obtain the following expression for the distribution expected or mean value:

$$E(t) = \mu \quad (2.49)$$

2.4.6 Gamma Distribution

This is another continuous random variable probability distribution and is quite flexible in fitting a wide range of problems including human errors. The distribution probability density function is defined by

$$f(t) = \frac{\lambda(\lambda t)^{\alpha-1} e^{-\lambda t}}{\Gamma(x)}, \quad t \geq 0, \lambda, \alpha \rangle 0 \quad (2.50)$$

where $\Gamma(\cdot)$ is the gamma function, λ is the distribution scale parameter, and α is the distribution shape parameter.

Using Equation (2.50) in Equation (2.20), we get the following cumulative distribution function:

$$F(t) = 1 - \sum_{j=0}^{\alpha-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!} \quad (2.51)$$

Substituting Equation (2.50) into Equation (2.27), we get the following expression for the distribution expected or mean value:

$$E(t) = \frac{\alpha}{\lambda} \tag{2.52}$$

It is to be noted that at $\alpha = 1$, the gamma distribution becomes the exponential distribution.

2.4.7 Rayleigh Distribution

This continuous random variable probability distribution is often used in the theory of sound and in reliability studies and is known after John Rayleigh (1842–1919), its originator [1]. The distribution probability density function is defined by

$$f(t) = \frac{2}{\beta^2} t e^{-\left(\frac{t}{\beta}\right)^2}, \quad t \geq 0, \beta \rangle 0 \tag{2.53}$$

where β is the distribution parameter.

Substituting Equation (2.53) into Equation (2.20), we obtain the following cumulative distribution function:

$$F(t) = 1 - e^{-\left(\frac{t}{\beta}\right)^2} \tag{2.54}$$

Using Equation (2.53) in Equation (2.27), we obtain the following equation for the distribution expected or mean value:

$$E(t) = \beta \Gamma\left(\frac{3}{2}\right) \tag{2.55}$$

where $\Gamma(\cdot)$ is the gamma function and is defined by

$$\Gamma(y) = \int_0^\infty t^{y-1} e^{-t} dt, \quad for\ y \rangle 0 \tag{2.56}$$

2.4.8 Weibull Distribution

This continuous random variable probability distribution can be used to represent many different physical phenomena and it was developed by W. Weibull, a Swedish mechanical engineering professor, in the early 1950s [21]. The distribution probability density function is expressed by

$$f(t) = \frac{\theta t^{\theta-1}}{\beta^\theta} e^{-\left(\frac{t}{\beta}\right)^\theta}, \quad t \geq 0, \theta, \beta \rangle 0 \tag{2.57}$$

where θ is the distribution shape parameter and β is the distribution scale parameter.

Using Equations (2.57) and (2.20), we get the following cumulative distribution function:

$$F(t) = 1 - e^{-\left(\frac{t}{\beta}\right)^{\theta}} \tag{2.58}$$

Substituting Equation (2.57) into Equation (2.27), we get the following expression for the distribution expected or mean value:

$$E(t) = \beta \Gamma\left(1 + \frac{1}{\theta}\right) \tag{2.59}$$

It is to be noted that for $\theta = 1$ and 2, the exponential and Rayleigh distributions are the special cases of the Weibull distribution, respectively.

2.5 SOLVING FIRST-ORDER DIFFERENTIAL EQUATIONS USING LAPLACE TRANSFORMS

Sometime in human reliability and error studies, the Markov method (described in Chapter 4), is used and it results in a system of linear first-order differential equations. The use of Laplace transforms is considered to be an effective approach to find solutions to these differential equations. The following example demonstrates the application of Laplace transforms in finding the solutions to a set of linear first-order differential equations.

Example 2.6

Assume that an engineering system can either be in three states: operating normally, failed due to hardware problems, or failed due to maintenance errors. The following set of differential equations describes the system:

$$\frac{dP_0(t)}{dt} + (\lambda + \lambda_m) P_0(t) = 0 \tag{2.60}$$

$$\frac{dP_1(t)}{dt} - \lambda P_0(t) = 0 \tag{2.61}$$

$$\frac{dP_2(t)}{dt} - \lambda_m P_0(t) = 0 \tag{2.62}$$

At $t = 0$, $P_0(0) = 1$, and $P_1(0) = P_2(0) = 0$, where $P_i(t)$ is the probability that the engineering system is in state i at time for $i = 0$ (operating normally), $i = 1$ (failed due to hardware problems), $i = 2$ (failed due to maintenance errors); λ is the constant failure rate of the system due to hardware problems; and λ_m is the constant failure rate of the system due to maintenance errors.

Find solutions to differential Equations (2.60)–(2.62) by using Laplace transforms.

Thus, taking Laplace transforms of Equations (2.60)–(2.62), using the given initial conditions, and then solving the resulting equations, we get

$$P_0(s) = \frac{1}{(s+\lambda+\lambda_m)} \tag{2.63}$$

$$P_1(s) = \frac{\lambda}{s(s+\lambda+\lambda_m)} \tag{2.64}$$

$$P_2(s) = \frac{\lambda_m}{s(s+\lambda+\lambda_m)} \tag{2.65}$$

where s is the Laplace transform variable, and $P_i(s)$ is the Laplace transform of the probability that the engineering system is in state i at time t, for $i = 0, 1, 2$.

By taking the inverse Laplace transforms of Equations (2.63)–(2.65), we obtain

$$P_0(t) = e^{-(\lambda+\lambda_m)t} \tag{2.66}$$

$$P_1(t) = \frac{\lambda}{\lambda+\lambda_m}(1-e^{-(\lambda+\lambda_m)t}) \tag{2.67}$$

$$P_2(t) = \frac{\lambda_m}{\lambda+\lambda_m}(1-e^{-(\lambda+\lambda_m)t}) \tag{2.68}$$

Thus, Equations (2.66)–(2.68) represent solutions to differential Equations (2.60)–(2.62).

2.6 PROBLEMS

1. Prove Equation (2.6).
2. Assume that a maintenance worker is performing a maintenance task composed of three independent steps: steps X, Y, and Z. The task will be accomplished correctly only if all the steps are performed correctly. The probabilities of performing steps X, Y, and Z correctly by the maintenance worker are 0.95, 0.75, and 0.99, respectively. Calculate the probability of accomplishing the task correctly by the maintenance worker.
3. In the above Problem No. 2, by using Equations (2.14) and (2.16) calculate the probability that the task will not be accomplished successfully by the maintenance worker.
4. Define *probability* mathematically.
5. Take the Laplace transform of the following function:

$$f(t) = te^{-\lambda t} \tag{2.69}$$

where λ is a constant and t is a time variable.

6. Obtain an expression for the hazard rate by using the following failure density function.

$$f(t) = \lambda e^{-\lambda t} \qquad t \geq 0,\ \lambda \rangle 0 \tag{2.70}$$

where λ is the distribution parameter and t is time.
7. Prove Equation (2.46).
8. What are the special case probability distributions of the Weibull distribution?
9. Assume that the constant error rate of maintenance personnel in performing a certain task is 0.001 errors/hour. Calculate the probability that the maintenance personnel will not make an error during a 6-hour mission.
10. Prove that the sum of Equations (2.62)–(2.65) is equal to $1/S$. Comment on the end result.

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